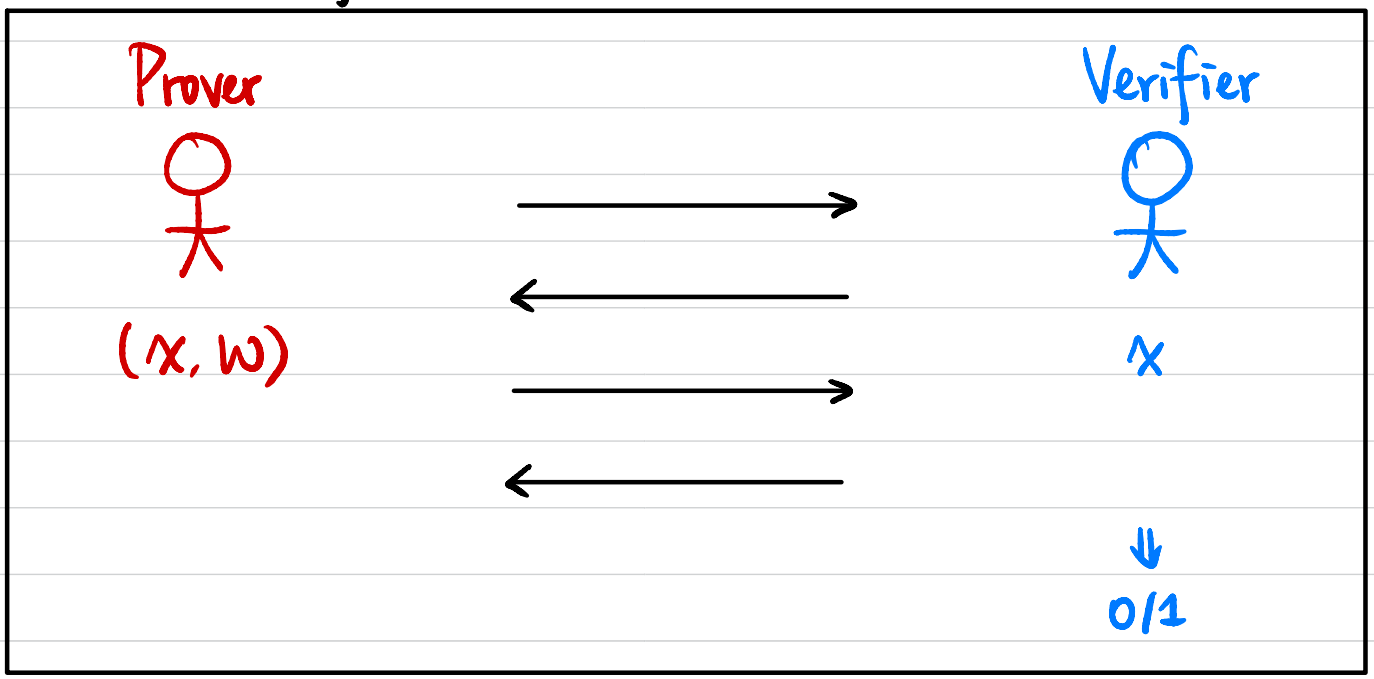
CSCI 1510

- Perfect ZKP for Diffie-Hellman Tuples (continued)
- Commitment Schemes
- ZKP for All NP
- Non-Interactive Zero-Knowledge Proofs

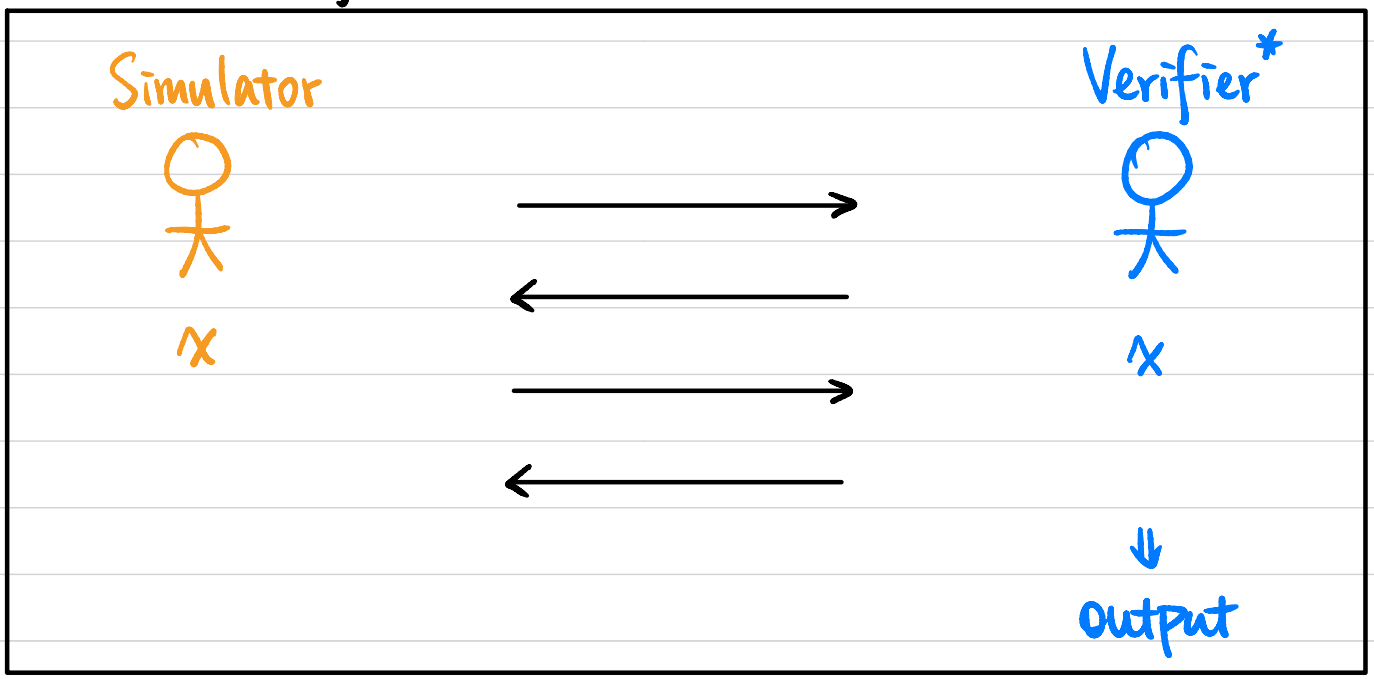
Zero-Knowledge Proof (ZKP)



