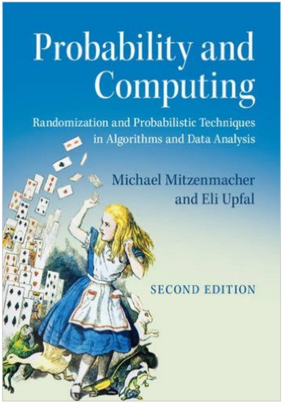
CS155/254: Probabilistic Methods in Computer Science

Chapter 13.1: Martingales



Hoeffding's Bound

Theorem

Let $X_1, \ldots, X_n$ be independent random variables with $\mathrm{E}[X_i] = \mu_i$ and $Pr(B_i \leq X_i \leq B_i + c_i) = 1$, then

$$Pr(|\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i| \geq \epsilon) \leq e^{-\frac{2\epsilon^2}{\sum_{i=1}^{n} c_i^2}}$$

Do we need independence?

Martingales

Definition

A sequence of random variables $Z_0, Z_1, \ldots$ is a *martingale* with respect to the sequence $X_0, X_1, \ldots$ if for all $n \geq 0$ the following hold:

1. Z_n is a function of $X_0, X_1, \ldots, X_n$;
2. $\mathrm{E}[|Z_n|] < \infty$;
3. $\mathrm{E}[Z_{n+1} | X_0, X_1, \ldots, X_n] = Z_n$;

Definition

A sequence of random variables $Z_0, Z_1, \ldots$ is a *martingale* when it is a martingale with respect to itself, that is

1. $\mathrm{E}[|Z_n|] < \infty$;
2. $\mathrm{E}[Z_{n+1} | Z_0, Z_1, \ldots, Z_n] = Z_n$;

Conditioning Defines a Probability Space

Let $(\Omega, Pr(\cdot))$ be a probability space.

Let B be an event in Ω, $Pr(B) > 0$.

We show that $(B, Pr(\cdot \mid B))$ is a probability space.

1. For any $E \subseteq B$,

$$0 \leq Pr(E \mid B) = \frac{Pr(E \cap B)}{Pr(B)} \leq 1$$

2. Let E_1 and E_2 be disjoint events in B,

$$\begin{aligned} Pr(E_1 \cup E_1 \mid B) &= \frac{Pr((E_1 \cup E_2) \cap B)}{Pr(B)} \\ &= \frac{Pr(E_1 \cap B)}{Pr(B)} + \frac{Pr(E_2 \cap B)}{Pr(B)} \\ &= Pr(E_1 \mid B) + Pr(E_2 \mid B) \end{aligned}$$

Conditional Expectation

Definition

$$E[Y \mid Z = z] = \sum_{y} y \Pr(Y = y \mid Z = z) \ ,$$

where the summation is over all y in the range of Y.

Note that $E[Y \mid Z]$ is a random variable (a function of Z)

Lemma

For any random variables X and Y,

$$E[X] = E_Y[E_X[X \mid Y]] = \sum_{y} \Pr(Y = y) E[X \mid Y = y] \ ,$$

where the sum is over all values in the range of Y.

Lemma

For any random variables X and Y,

$$\mathrm{E}[X] = \mathrm{E}_Y[\mathrm{E}_X[X \mid Y]] = \sum_y \Pr(Y = y)\mathrm{E}[X \mid Y = y] \ ,$$

where the sum is over all values in the range of Y.

Proof.

$$\begin{aligned}
& \sum_y \Pr(Y = y)\mathrm{E}[X \mid Y = y] \\
= \ & \sum_y \Pr(Y = y) \sum_x x \Pr(X = x \mid Y = y) \\
= \ & \sum_x \sum_y x \Pr(X = x \mid Y = y) \Pr(Y = y) \\
= \ & \sum_x \sum_y x \Pr(X = x \cap Y = y) = \sum_x x \Pr(X = x) = \mathrm{E}[X].
\end{aligned}$$

□

Example

Y - the number of students attending class, $Y \sim B(n, p)$

X - the number of questions asked in class is $X|_{Y=y} \sim B(\lfloor\sqrt{y}\rfloor, q)$.

