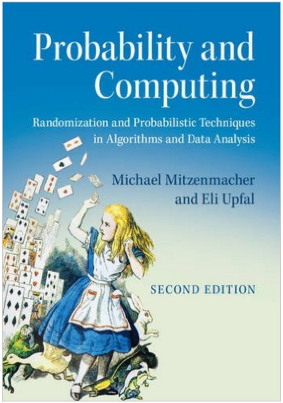
CS155/254: Probabilistic Methods in Computer Science

Chapter 13.2: Martingale's Large Deviation Bound



Martingales

Definition

A sequence of random variables $Z_0, Z_1, \ldots$ is a *martingale* with respect to the sequence $X_0, X_1, \ldots$ if for all $n \geq 0$ the following hold:

1. Z_n is a function of $X_0, X_1, \ldots, X_n$;
2. $\mathrm{E}[|Z_n|] < \infty$;
3. $\mathrm{E}[Z_{n+1}|X_0, X_1, \ldots, X_n] = Z_n$;

Definition

A sequence of random variables $Z_0, Z_1, \ldots$ is a *martingale* when it is a martingale with respect to itself, that is

1. $\mathrm{E}[|Z_n|] < \infty$;
2. $\mathrm{E}[Z_{n+1}|Z_0, Z_1, \ldots, Z_n] = Z_n$;

Martingale Stopping Theorem

Theorem

If $Z_0, Z_1, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$ and if T is a stopping time for $X_1, X_2, \ldots$ then (if T is finite),

$$\mathrm{E}[Z_T] = \mathrm{E}[Z_0]$$

whenever one of the following holds:

1. *there is a constant c such that, for all i, $|Z_i| \leq c$;*
2. *T is bounded;*
3. *$\mathrm{E}[T] < \infty$, and there is a constant c such that $\mathrm{E}\big[|Z_{i+1} - Z_i| \big| X_1, \ldots, X_i\big] < c$.*

Compound Stochastic Process

Examples:

1. Two stages game:
   1. roll one die; let X be the outcome;
   2. roll X standard dice; your gain Z is the sum of the outcomes of the X dice.

   What is your expected gain?

2. A couple expects to have X children, $X \sim G(p)$. They expect each of the children to have a number of children distributed $G(r)$.
   What is their expected number of grandchildren?

Wald's Equation

Theorem

Let $X_1, X_2, \dots$ be nonnegative, independent, identically distributed random variables with distribution X. Let T be a stopping time for this sequence. If T and X have bounded expectations, then

$$\mathrm{E}\left[\sum_{i}^{T} X_i\right] = \mathrm{E}[T]\mathrm{E}[X] \ .$$

Note that T is not independent of $X_1, X_2, \dots$.
Corollary of the martingale stopping theorem.

Proof

For $i \geq 1$, let $Z_i = \sum_{j=1}^{i}(X_j - \mathrm{E}[X])$.

The sequence $Z_1, Z_2, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$.

1. Z_i is determined by $X_1, \ldots, X_i$
2. $E[|Z_i|] = E[|\sum_{j=1}^{i}(X_j - E[X])|] = \leq 2iE[|X|]$
3. $E[Z_{i+1} - Z_i \mid X_0, X_1, \ldots, X_i] = E[X_{j+1} - E[X]] = 0$

$\mathrm{E}[Z_1] = 0$, $\mathrm{E}[T] < \infty$, and

$$\mathrm{E}\left[|Z_{i+1} - Z_i| \mid X_1, \ldots, X_i\right] = \mathrm{E}[|X_{i+1} - \mathrm{E}[X]|] \leq 2\mathrm{E}[X] \ .$$

We can apply the martingale stopping theorem to compute

$$\mathrm{E}[Z_T] = \mathrm{E}[Z_1] = 0 \ .$$

We can apply the martingale stopping theorem to compute

$$\mathsf{E}[Z_T] = \mathsf{E}[Z_1] = 0 \ .$$

$$\begin{aligned} 0 &= \mathsf{E}[Z_T] = \mathsf{E}\left[\sum_{j=1}^{T}(X_j - \mathsf{E}[X])\right] = \mathsf{E}\left[\sum_{j=1}^{T} X_j - T\mathsf{E}[X]\right] \\ &= \mathsf{E}\left[\sum_{j=1}^{T} X_j\right] - \mathsf{E}[T] \cdot \mathsf{E}[X] = 0, \end{aligned}$$

Examples

Two stages game:

1. roll one die; let X be the outcome;
2. roll X standard dice; your gain Z is the sum of the outcomes of the X dice.

