Introduction to Computer Vision

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Motion estimation

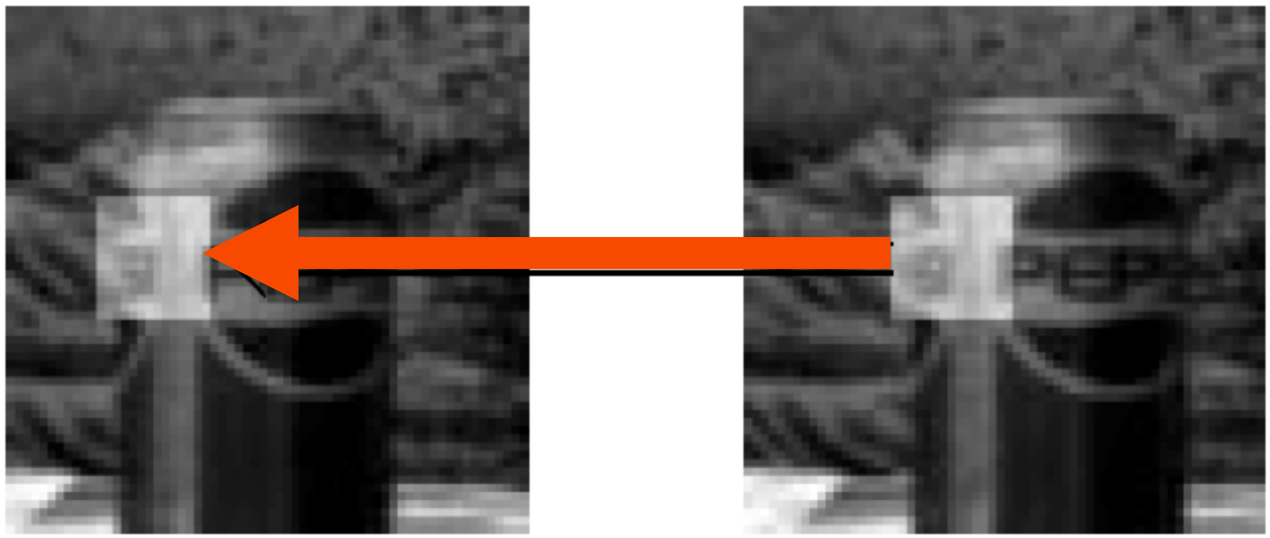
Goals

- Motion estimation
  - Brightness constancy
  - SSD matching
  - Optical flow constraint equation
  - Aperture problem
  - Spatial coherence and parametric motion
  - Optimization

Readings

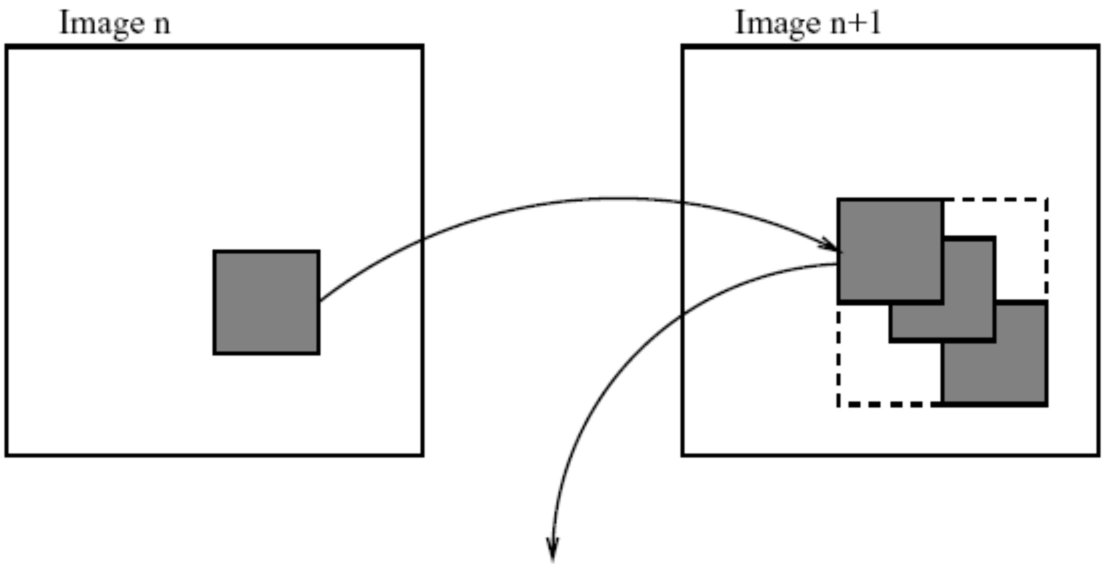
- Szeliksi: 8.1, 8.2, 8.4

Brightness Constancy Assumption

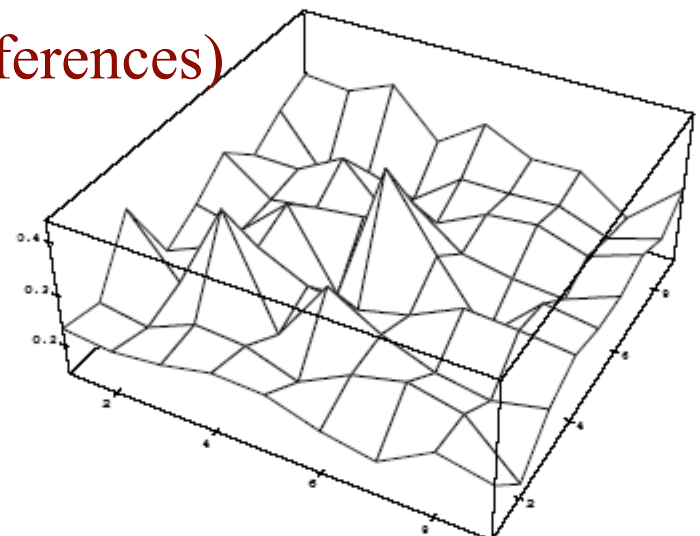


$$I(x+u, y+v, t+1) = I(x, y, t)$$

(assumption)



SSD surface
(sum of squared differences)



$$E_{SSD}(u,v) = \sum_{x,y \in R} (I(x+u, y+v, t+1) - I(x,y,t))^2$$

How can we optimize over u,v?

$$E_{SSD}(u,v) = \sum_{x,y \in R} \left(I(x+u, y+v, t+1) - I(x,y,t) \right)^2$$

How do we find the minimum of $E_{SSD}(u,v)$?