All events are independent

$E[X \mid Y = y] = q\lfloor\sqrt{y}\rfloor$ - a constant

$E[X \mid Y] = q\lfloor\sqrt{Y}\rfloor$ - a random variable

$$\begin{aligned} E[X] = E_Y[E_X[X \mid Y]] &= E_Y[q\lfloor\sqrt{Y}\rfloor] \\ &= qE[\lfloor\sqrt{Y}\rfloor] \leq q\sqrt{E[Y]} = q\sqrt{np} \end{aligned}$$

Martingales

Definition

A sequence of random variables $Z_0, Z_1, \ldots$ is a *martingale* with respect to the sequence $X_0, X_1, \ldots$ if for all $n \geq 0$ the following hold:

1. Z_n is a function of $X_0, X_1, \ldots, X_n$;
2. $\mathrm{E}[|Z_n|] < \infty$;
3. $\mathrm{E}[Z_{n+1} | X_0, X_1, \ldots, X_n] = Z_n$;

Definition

A sequence of random variables $Z_0, Z_1, \ldots$ is a *martingale* when it is a martingale with respect to itself, that is

1. $\mathrm{E}[|Z_n|] < \infty$;
2. $\mathrm{E}[Z_{n+1} | Z_0, Z_1, \ldots, Z_n] = Z_n$;

Martingale Example

A series of fair games ($E[\text{gain}] = 0$), not necessarily independent..

Game 1: bet $\$1$.

Game $i > 1$: bet 2^i if won in round $i - 1$; bet i otherwise.

$X_i =$ amount won in ith game. ($X_i < 0$ if ith game lost).

$Z_i =$ total winnings at end of ith game.

Example

$X_i =$ amount won in ith game. ($X_i < 0$ if ith game lost).

$Z_i =$ total winnings at end of ith game.

$Z_1, Z_2, \ldots$ is martingale with respect to $X_1, X_2, \ldots$

$\mathrm{E}[X_i] = 0.$

$\mathrm{E}[Z_i] = \sum_{j=1}^{i} \mathrm{E}[X_j] = 0 < \infty.$

$\mathrm{E}[Z_{i+1} | X_1, X_2, \ldots, X_i] = Z_i + \mathrm{E}[X_{i+1}] = Z_i.$

Efficient Market Hypothesis

The efficient markets hypothesis (EMH) maintains that market prices fully reflect all available information. Samuelson (1965), Fama (1963);

For simplicity assume an asset that is paying no dividend, and assume 0 interest rate (so value is not discounted in time).

Let X_t be the price of a unit asset at time t.
If I know that at time $t+1$ the price will be $X_{t+1} = c$, I will not sale the asset now for less than c.
If I know that at time $t+1$ the price will be $X_{t+1} = c$, I will not buy the asset now for more than c.

$$X_t = E[X_{t+1} \mid X_0, \dots X_t]$$

$X_0, X_1, \dots, X_t,$ is a martingale.

Gambling Strategies

I play series of fair games (win with probability $1/2$).

Game 1: bet $\$1$.

Game $i > 1$: bet 2^i if I won in round $i - 1$; bet i otherwise.

$X_i =$ amount won in ith game. ($X_i < 0$ if ith game lost).

$Z_i =$ total winnings at end of ith game.

Assume that (before starting to play) I decide to quit after k games: what are my expected winnings?

Lemma

Let $Z_0, Z_1, Z_{2,} \ldots$ *be a martingale with respect to* $X_0, X_1, \ldots$. *For any fixed* n,

$$\mathrm{E}_{X[0:n]}[Z_n] = \mathrm{E}_{X_0}[Z_0] \quad .$$

($X[0:i] = X_0, \ldots, X_i$*)*

Proof.