What is your expected gain?

$Y_i =$ outcome of ith die in second stage.

$$\mathrm{E}[Z] = \mathrm{E}\left[\sum_{i=1}^{X} Y_i\right] .$$

X is a stopping time for $Y_1, Y_2, \ldots$.

By Wald's equation:

$$\mathrm{E}[Z] = \mathrm{E}[X]\mathrm{E}[Y_i] = \left(\frac{7}{2}\right)^2 .$$

Examples

A couple expect to have X children, $X \sim G(p)$. They expect each of their children to have a number of children distributed $G(r)$. What is their expected number of grandchildren?

$$\frac{1}{p} \cdot \frac{1}{r}$$

Example: a k-run

We flip a coin with probability p for head, $q = (1 - p)$ for tail, until we get a consecutive sequence of k heads. What's the expected number of times we flip the coin?

- A **switch** is a head followed by a tail.
- A **segment** is a sequence of flips till the first switch or consecutive sequence of k heads.
- Let X_i be the number of flips in the i segment.
- Let T be the first i with k heads.
- Expected number of flips till (including) the first head - $\sum_{j \geq 1} jq^{j-1}p$.
- Expected number of following flips till a switch before $k-2$ flips - $\sum_{j=1}^{k-2} jp^{j-1}q$

$$E[X_i] = \sum_{j \geq 1} jq^{j-1}p + \sum_{j=1}^{k-2} jp^{j-1}q + (k-1)p^{(k-2)}$$

- Let X_i be the number of flips in the i segment.

$$\mathrm{E}[X_i] = \sum_{j \geq 1} j q^{j-1} p + \sum_{j=1}^{k-2} j p^{j-1} q + (k-1) p^{(k-2)}$$

- Let T be the first i with k heads.
- The probability that a segment ends with k heads is p^{k-1} ($k-1$ heads following the first head).

$$\mathrm{E}[T] = p^{-(k-1)}$$

- The expected number of coin flips is $\mathrm{E}[X_i] E[T] \leq (\frac{1}{p} + \frac{1}{q}) \frac{1}{p^{k-1}}$

Hoeffding's Bound

Theorem

Let $X_1, \ldots, X_n$ *be* **independent** *random variables with* $\mathrm{E}[X_i] = \mu_i$ *and* $\Pr(B_i \leq X_i \leq B_i + c_i) = 1$, *then*

$$\Pr\left(\left| \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i \right| \geq \epsilon \right) \leq 2e^{-\frac{2\epsilon^2}{\sum_{i=1}^{n} c_i^2}}$$

Do we need independence?

Tail Inequalities

Theorem (Azuma-Hoeffding Inequality)

Let $Z_0, Z_1, \ldots, Z_n$ be a martingale (with respect to $X_1, X_2, \ldots$) such that $|Z_k - Z_{k-1}| \leq c_k$. Then, for all $t \geq 0$ and any $\lambda > 0$,

$$\Pr(|Z_t - Z_0| \geq \lambda) \leq 2e^{-\lambda^2/(2\sum_{k=1}^{t} c_k^2)} \quad .$$

The following corollary is often easier to apply.

Corollary

Let $X_0, X_1, \ldots$ be a martingale such that for all $k \geq 1$,

$$|X_k - X_{k-1}| \leq c \quad .$$

Then for all $t \geq 1$ and $\lambda > 0$,

$$\Pr\left(|X_t - X_0| \geq \lambda c\sqrt{t} \right) \leq 2e^{-\lambda^2/2} \quad .$$

Example

Assume that you play a sequence of n fair games, where the bet $b_i \leq B$ in game i depends on the outcome of previous games.

Let Z_n be the accumulated gain/loss after the n-th game.