Let (P, V) be a pair of PPT interactive machines. (P, V) is a **zero-knowledge proof system** for a language L with associated relation R_L if

- **Completeness:** $\forall (x, w) \in R_L, \quad \Pr[\, P(x,w) \longleftrightarrow V(x) \text{ outputs } 1 \,] = 1.$
- **Soundness:** $\forall x \notin L, \quad \forall \underset{\text{argument}}{(\text{PPT})} P^*, \quad \Pr[\, P^*(x) \longleftrightarrow V(x) \text{ outputs } 1 \,] \leq \text{negl}(n).$
- **Zero-Knowledge?**

Zero-Knowledge Proof (ZKP)



- **Zero-Knowledge:** $\forall PPT\ V^*,\ \exists PPT\ S\ \ s.t.\ \ \forall (x,w) \in R_L,$

$$\text{Output}_{V^*}[P(x,w) \longleftrightarrow V^*(x)] \approx S(x)$$

↑

perfect / statistical / computational

$\equiv$ $\quad$ $\overset{s}{\approx}$ $\quad$ $\overset{c}{\approx}$

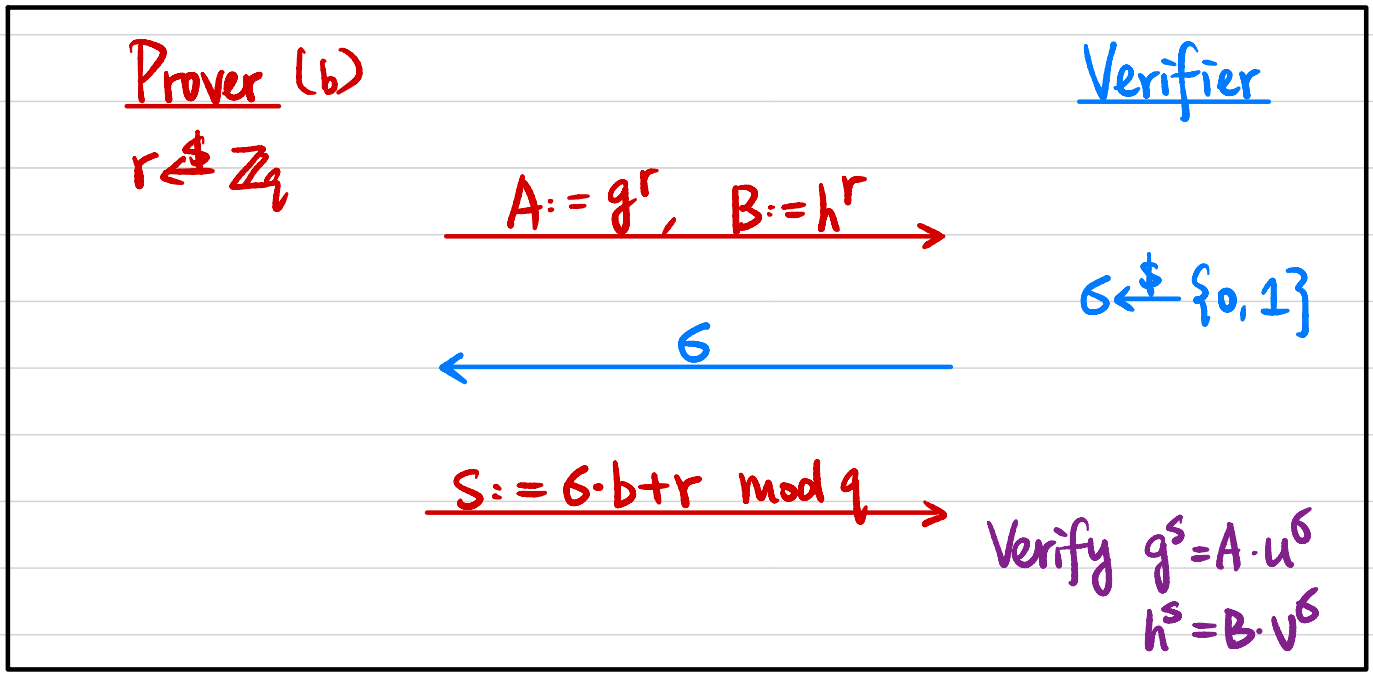
Perfect ZKP for Diffie-Hellman Tuples

Input: Cyclic group G of order q, generator g, h, u, v

$h = g^a$, $u = g^b$, $v = g^{ab}$

Witness: b

Statement: $\exists\, b \in \mathbb{Z}_q$ s.t. $u = g^b \wedge v = h^b$



Completeness ?