Since Z_i is a martingale $\mathrm{E}_{X_i}[Z_i | X_0, X_1, \ldots, X_{i-1}] = Z_{i-1}$.
Then

$$\mathrm{E}_{X[0:i-1]}[Z_{i-1}] = \mathrm{E}_{X[0:i-1]}[\mathrm{E}_{X_i}[Z_i | X_0, X_1, \ldots, X_{i-1}]] = \mathrm{E}_{X[0:i]}[Z_i]$$

Thus,

$$\mathrm{E}_{X[0:n]}[Z_n] = \mathrm{E}_{X[0:n-1]}[Z_{n-1}] = \ldots, = \mathrm{E}[Z_0]$$

□

Gambling Strategies

I play series of fair games (win with probability $1/2$).

Game 1: bet $\$1$.

Game $i > 1$: bet 2^i if I won in round $i-1$; bet i otherwise.

$X_i =$ amount won in ith game. ($X_i < 0$ if ith game lost).

$Z_i =$ total winnings at end of ith game.

Assume that (before starting to gamble) we decide to quit after k games: what are my expected winnings?

$E[Z_k] = E[Z_1] = 0$.

A Different Strategy

Same gambling game. What happens if I:

- play a random number of games?
- decide to stop only when I have won $1000?

Stopping Time

Definition

A non-negative, integer *random variable* T is a *stopping time* for the sequence $Z_0, Z_1, \ldots$ if the event "$T = n$" depends only on the value of random variables $Z_0, Z_1, \ldots, Z_n$.

Intuition: corresponds to a strategy for determining when to stop a sequence based only on values seen so far.

In the gambling game:

- *first time I win 10 games in a row*: is a stopping time;
- *the last time when I win*: is not a stopping time.

Consider again the gambling game: let T be a stopping time.

$Z_i =$ total winnings at end of ith game.

What are my winnings at the stopping time, i.e. $\mathrm{E}[Z_T]$?

Fair game: $\mathrm{E}[Z_k] = \mathrm{E}[Z_0] = 0$?

"$T =$first time my total winnings are at least \$1000" is a stopping time, and $\mathrm{E}[Z_T] > 1000$...

Martingale Stopping Theorem

Theorem

If $Z_0, Z_1, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$ and if T is a stopping time for $X_1, X_2, \ldots$ then (if T is finite),

$$\mathrm{E}[Z_T] = \mathrm{E}[Z_0]$$

whenever one of the following holds:

1. *there is a constant c such that, for all i, $|Z_i| \leq c$;*
2. *T is bounded;*
3. *$\mathrm{E}[T] < \infty$, and there is a constant c such that $\mathrm{E}\big[|Z_{i+1} - Z_i| \big| X_1, \ldots, X_i\big] < c$.*

Proof of Martingale Stopping Theorem (Sketch)

Define a sequence $Y_0, Y_1, \ldots$ such that

$$Y_i = \begin{cases} Z_i & \text{if } T > i \\ Z_T & \text{if } T \leq i \end{cases}$$

Lemma

The sequence $Y_0, Y_1, \ldots$ is a martingale with respect to $Z_0, Z_1, \ldots$.

Proof.

1. Y_n is determined by $Z_0, \ldots, Z_n$.
2. $E[|Y_n|] \leq \max_{0 \leq i \leq n} E[|X_i|] \leq \sum_{i=1}^{n} E[|X_i|] < \infty$
3. $\mathrm{E}[Y_{n+1}|Z_0, Z_1, \ldots, Z_n] = Y_n + E_{Z_{n+1}}[(Y_{n+1} - Y_n)1_{(T>n)}] = Y_n + E_{Z_{n+1}}[(Z_{n+1} - Z_n)]Pr(T > n) = Y_n$;

   Since $Pr(T > n)$ is independent of Z_{n+1}, and $E[(Z_{n+1} - Z_n)] = 0$.

□

Since $Y_0, Y_1, \ldots$ is a martingale, for any $n \geq 0$, $E[Y_n] = E[Z_0]$, and

$$\lim_{n\to\infty} E[Y_n] = E[Y_0] = E[Z_0].$$

Since T is finite, $Z_t = \lim_{n\to\infty} Z_{\min(n,T)} = lim_{n\to\infty} Y_n$.
We want to show that $E[Z_T] = \lim_{n\to\infty} E[Y_n] = E[Z_0]$.