We know that $E[Z_n] = 0$. We'll prove:

$$\Pr(|Z_n| \geq \lambda) \leq 2e^{-2\lambda^2/nB^2}$$

Tail Inequalities: A More General Form

Theorem (Azuma-Hoeffding Inequality)

Let $Z_0, Z_1, \ldots,$ *be a martingale with respect to* $X_0, X_1, X_2, \ldots,$ *such that*

$$B_k \leq Z_k - Z_{k-1} \leq B_k + c_k \ ,$$

for some constants c_k *and for some random variables* B_k *that may be functions of* $X_0, X_1, \ldots, X_{k-1}$*. Then, for any* $t \geq 0$ *and* $\lambda > 0$*,*

$$\Pr(|Z_t - Z_0| \geq \lambda) \leq 2e^{-2\lambda^2/(\sum_{k=1}^{t} c_k^2)} \ .$$

Proof

Let $X^k = X_0, \ldots, X_k$ and $Y_i = Z_i - Z_{i-1}$.

Since $\mathrm{E}[Z_i \mid X^{i-1}] = Z_{i-1}$,

$$\mathrm{E}[Y_i \mid X^{i-1}] = \mathrm{E}[Z_i - Z_{i-1} \mid X^{i-1}] = 0 \ .$$

Since $\Pr(B_i \le Y_i \le B_i + c_i \mid X^{i-1}) = 1$, by Hoeffding's Lemma:

$$\mathrm{E}[e^{\beta Y_i} \mid X^{i-1}] \le e^{\beta^2 c_i^2/8} \ .$$

Lemma

(Hoeffding's Lemma) Let X be a random variable such that $\Pr(X \in [a,b]) = 1$ and $\mathrm{E}[X] = 0$. Then for every $\lambda > 0$,

$$\mathrm{E}[e^{\lambda X}] \le e^{\lambda^2 (a-b)^2/8}.$$

Proof of the Lemma

Lemma

(Hoeffding's Lemma) Let X be a random variable such that $\Pr(X \in [a, b]) = 1$ and $\mathrm{E}[X] = 0$. Then for every $\lambda > 0$,

$$\mathrm{E}[e^{\lambda X}] \leq e^{\lambda^2 (a-b)^2/8}.$$

Since $f(x) = e^{\lambda x}$ is a convex function, for any $\alpha \in (0, 1)$ and $x \in [a, b]$,

$$f(X) \leq \alpha f(a) + (1-\alpha) f(b) \ .$$

Thus, for $\alpha = \frac{b-x}{b-a} \in (0, 1)$,

$$e^{\lambda x} \leq \frac{b-x}{b-a} e^{\lambda a} + \frac{x-a}{b-a} e^{\lambda b} \ .$$

Taking expectation, and using $\mathrm{E}[X] = 0$, we have

$$\mathrm{E}\left[e^{\lambda X}\right] \leq \frac{b}{b-a} e^{\lambda a} - \frac{a}{b-a} e^{\lambda b} \leq e^{\lambda^2 (b-a)^2/8} \ .$$

Proof of Azuma-Hoeffding Inequality

$$\mathrm{E}\left[e^{\beta Y_i} \;\middle|\; X^{i-1}\right] \leq e^{\beta^2 c_i^2/8} \quad .$$

$$\begin{aligned}
\mathrm{E}_{X^n}\left[e^{\beta \sum_{i=1}^n Y_i}\right] &= \mathrm{E}_{X^{n-1}}\left[\mathrm{E}_{X_n}\left[e^{\beta \sum_{i=1}^n Y_i} \;\middle|\; X^{n-1}\right]\right] \\
&= \mathrm{E}_{X^{n-1}}\left[e^{\beta \sum_{i=1}^{n-1} Y_i} \mathrm{E}_{X_n}\left[e^{\beta Y_n} \mid X^{n-1}\right]\right] \\
&\leq e^{\beta^2 c_n^2/8} \mathrm{E}_{X^{n-1}}\left[e^{\beta \sum_{i=1}^{n-1} Y_i}\right] \\
&\leq e^{\beta^2 \sum_{i=1}^n c_i^2/8}
\end{aligned}$$

In the second inequality we use the fact that X^{n-1} determines the values of $Y_1, \ldots, Y_{n-1}$