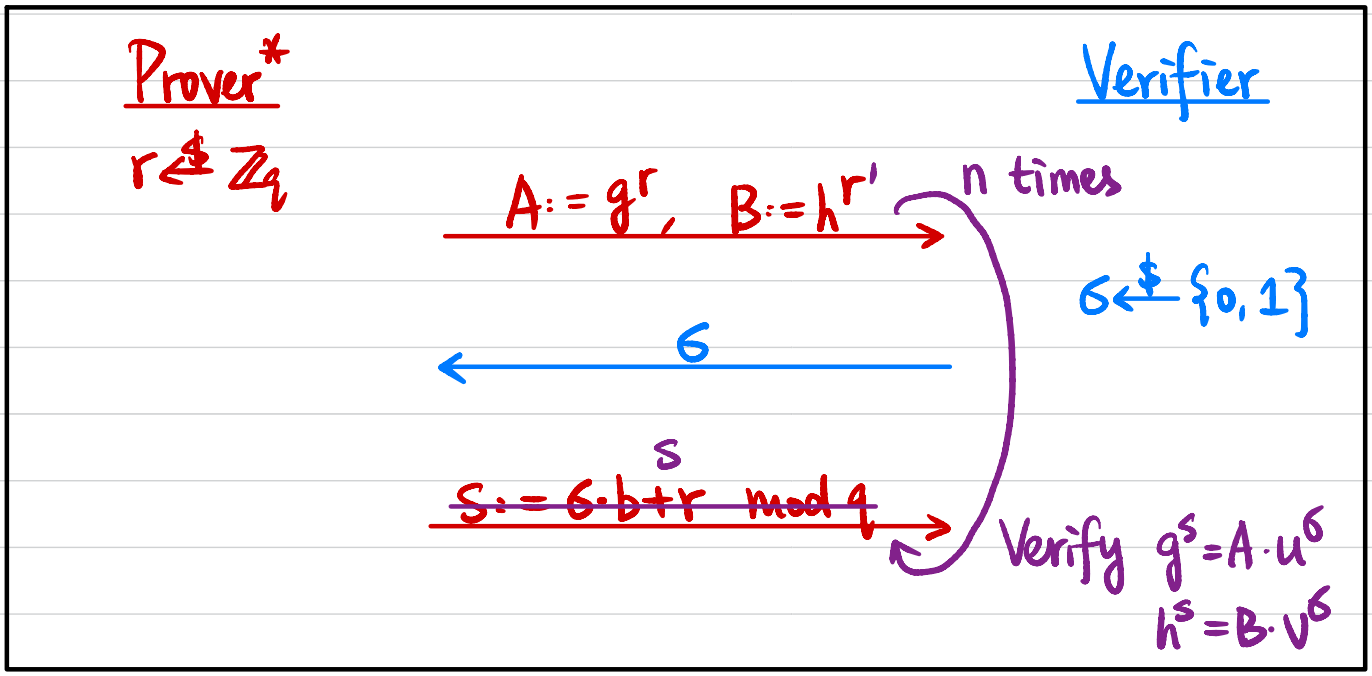
If $\sigma = 0 \Rightarrow s = r \Rightarrow g^s = A \quad h^s = B$

If $\sigma = 1 \Rightarrow s = b + r \Rightarrow g^s = A \cdot u \quad h^s = B \cdot v$

Soundness? $(g, h, u, v) \notin L$ ($v = h^{b'}$, $b \neq b'$)

$h = g^a$, $u = g^b$, $v = g^c$

$\forall x \notin L, \forall p^*, \Pr[p^*(x) \longleftrightarrow V(x) \text{ outputs } 1] \leq \text{negl}(n)$



Soundness error 2^{-n}.

$g^s = A \cdot u^\sigma \Leftrightarrow g^s = g^r \cdot (g^b)^\sigma = g^{r+b\cdot\sigma} \Leftrightarrow s = r + b\cdot\sigma \bmod q$

$h^s = B \cdot v^\sigma \Leftrightarrow h^s = h^{r'} \cdot (h^{b'})^\sigma = h^{r'+b'\cdot\sigma} \Leftrightarrow s = r' + b'\cdot\sigma \bmod q$