What is the problem with the equation in this form?

Brightness Constancy

$$E_{SSD}(u,v) = \sum_{x,y \in R} (I(x+u, y+v, t+1) - I(x,y,t))^2$$

$$\cancel{I(x,y,t)} + dx\frac{\partial}{\partial x}I(x,y,t) + dy\frac{\partial}{\partial y}I(x,y,t) + dt\frac{\partial}{\partial t}I(x,y,t) - \cancel{I(x,y,t)} = 0$$

Divide through by *dt*

$$u\frac{\partial}{\partial x}I(x,y,t) + v\frac{\partial}{\partial y}I(x,y,t) + \frac{\partial}{\partial t}I(x,y,t) = 0$$

Approximating SSD

$$E_{SSD}(u,v) = \sum_{x,y \in R} (I(x+u, y+v, t+1) - I(x,y,t))^2$$

$$I_x(x,y,t)u + I_y(x,y,t)v + I_t(x,y,t)$$

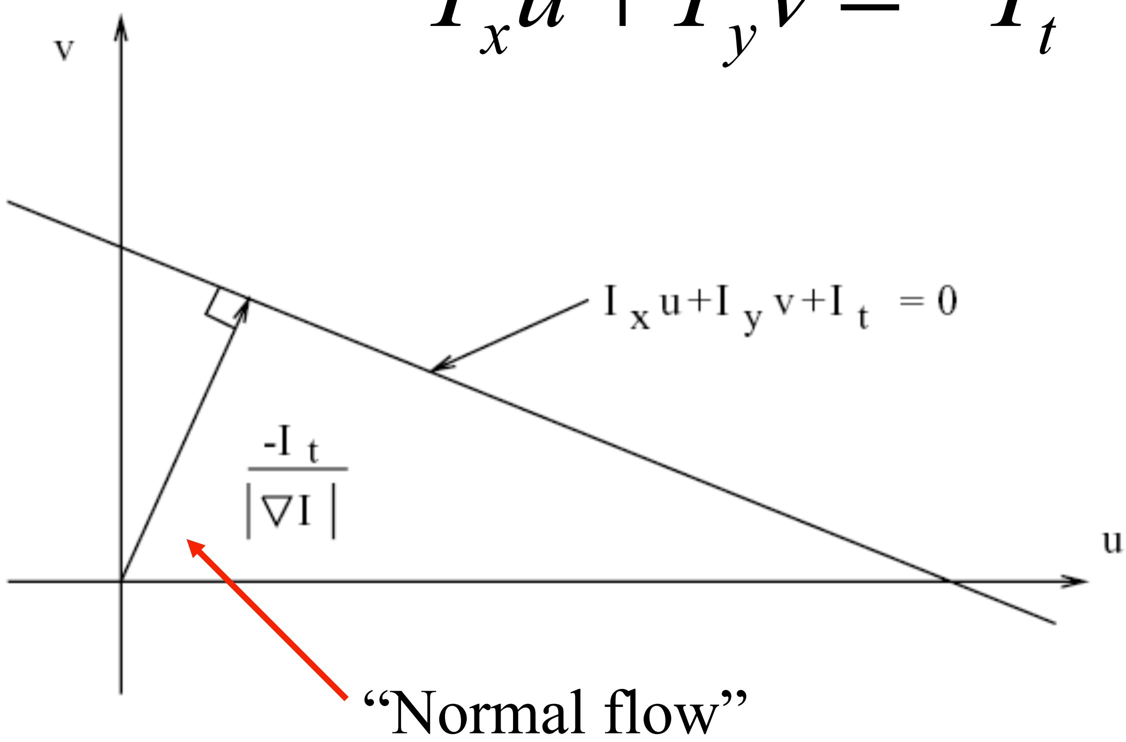
"Optical flow constraint equation"