We use a simple version of the Uniform Convergence Theorem:

Theorem

Let $W_0, W_1, \ldots$ be a sequence of random variables such that $\lim_{n\to\infty} W_n = W$ (pointwise), and $max_i|W_i| \leq M$, where M is either a constant or a random variable with $E[|M|] < \infty$, then

$$\lim_{n\to\infty} E[W_n] = E[W].$$

Proof of Martingale Stopping Theorem (Sketch)

Since T is finite, $\lim_{n\to\infty} Y_n = \lim_{n\to\infty} Z_{\min(n,T)} = Z_T$.

We need to show that $|Y_n| \leq M$.

1. there is a constant c such that, for all i, $|Z_i| \leq c$ - $|Y_n| \leq max_{0\leq i\leq n}|Z_i| \leq c$, $c = M < \infty$.
2. T is bounded - $|Y_n| \leq max_{0\leq i\leq \max T}|Z_i| \leq M < \infty$
3. $\mathrm{E}[T] < \infty$, and there is a constant c such that $\mathrm{E}\left[|Z_{i+1} - Z_i|\middle|X_1,\ldots,X_i\right] < c$

$$Y_n = Z_0 + \sum_{i=1}^{\infty}(Z_{i+1} - Z_i)1_{i\leq T} \leq |Z_0| + \sum_{i=1}^{\infty}|Z_{i+1} - Z_i|1_{i\leq T} = M.$$

$$\begin{aligned} E[|M|] &= E[|Z_0|] + \sum_{i=1}^{\infty} \mathrm{E}[\mathrm{E}\left[|Z_{i+1} - Z_i|\middle|X_1,\ldots,X_i\right] 1_{i\leq T}] \\ &\leq E[|Z_0|] + c\sum_{i=1}^{\infty} Pr(T\geq i) \\ &\leq E[|Z_0|] + cE[T] < \infty \end{aligned}$$

Martingale Stopping Theorem Applications

We play a sequence of fair game with the following stopping rules:

1. T is bounded, $\mathsf{E}[Z_T] = \mathsf{E}[Z_0]$.
2. T is the first time we made \$1000: $\mathsf{E}[T]$ is unbounded.
3. We double until the first win. $\mathsf{E}[T] = 2$ but $\mathsf{E}\left[|Z_{i+1} - Z_i| \middle| X_1, \ldots, X_i\right]$ is unbounded.

Example: The Gambler's Ruin

- Consider a sequence of independent, fair 2-player gambling games.
- In each round, each player wins or loses \$1 with probability $\frac{1}{2}$.
- $X_i =$ amount won by player 1 on ith round.
  - If player 1 has lost in round i: $X_i < 0$.
- $Z_i =$ total amount won by player 1 after ith rounds.
  - $Z_0 = 0$.
- Game ends when one player runs out of money
  - Player 1 must stop when she loses net ℓ_1 dollars ($Z_t = -\ell_1$)
  - Player 2 terminates when she loses net ℓ_2 dollars ($Z_t = \ell_2$).
- $q =$ probability game ends with player 1 winning ℓ_2 dollars.

Example: The Gambler's Ruin

- $T =$ first time player 1 wins ℓ_2 dollars or loses ℓ_1 dollars.
  - T is a stopping time for $X_1, X_2, \ldots$.
- $Z_0, Z_1, \ldots$ is a martingale.
  - Z_i's are bounded.
- Martingale Stopping Theorem: $\mathrm{E}[Z_T] = \mathrm{E}[Z_0] = 0$.

$$\mathrm{E}[Z_T] = q\ell_2 - (1-q)\ell_1 = 0$$

$$q = \frac{\ell_1}{\ell_1 + \ell_2}$$

Example: A Ballot Theorem

- Candidate **A** and candidate **B** run for an election.
  - Candidate **A** gets a votes.
  - Candidate **B** gets b votes.
  - $a > b$.
- Votes are counted in *random order*:
  - chosen from all permutations on $n = a + b$ votes.
- What is the probability that **A** is always ahead in the count?