$r - r' = (b - b')\cdot\sigma$

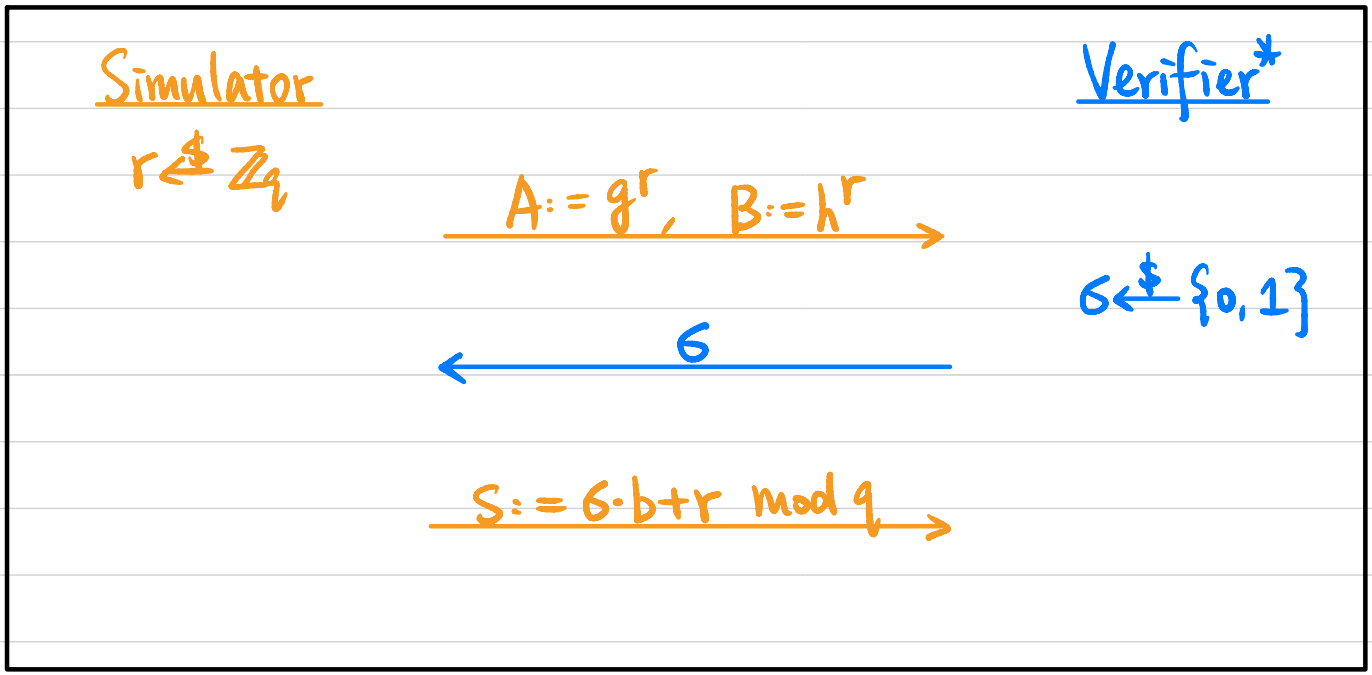
If $r = r' \Rightarrow$ caught by V if $\sigma = 1$

If $r \neq r' \Rightarrow$ caught by V if $\sigma = 0$.

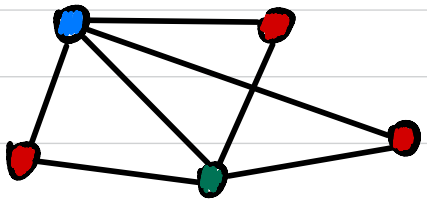
Zero-Knowledge?

$$\forall \text{PPT } V^*,\ \exists \text{PPT } S \quad \text{s.t.} \quad \forall (x, w) \in R_L,$$

$$\text{Output}_{V^*}[P(x, w) \longleftrightarrow V^*(x)] \equiv S(x)$$

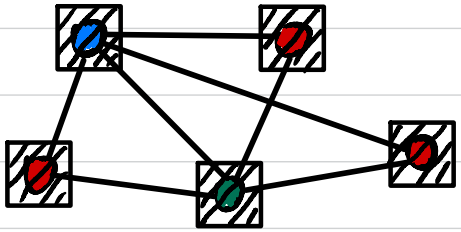


ZKP for Graph 3-Coloring (All NP)