$$I_x u + I_y v + I_t = 0$$

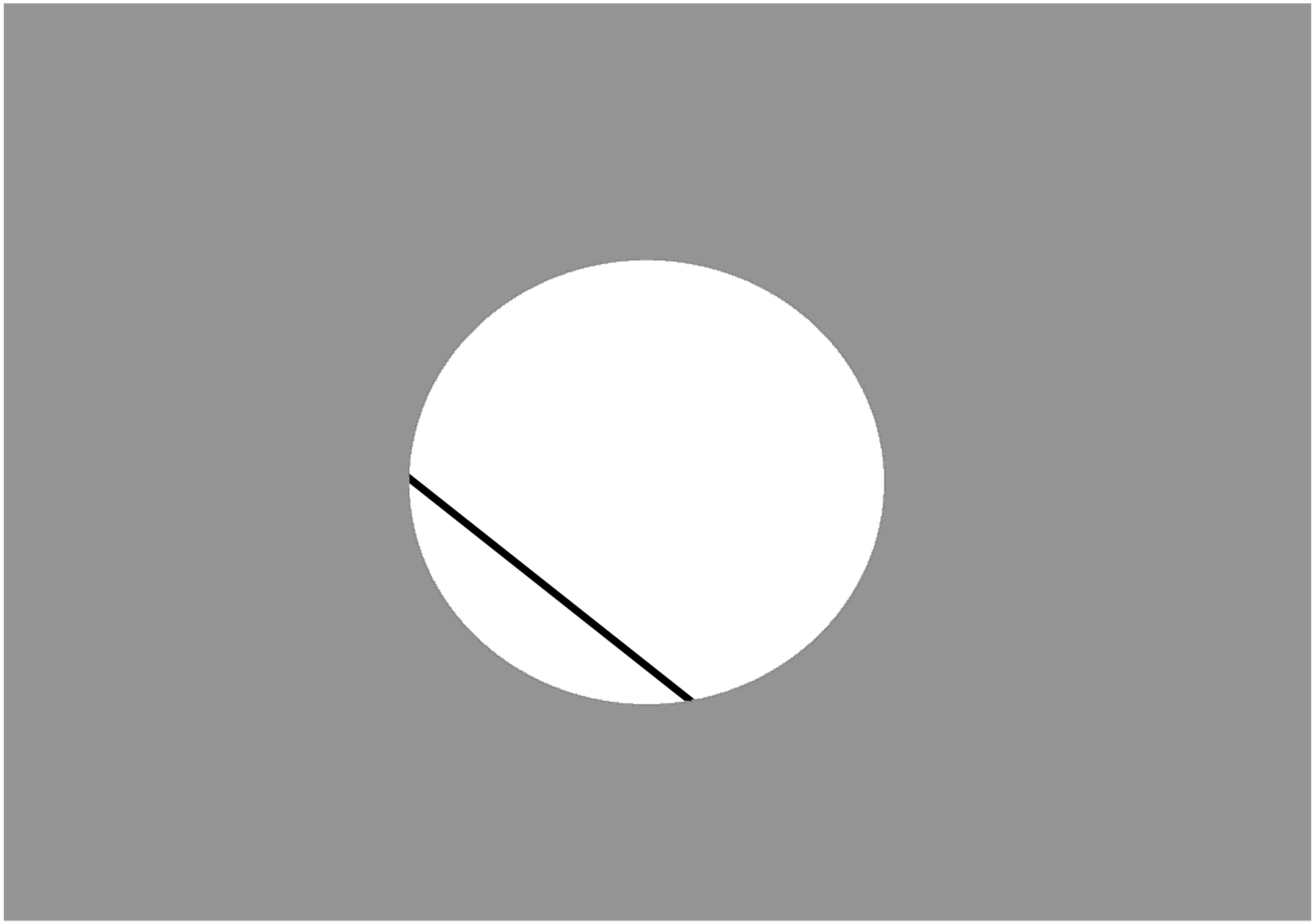
OFCE

At a single image pixel, we get a line:

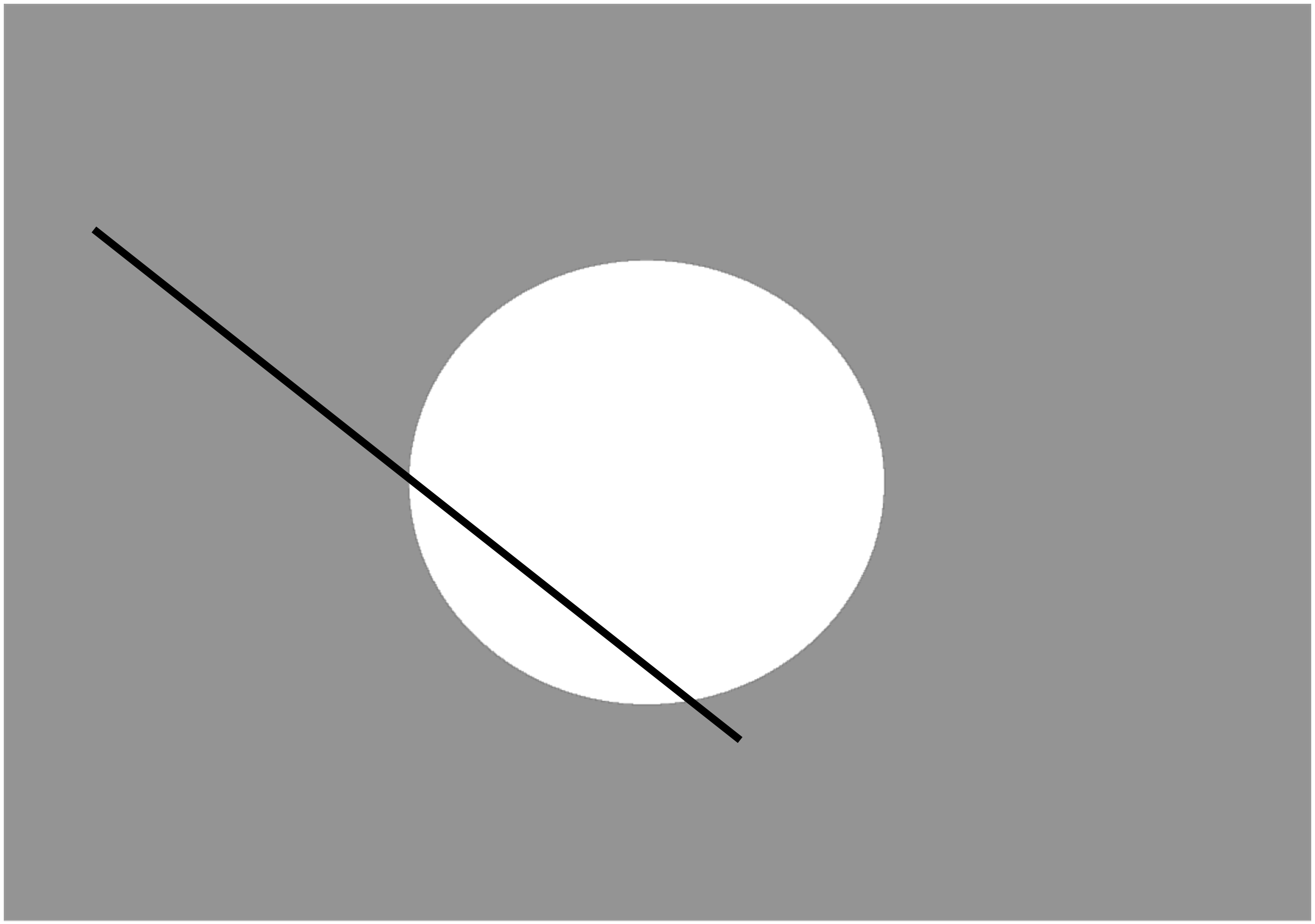
$$I_x u + I_y v = -I_t$$



Aperture Problem



Aperture Problem



Aperture problem

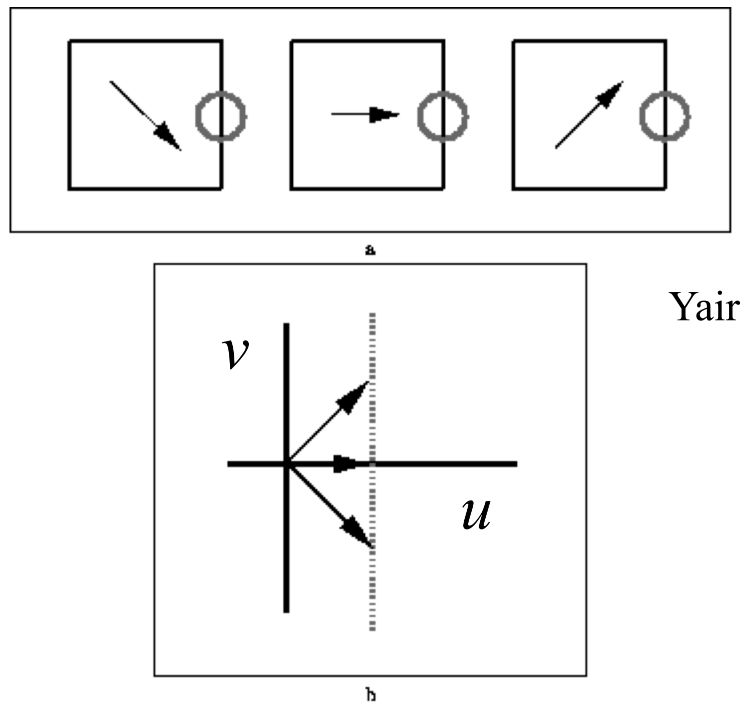
$$I_x u + I_y v = -I_t$$

$$I_x u + 0v = -I_t$$

$$I_x u = -I_t$$

$$u = -I_t / I_x$$

v could be anything



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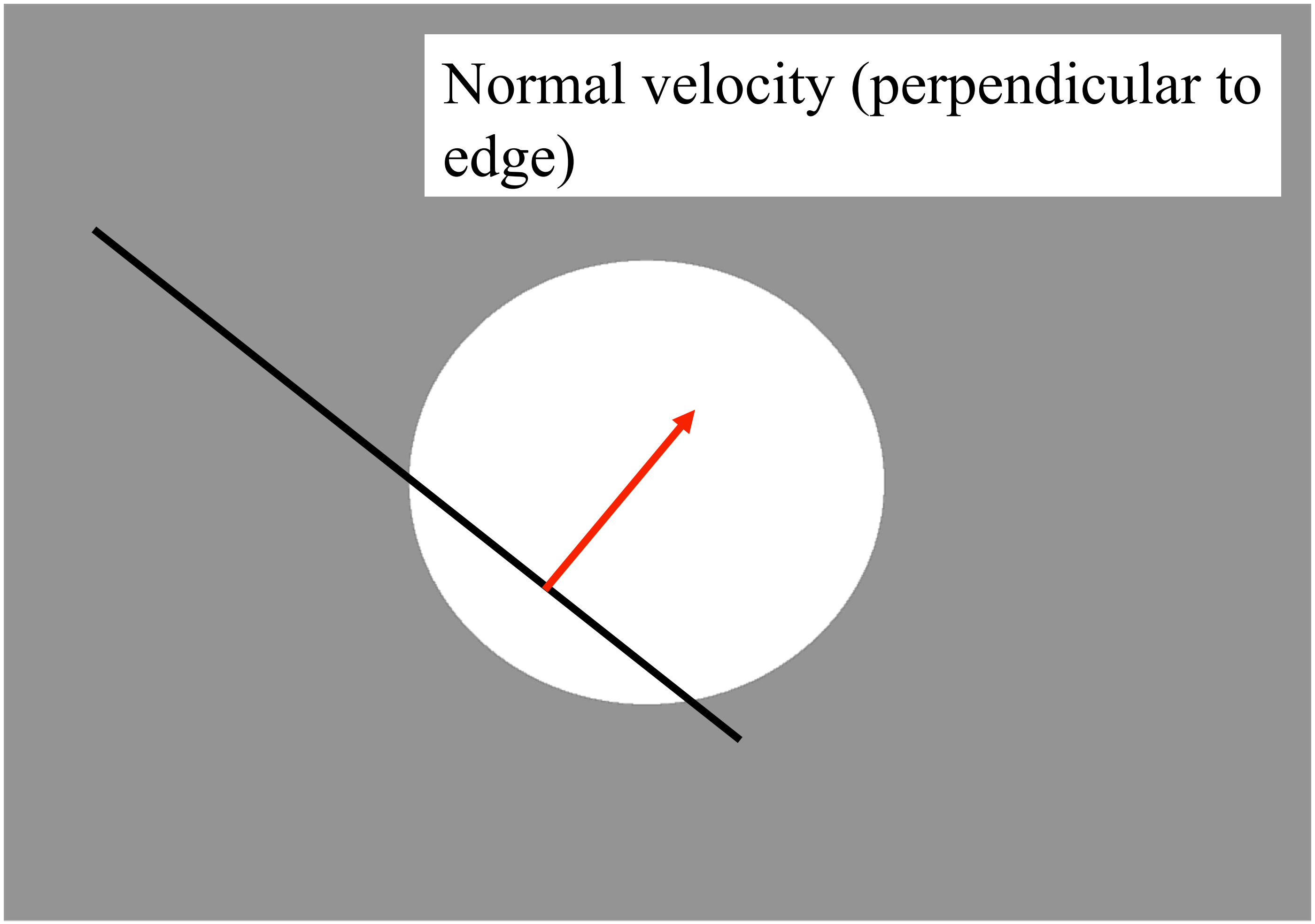
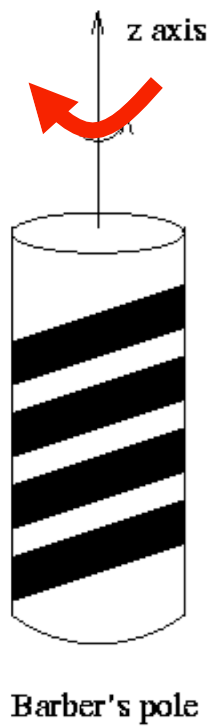
Aperture Problem