Example: A Ballot Theorem

- $S_i =$ number of votes **A** is leading by after i votes counted
  - If **A** is trailing: $S_i < 0$.
  - $S_n = a - b$.
- For $0 \leq k \leq n-1$: $X_k = \frac{S_{n-k}}{n-k}$.
- Consider $X_0, X_1, \ldots, X_n$.
  - This sequence goes backward in time!

$$E[X_k | X_0, X_1, \ldots, X_{k-1}] = ?$$

Example: A Ballot Theorem

$$E[X_k | X_0, X_1, \ldots, X_{k-1}] = ?$$

- Conditioning on $X_0, X_1, \ldots, X_{k-1}$: equivalent to conditioning on $S_n, S_{n-1}, \ldots, S_{n-(k-1)}$,
- $a_i =$ number of votes for **A** after first i votes are counted.
- $(n-k+1)$th vote: random vote among these first $n-k+1$ votes.

$$S_{n-k} = \begin{cases} S_{n-(k-1)} + 1 & \text{if } (n-k+1)\text{th vote is for } \mathbf{B} \\ S_{n-(k-1)} - 1 & \text{if } (n-k+1)\text{th vote is for } \mathbf{A} \end{cases}$$

$$S_{n-k} = \begin{cases} S_{n-(k-1)} + 1 & \text{with prob. } \frac{n-k+1-a_{n-(k-1)}}{n-(k-1)} \\ S_{n-(k-1)} - 1 & \text{with prob. } \frac{a_{n-(k-1)}}{n-(k-1)} \end{cases}$$

$$
\begin{aligned}
\mathrm{E}[S_{n-k}|S_{n-(k-1)}] &= (S_{n-(k-1)}+1)\frac{n-k+1-a_{n-(k-1)}}{(n-k+1)} \\
&+ (S_{n-(k-1)}-1)\frac{a_{n-(k-1)}}{(n-k+1)} \\
&= S_{n-(k-1)}\frac{n-k}{n-(k-1)}
\end{aligned}
$$

(Since $2a_{n-(k-1)} - n - k + 1 = a_{n-(k-1)} - b_{n-(k-1)} = S_{n-(k-1)}$)
(Since
$n - k + 1 - 2a_{n-(k-1)} - = b_{n-(k-1)} - a_{n-(k-1)} = -S_{n-(k-1)}$)

$$
\begin{aligned}
\mathrm{E}[X_k|X_0, X_1, \ldots, X_{k-1}] &= \mathrm{E}\left[\left.\frac{S_{n-k}}{n-k}\right| S_n, \ldots, S_{n-(k-1)}\right] \\
&= \frac{S_{n-(k-1)}}{n-(k-1)} \\
&= X_{k-1}
\end{aligned}
$$

$\implies \quad X_0, X_1, \ldots, X_n$ is a martingale.

Example: A Ballot Theorem

$$T = \begin{cases} \min\{k < n-1 : X_k = 0\} & \text{if such } k \text{ exists} \\ n-1 & \text{otherwise} \end{cases}$$

- T is a stopping time.
- T is bounded.
- Martingale Stopping Theorem:

$$\mathrm{E}[X_T] = \mathrm{E}[X_0] = \frac{\mathrm{E}[S_n]}{n} = \frac{a-b}{a+b} \; .$$

Two cases:

1. **A** leads throughout the count.
2. **A** does not lead throughout the count.

1. **A** leads throughout the count.

For $0 \leq k \leq n-1$: $S_{n-k} > 0$, then $X_k > 0$.

$T = n - 1$.

$X_T = X_{n-1} = S_1$.

A gets the first vote in the count: $S_1 = 1$, then $X_T = 1$.

2. **A** does not lead throughout the count.

For some k: $S_k = 0$. Then $X_k = 0$.

$T = k < n - 1$.

$X_T = 0$.

Example: A Ballot Theorem

Putting all together:

1. **A** leads throughout the count: $X_T = 1$.
2. **A** does not lead throughout the count: $X_T = 0$

$$E[X_T] = \frac{a-b}{a+b} = 1 * \Pr(\text{Case 1}) + 0 * \Pr(\text{Case 2}) \quad .$$

That is

$$\Pr(\textbf{A} \text{ leads throughout the count}) = \frac{a-b}{a+b} \quad .$$