$Y_i = Z_i - Z_{i-1}$ and $\mathrm{E}[e^{\beta \sum_{i=1}^{n} Y_i}] \leq e^{\beta^2 \sum_{i=1}^{n} c_i^2/8}$

$$\begin{aligned} \Pr(Z_t - Z_0 \geq \lambda) &= \Pr\left(\sum_{i=1}^{t} Y_i \geq \lambda\right) \leq \frac{\mathrm{E}[e^{\beta \sum_{i=1}^{t} Y_i}]}{e^{\beta\lambda}} \\ &\leq e^{-\lambda\beta} e^{\beta^2 \sum_{i=1}^{t} c_i^2/8} \end{aligned}$$

For $\beta = \frac{4\lambda}{\sum_{i=1}^{t} c_i^2}$ we get:

$$\Pr(|Z_t - Z_0| \geq \lambda) \leq 2e^{-2\lambda^2/(\sum_{k=1}^{t} c_k^2)}$$

Theorem (Azuma-Hoeffding Inequality)

Let $Z_0, Z_1, \ldots, Z_n$ be a martingale (with respect to $X_1, X_2, \ldots$) such that $|Z_k - Z_{k-1}| \leq c_k$. Then, for all $t \geq 0$ and any $\lambda > 0$,

$$\Pr(|Z_t - Z_0| \geq \lambda) \leq 2e^{-\lambda^2/(2\sum_{k=1}^{t} c_k^2)} \quad .$$

Example

Assume that you play a sequence of n fair games, where the bet b_i in game i depends on the outcome of previous games. Let $B = \max_i b_i$. The probability of winning or losing more than λ is bounded by

$$\Pr(|Z_n| \geq \lambda) \leq 2e^{-2\lambda^2/nB^2}$$

$$\Pr(|Z_n| \geq \lambda B\sqrt{n}) \leq 2e^{-2\lambda^2}$$

$$\Pr\left(|Z_n| \geq \lambda\sqrt{\sum_{i=1}^{n} b_i^2}\right) \leq 2e^{-2\lambda^2}$$

Application: Balls and Bins

We place m balls independently and uniformly at random into n bins.

Let $X_i = 1$ if bin i is empty after all the balls were placed, otherwise $X_i = 0$.

$$E[X_i] = Pr(X_i = 1) = \left(1 - \frac{1}{n}\right)^m$$

Let $F = \sum_{i=1}^{n} X_i$ be the number of empty bins after the m balls are thrown. We know that

$$\mathrm{E}[F] = n\left(1 - \frac{1}{n}\right)^m \quad ,$$

but the events for different bins are not independent.

Formulating the process as a (Doob) martingale we'll get

$$\Pr(|F - \mathrm{E}[F]| \geq \epsilon) \leq 2\mathrm{e}^{-2\epsilon^2/m}$$

Doob Martingale

Let $X_1, X_2, \ldots, X_n$ be sequence of random variables. Let $Y = f(X_1, \ldots, X_n)$ be a random variable with $\mathrm{E}[|Y|] < \infty$.

For $i = 0, 1, \ldots, n$, let

$$\begin{aligned} Z_0 &= \mathrm{E}[Y] = \mathrm{E}_{X[1,n]}[f(X_1, \ldots, X_n] \\ Z_i &= \mathrm{E}_{X[i+1,n]}[Y | X_1 = x_1, X_2 = x_2, \ldots, X_i = x_i] \\ Z_n &= \mathrm{E}[Y | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n] = f(x_1, \ldots, x_n) \end{aligned}$$

Theorem

$Z_0, Z_1, \ldots, Z_n$ *is martingale with respect to* $X_1, X_2, \ldots, X_n$.

Proof

$Y = f(X_1, \ldots, X_n)$, $Z_0 = \mathrm{E}[Y]$,
$Z_i = \mathrm{E}_{X[i+1,n]}[Y | X_1 = x_1, \ldots, X_i = x_i]$,

$Z_1, Z_2. \ldots, Z_n$ is a martingale iff
(1) $E[|Z_i] < \infty$, and
(2) $\mathrm{E}_{X_{[i+1,n]}}[Z_{i+1} | X_1 = x_1, \ldots, X_i = x_i] = Z_i$.