Aperture Problem

Barber pole illusion

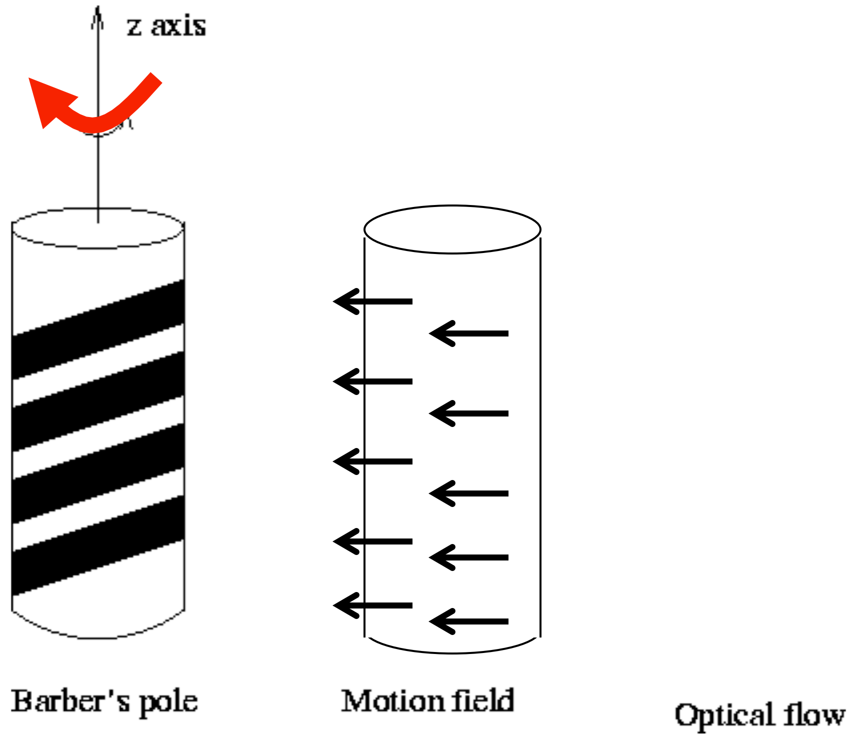


What is the motion field?

http://www.dai.ed.ac.uk/CVonline/LOCAL_COPIES/OWENS/LECT12/node4.html

Aperture Problem

Barber pole illusion



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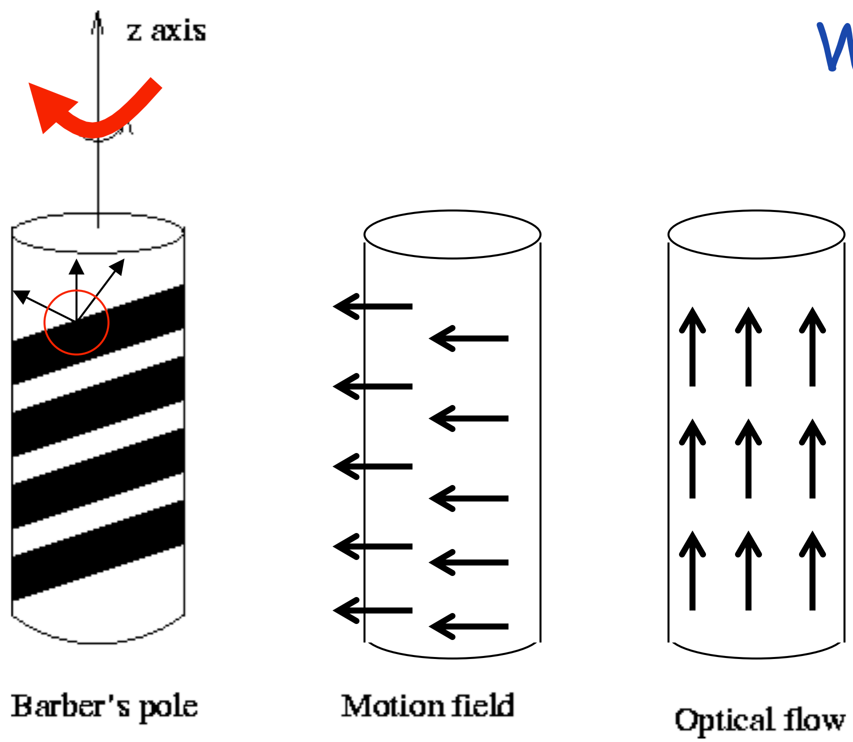
Aperture Problem

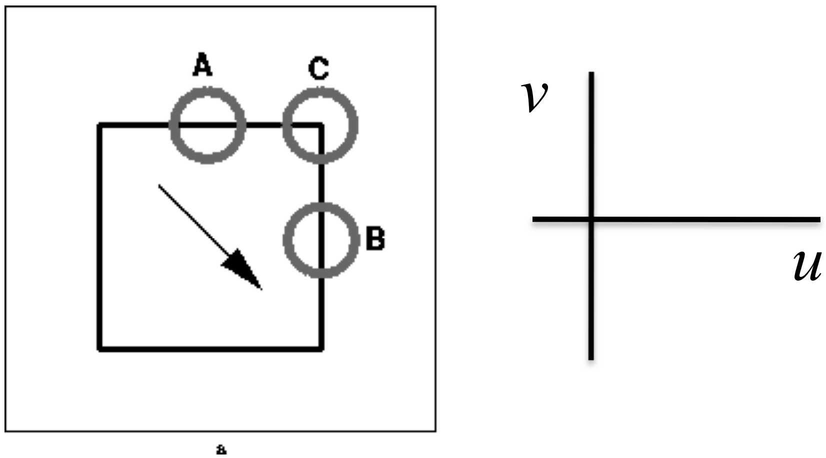
Barber pole illusion



Why?

http://www.dai.ed.ac.uk/CVonline/LOCAL_COPIES/OWENS/LECT12/node4.html

Multiple constraints



What are the constraint lines?