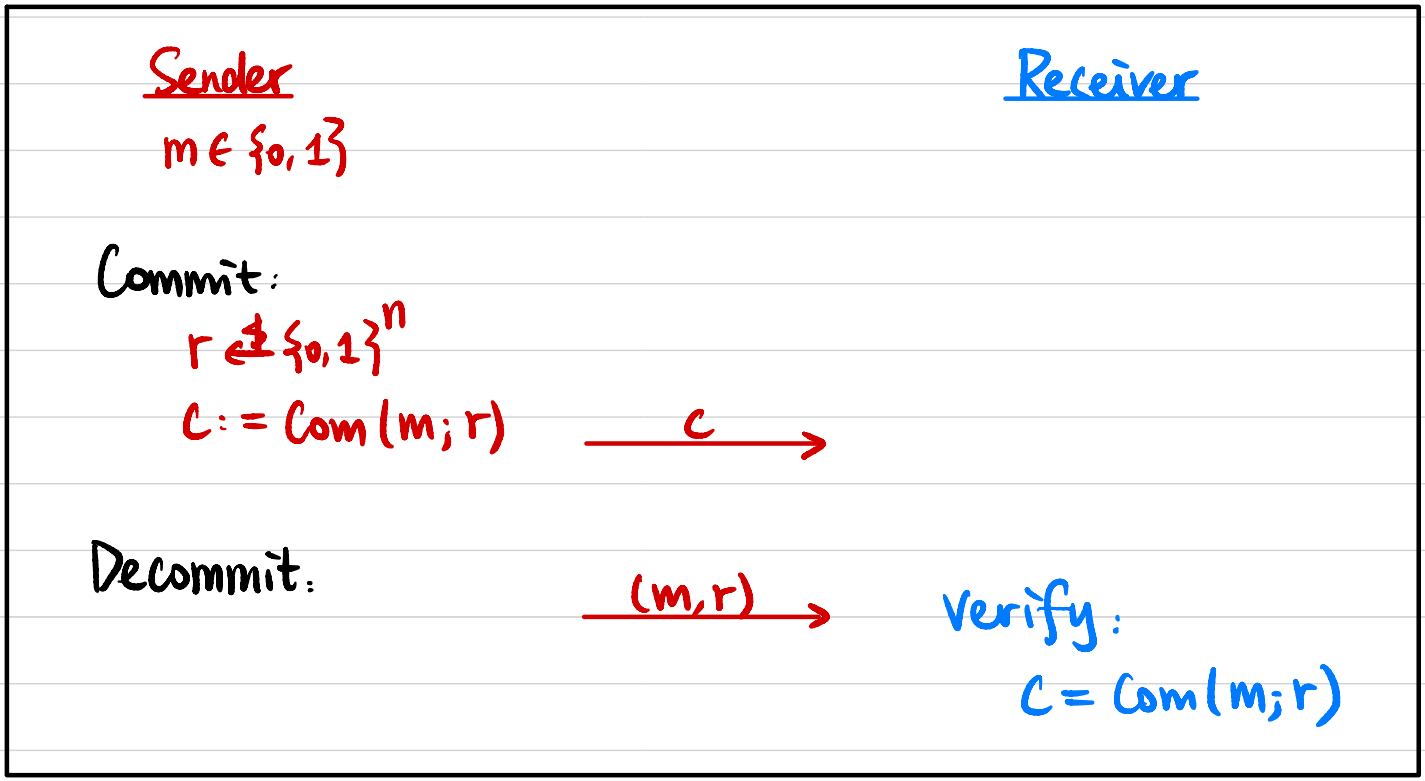


NP language $L = \{ G : G \text{ has 3-coloring} \}$

NP relation $R_L = \{ (G, 3COL) \}$



Commitment Scheme



Commitment Scheme

Def A non-interactive perfectly binding commitment scheme is a PPT algorithm Com statisfying:

- **Perfectly Binding:** $\forall r, s \in \{0,1\}^n$, $\mathrm{Com}(0; r) \neq \mathrm{Com}(1; s)$
- **Computationally Hiding:** $\mathrm{Com}(0; U_n) \overset{c}{\approx} \mathrm{Com}(1; U_n)$

A decommitment of a commitment value c is (b, r) s.t. $c = \mathrm{Com}(b; r)$.

Can a commitment scheme be both perfectly binding & perfectly hiding ?

Perfectly Binding Commitment Scheme

Assume one-way permutations exist.

Let $f: \{0,1\}^n \to \{0,1\}^n$ be a OWP and $hc: \{0,1\}^n \to \{0,1\}$ be a hard-core predicate of f.

$$Com(b; r) := (f(r), hc(r) \oplus b)$$

- Perfectly Binding?

- Computationally Hiding?

ZKP for Graph 3-Coloring

Input: $G = (V, E)$

Witness: $\phi: V \to \{0,1,2\}$

Given a perfectly binding commitment scheme Com.

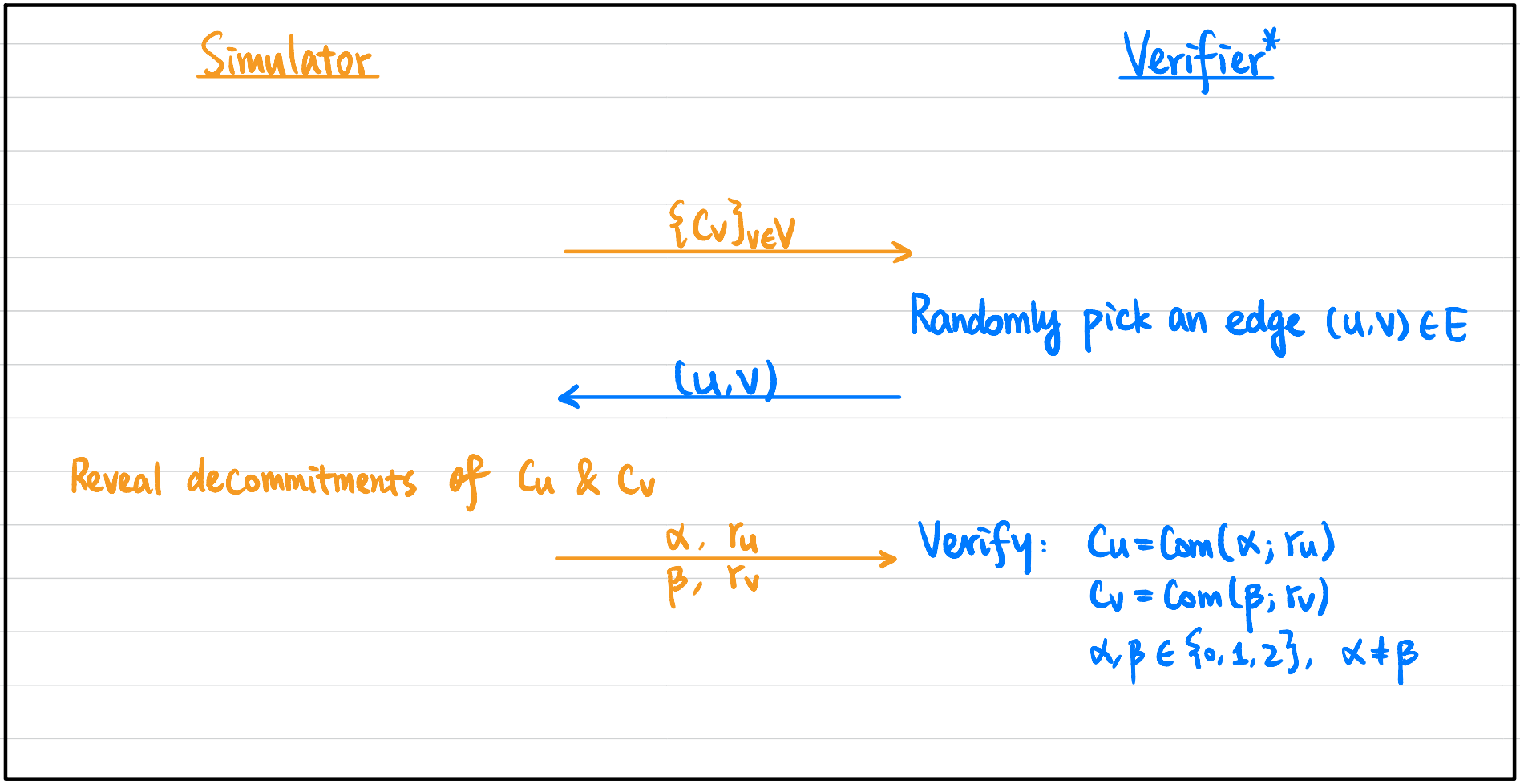
Prover		Verifier
Randomly sample $\pi: \{0,1,2\} \to \{0,1,2\}$ $\forall v \in V$, $r_v \xleftarrow{\$} \{0,1\}^n$, $c_v := \mathsf{Com}(\pi(\phi(v)); r_v)$		
	$\xrightarrow{\{c_v\}_{v \in V}}$	
		Randomly pick an edge $(u,v) \in E$
	$\xleftarrow{(u,v)}$	
Reveal decommitments of c_u & c_v		
$\alpha = \pi(\phi(u)),\ r_u$ $\beta = \pi(\phi(v)),\ r_v$	$\xrightarrow{\quad}$	Verify: $c_u = \mathsf{Com}(\alpha; r_u)$ $c_v = \mathsf{Com}(\beta; r_v)$ $\alpha, \beta \in \{0,1,2\},\ \alpha \neq \beta$

Completeness ?

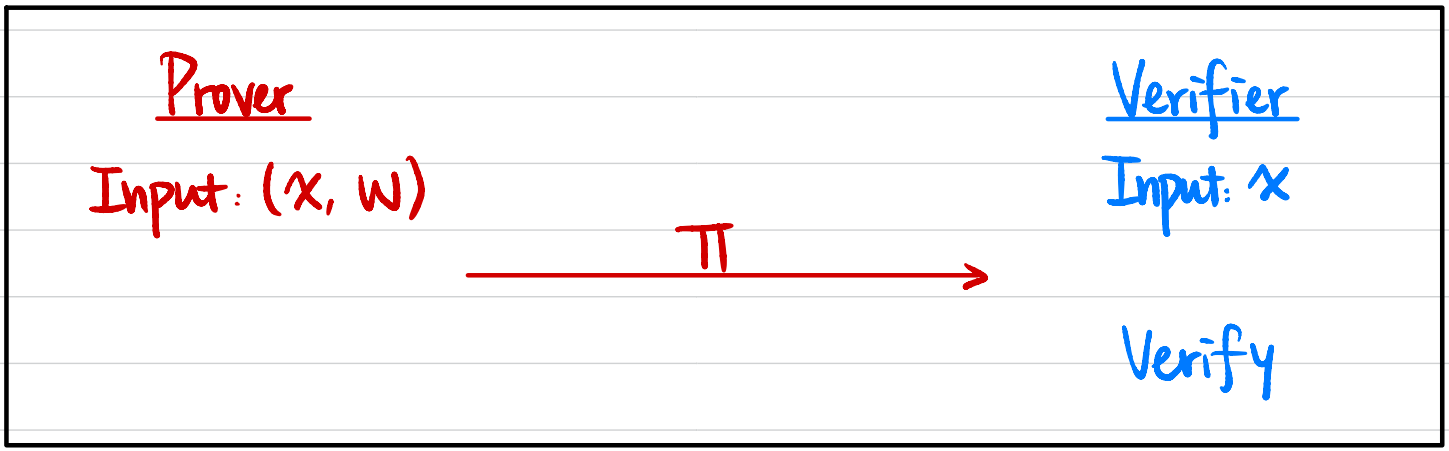
Soundness ?

Zero-Knowledge?

$$\forall \text{PPT } V^*,\ \exists \text{PPT } S \quad \text{s.t.} \quad \forall (x, w) \in R_L,$$
$$\text{Output}_{V^*}[P(x,w) \longleftrightarrow V^*(x)] \overset{c}{\approx} S(x)$$