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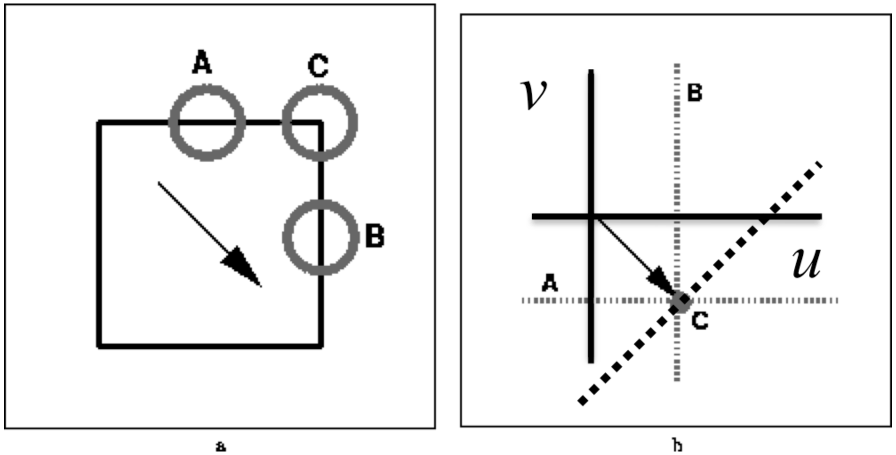
Multiple constraints



Combine constraints to get an estimate of velocity.

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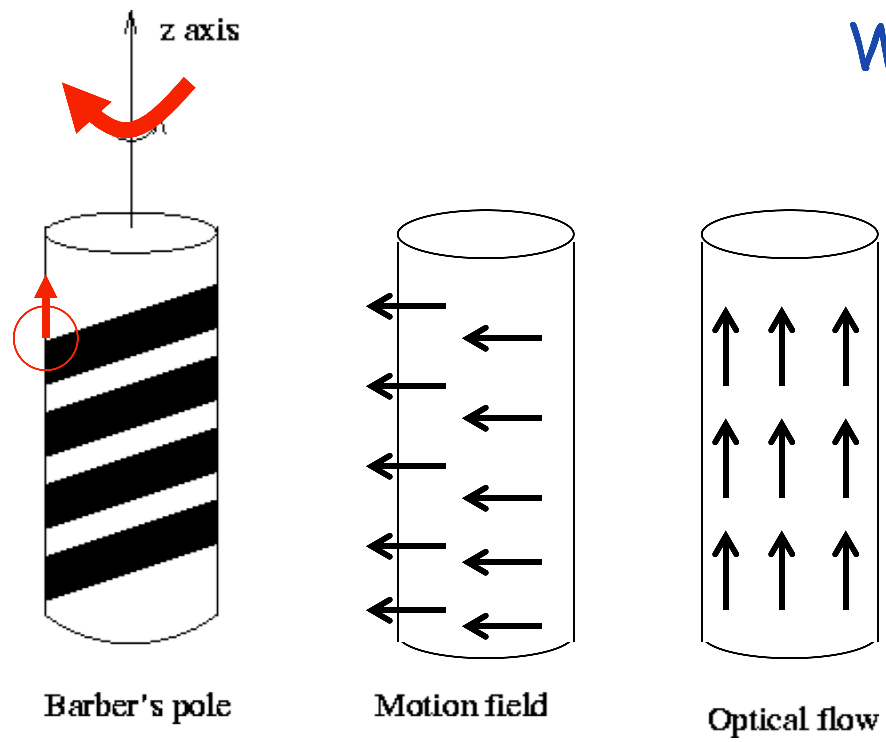
Aperture Problem

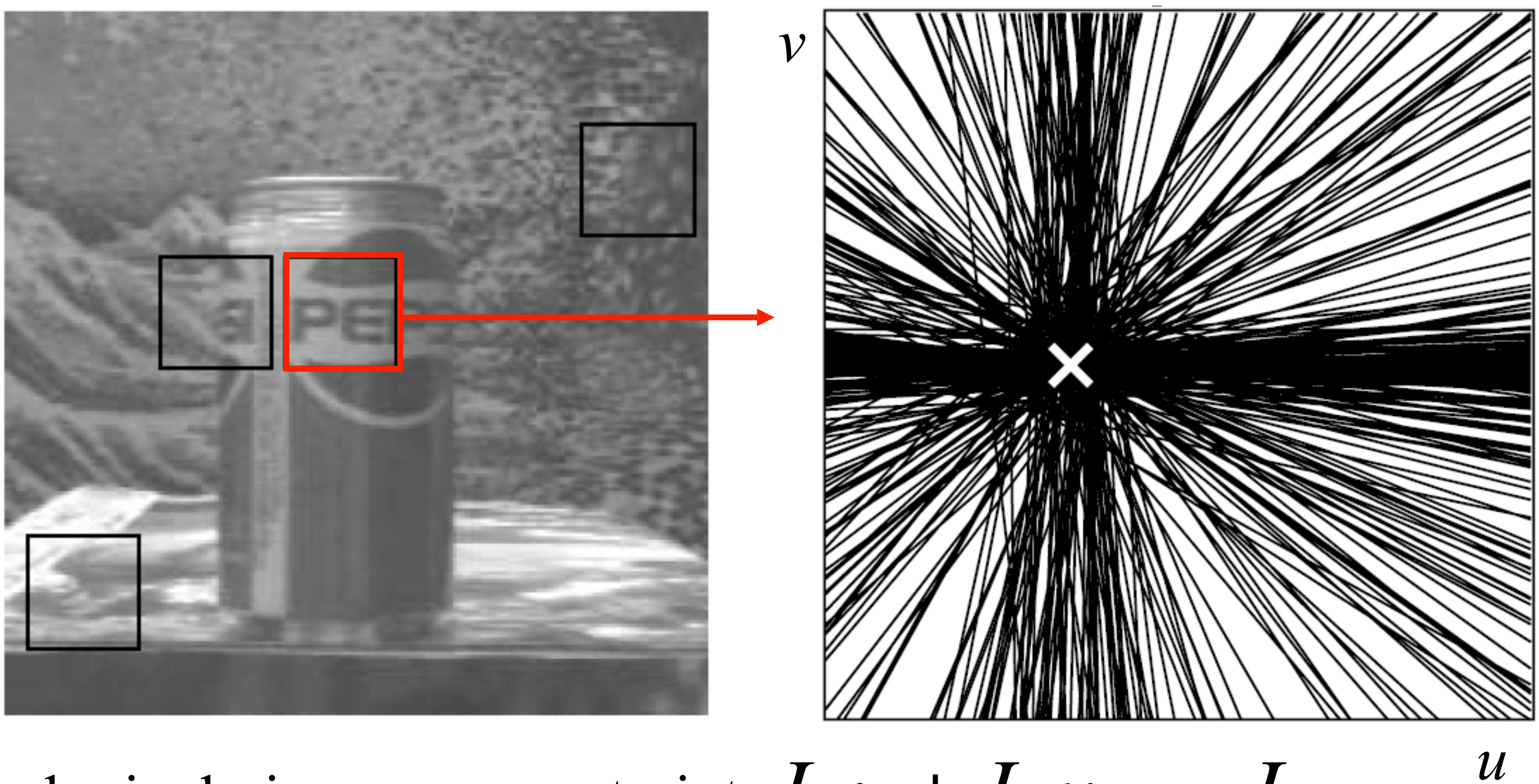
Barber pole illusion



Why?