Non-Interactive Zero-Knowledge (NIZK) Proof

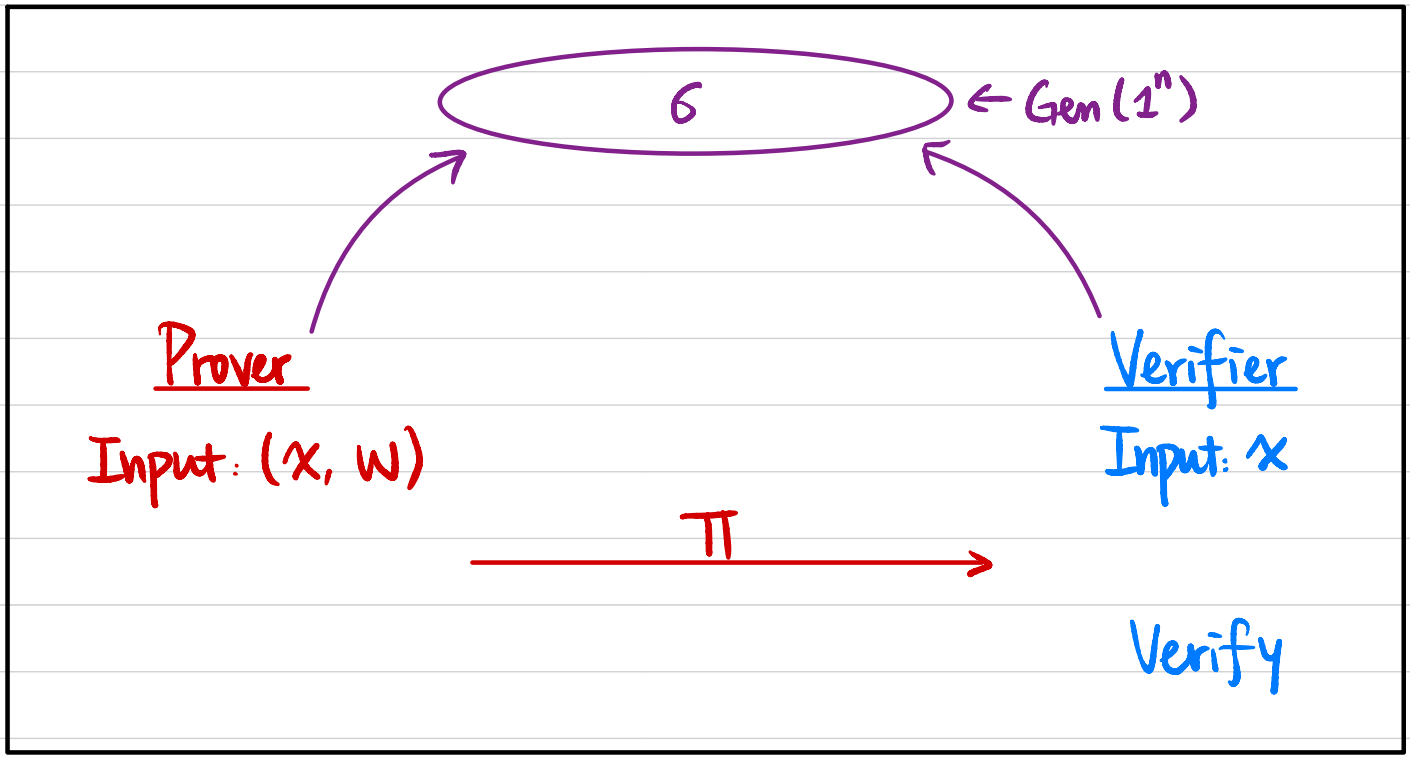


- **Completeness:** $\forall (x,w) \in R_L, \quad Pr[\, P(x,w) \longrightarrow V(x) \text{ outputs } 1\,] = 1.$
- **Soundness:** $\forall x \notin L, \ \forall P^*, \quad Pr[\, P^*(x) \longrightarrow V(x) \text{ outputs } 1\,] \leq negl(n)$
- **Zero-Knowledge:** $\forall PPT\ V^*, \ \exists PPT\ S \quad s.t. \quad \forall (x,w) \in R_L,$

$$Output_{V^*}[P(x,w) \longrightarrow V^*(x)] \simeq S(x)$$

Is it possible ?

Model 1: Common Random String / Common Reference String (CRS)



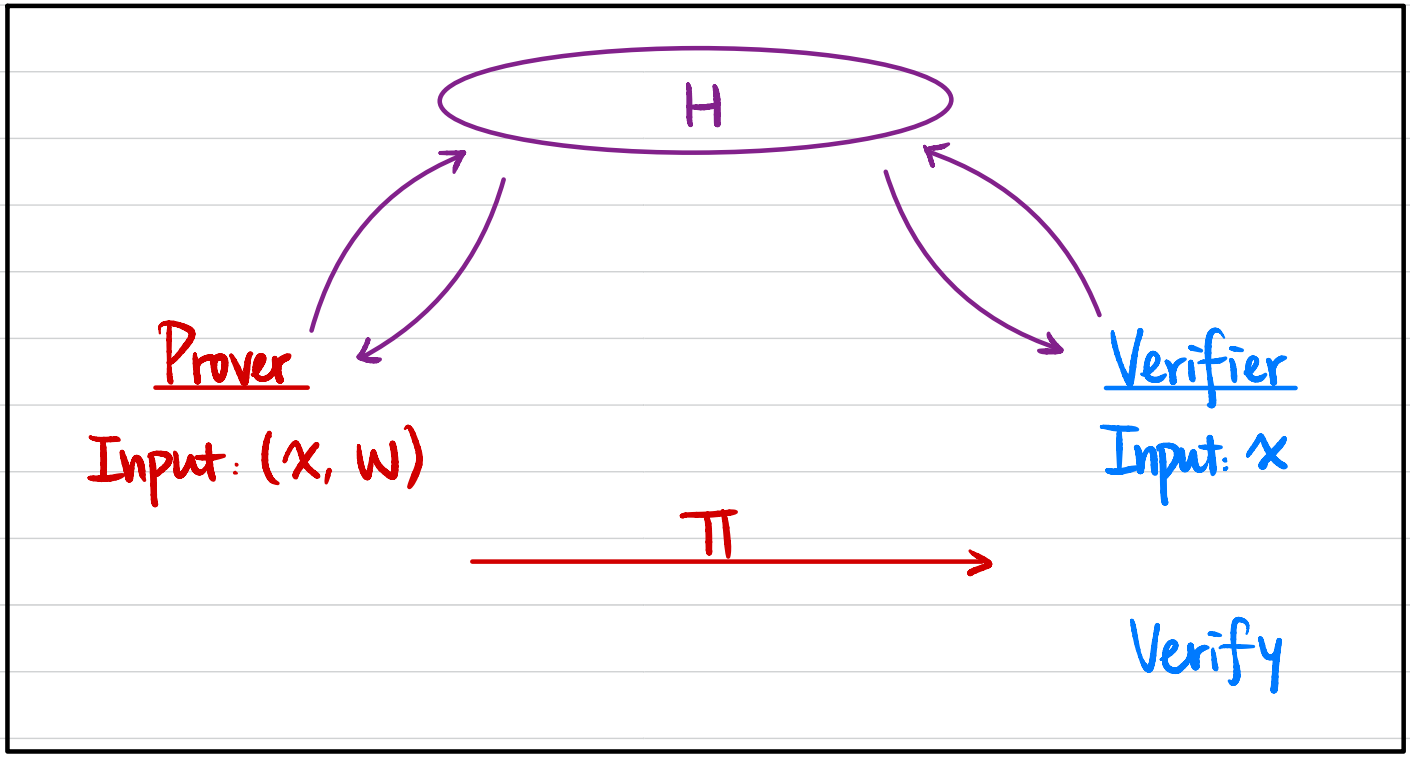
$S(x)$ generates both (σ, π)