http://www.dai.ed.ac.uk/CVonline/LOCAL_COPIES/OWENS/LECT12/node4.html

Multiple Constraints



Each pixel gives us a constraint: $I_x u + I_y v = -I_t$

Area-Based Flow Estimation

Spatial smoothness assumption

The flow is the same at every pixel in some neighborhood *R*

$$E(u,v) = \sum_{x,y \in R} (I_x(x,y,t)u + I_y(x,y,t)v + I_t(x,y,t))^2$$

How do we solve for *u* and *v*?

Optimization

$$E(u,v) = \sum_{x,y \in R} (I_x(x,y,t)u + I_y(x,y,t)v + I_t(x,y,t))^2$$

Differentiate with respect to *u* and *v* and set this to zero.

$$\frac{\partial E}{\partial u} = \sum_R (I_x u + I_y v + I_t) I_x = 0$$

$$\frac{\partial E}{\partial v} = \sum_R (I_x u + I_y v + I_t) I_y = 0$$

Optimization

$$\frac{\partial E}{\partial u} = \sum_R (I_x u + I_y v + I_t) I_x = 0$$

$$\frac{\partial E}{\partial v} = \sum_R (I_x u + I_y v + I_t) I_y = 0$$

Rearranging the terms into something simple to solve?
Gather terms.

Optimization

$$\frac{\partial E}{\partial u} = \sum_R (I_x u + I_y v + I_t) I_x = 0$$

$$\frac{\partial E}{\partial v} = \sum_R (I_x u + I_y v + I_t) I_y = 0$$

Rearranging the terms:

$$\left[\sum_R I_x^2\right] u + \left[\sum_R I_x I_y\right] v = -\sum_R I_x I_t$$

$$\left[\sum_R I_x I_y\right] u + \left[\sum_R I_y^2\right] v = -\sum_R I_y I_t$$

Optimization

$$\left[\sum_R I_x^2\right]u + \left[\sum_R I_x I_y\right]v = -\sum_R I_x I_t$$

$$\left[\sum_R I_x I_y\right]u + \left[\sum_R I_y^2\right]v = -\sum_R I_y I_t$$

Can I rewrite this?

Optimization

$$\left[\sum_R I_x^2\right]u + \left[\sum_R I_x I_y\right]v = -\sum_R I_x I_t$$

$$\left[\sum_R I_x I_y\right]u + \left[\sum_R I_y^2\right]v = -\sum_R I_y I_t$$

System of 2 equations in 2 unknowns:

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$$

Symmetric positive definite

Positive definite

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} > 0$$

for all non-zero real vectors $[u\ v]^T$

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ua + vb & ub + vc \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} =$$

$$u^2 a + v^2 b + u^2 b + v^2 c$$

Look familiar?

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$$

This is just the structure tensor from assignment 1!

The eigenvalues tell us about the local image structure.

They also tell us how well we can estimate the flow in both directions

Look familiar?

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$$

$$\nabla I = \begin{bmatrix} I_x \\ I_y \end{bmatrix} \qquad \sum \nabla I \nabla I^T$$

Optimization: solve for u,v

System of 2 equations in 2 unknowns:

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$$

$$\mathbf{Au} = \mathbf{b}$$

(Very very useful! Template for solving many problems.)

Algorithm

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \quad \nabla I = \begin{bmatrix} I_x \\ I_y \end{bmatrix}$$

$$\mathbf{u} = -\left(\sum \nabla I \nabla I^T\right)^{-1} \sum \nabla I I_t$$

For this to work, the structure tensor must be invertible.

Solving for **u**

$$\left(\sum \nabla I \nabla I^T\right)\mathbf{u} = -\sum \nabla I I_t$$

What happens if

* the region is homogeneous?
* there is a single edge?
* a corner
* eigenvalues

Rank of A < 2. ie, 1 or more eigenvalues = 0.

Pseudo-inverse

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

$$\mathbf{A}^T\mathbf{A}\mathbf{u} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{u} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

Pseudo-inverse

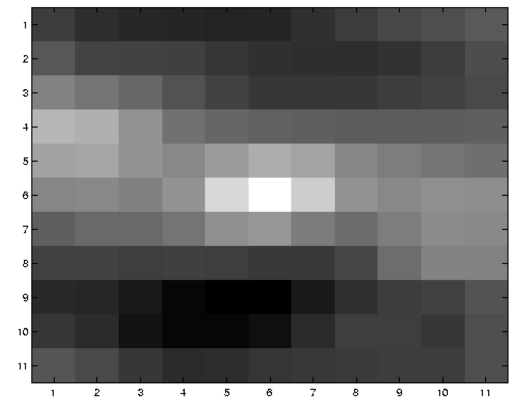
$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