- **Zero-Knowledge:** $\forall PPT\ V^*,\ \exists PPT\ S$ s.t. $\forall (x, w) \in R_L$,

$$Output_{V^*}\left[\sigma \leftarrow Gen(1^n),\ P(x, w, \sigma) \longrightarrow V^*(x, \sigma)\right] \simeq S(x)$$

Alternatively: $\left(\sigma \leftarrow Gen(1^n),\ P(x, w, \sigma)\right) \simeq S(x)$

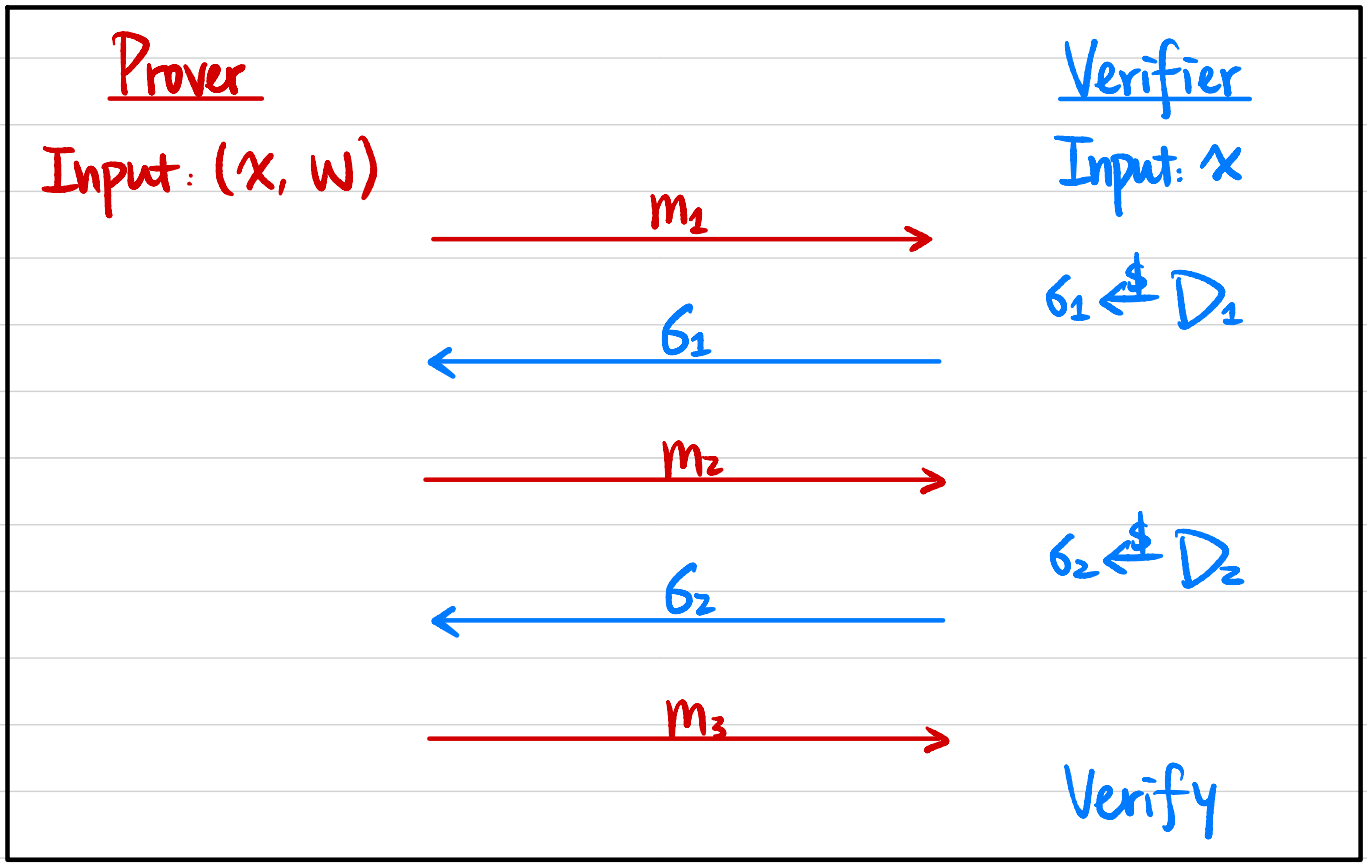
Model 2: Random Oracle Model



S controls input/output behavior of RO

Fiat-Shamir Heuristic

Public-Coin Honest-Verifier ZK (HVZK) $\Rightarrow$ NIZK in the RO model



$\sigma_1 := H(x \| m_1)$

$\sigma_2 := H(x \| m_1 \| m_2)$