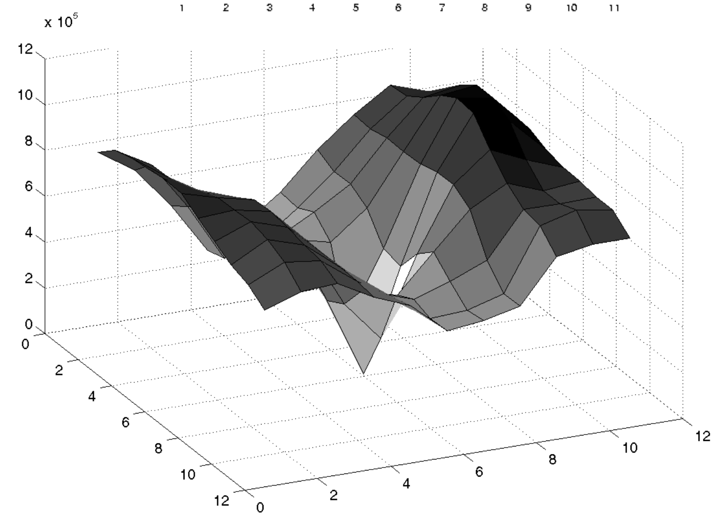
$$\mathbf{A}^T\mathbf{A}\mathbf{u} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{u} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

pinv(A) in Matlab

SSD Surface – Textured area

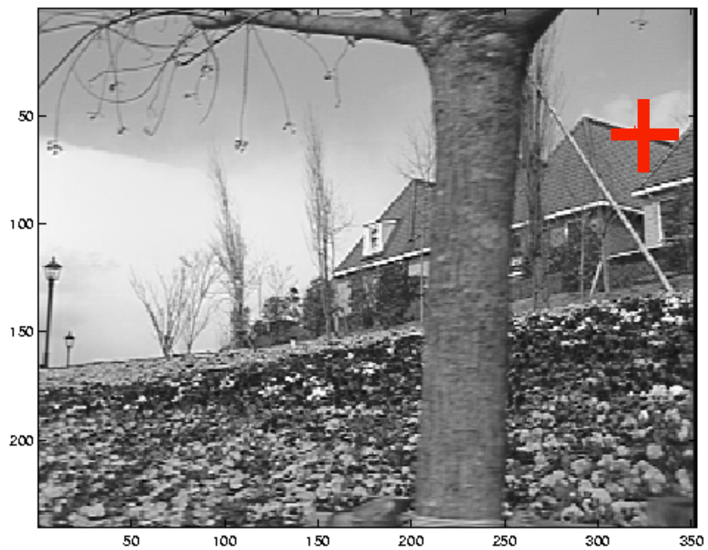


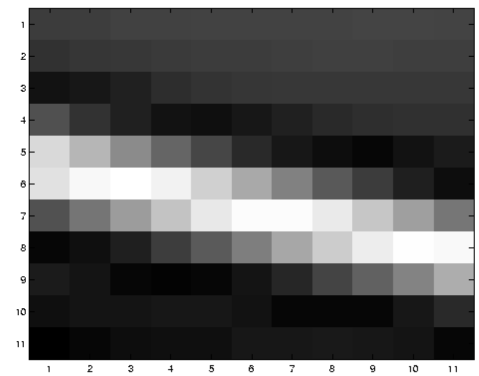


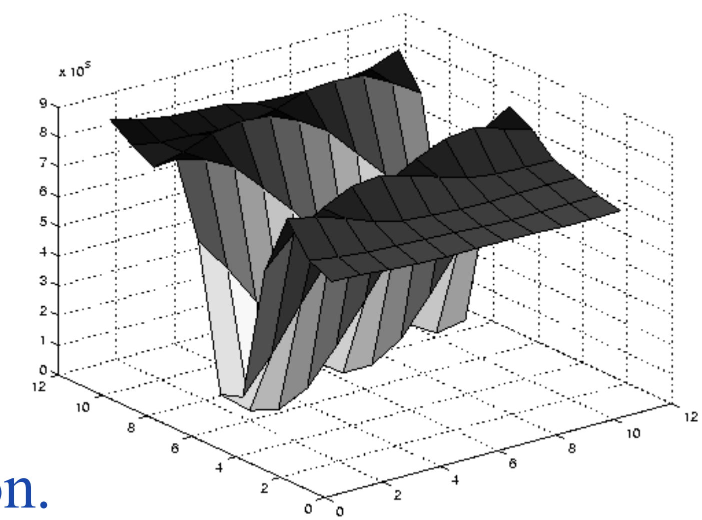
$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix}$$

Gradients in *x* and *y*.

SSD Surface -- Edge



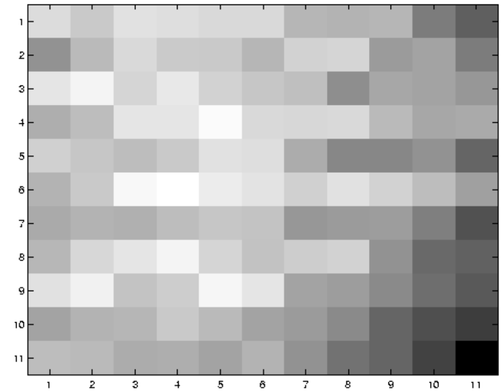


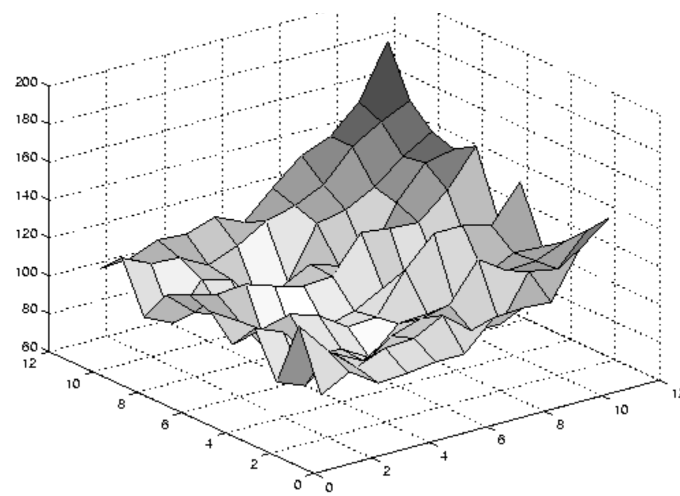


$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix}$$

Gradients oriented in one direction.

SSD Surface – homogeneous area





$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix}$$

Weak gradients everywhere.

SSD Surface – Surface Boundary

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_y I_x & \sum I_y^2 \end{bmatrix}$$

Gradients in *x* and *y*.

?

Translational Model

What's wrong with the translational assumption (ie constant motion within a region R)?

How can we generalize it?