(1) $E[|Z_i|] = E[|E[Y \mid X_, \ldots, X_i]|] \leq E[E[|Y| \mid X_1, \ldots, X_i]] = E[|Y|] < \infty$,
Jensen's Inequality: If $f(x)$ is convex then $f(E[X]) \leq E[f(X)]$.

$$\begin{aligned} (2) \quad & \mathrm{E}_{X_{[i+1,n]}}[Z_{i+1} | x_1, x_2, \ldots, x_i] \\ = \quad & \mathrm{E}_{X_{i+1}, X_{[i+2,n]}}[\mathrm{E}_{X[i+2,n]}[Y | X_1, \ldots, X_{i+1}] \mid x_1, ..., x_i] \\ = \quad & \mathrm{E}_{X[i+1,n]}[Y | x_1, x_2, \ldots, x_i] = Z_i \end{aligned}$$

Past: $P = x_1, \ldots, x_i$. Future: $F = X_{i+2}, \ldots, X_n$

$\mathrm{E}_{X_{i+1},F}[Z_{i+1} \mid P] = \mathrm{E}_{X_{i+1},F}[\mathrm{E}_F[Y | P, X_{i+1}] | P] = \mathrm{E}_{X[i+1,n]}[Y | P] = Z_i$

Simple Example

$Y = f(X_1, \ldots, X_n) = \sum_{i=1}^n X_i, \qquad X_i$ independent $\sim U[0,1]$.

$$
\begin{aligned}
Z_0 &= \mathrm{E}[Y] = \mathrm{E}_{X[1,n]} f(X_1, \ldots, X_n)] = \mathrm{E}[\sum_{i=1}^{n} X_i] = n/2 \\
Z_i &= \mathrm{E}_{X[i+1,n]}[Y | x_1, \ldots, x_i] \\
&= \sum_{j=1}^{i} x_j + \mathrm{E}[\sum_{j=i}^{n} X_i] = \sum_{j=1}^{i} x_j + (n-i)/2 \\
Z_n &= \mathrm{E}[Y | x_1, \ldots, x_n] = f(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}_{X_{i+1}}[Z_{i+1} | x_1, \ldots, x_i] &= \mathrm{E}_{X_{i+1}}\left[\sum_{j=1}^{i+1} X_i + \frac{n-i-1}{2} | x_1, \ldots, x_i\right] \\
&= \sum_{j=1}^{i} x_i + \frac{n-i}{2} = Z_i
\end{aligned}
$$

Example: Polya's Urn

- Start with m balls, r red, $m - r$ blue.
- Repeat n times:
  1. Pick a ball uniformly at random, check its color and return it to the urn.
  2. If red, add a new red ball, else add a new blue ball.

Let $X_i = 1$ if we add a red ball at step i, else $X_i = 0$

We want to estimate the number of new red balls among the n new balls, starting with ratio r/m

$$S_n\left(\frac{r}{m}\right) = \sum_{i=1}^{n} X_i = f(X_1, \ldots, X_n)$$

Claim: $E[S_n(\frac{r}{m})] = n\frac{r}{m}$.

On "average" the ratio doesn't change: $\frac{r+n\frac{r}{m}}{m+n} = \frac{r(1+\frac{n}{m})}{m(1+\frac{n}{m})} = \frac{r}{m}$

Example: Polya's Urn

Start with M balls, R red, $M-R$ blue. Repeat n times: pick a ball uniformly at random. Return it to the urn. If red add a red ball, else add a blue ball.

$X_i = 1$ if we add a red ball in step i, else $X_i = 0$.

$$S_n(r/m) = \sum_{i=1}^{n} X_i = f(X_1, \ldots, X_n)$$

Claim: $\mathrm{E}[S_n(\frac{r}{m})] = n\frac{r}{m}$.

Proof: By induction on $t \geq 0$, that $\mathrm{E}[S_t] = tr/m$.

$$\mathrm{E}[S_{t+1} \mid S_t] = S_t + \frac{r + S_t}{m+t}$$

$$\mathrm{E}[S_{t+1}] = \mathrm{E}[\mathrm{E}[S_{t+1} \mid S_t]] = \mathrm{E}\left[S_t + \frac{r+S_t}{m+t}\right]$$

$$= t\frac{r}{m} + \frac{r + tr/m}{m+t} = t\frac{r}{m} + \frac{r(1+t/m)}{m(1+t/m)} = (t+1)\frac{r}{m}$$

Example: Polya's Urn

$X_i = 1$ if added a red ball in step i, else $X_i = 0$,

$S_n(\frac{r}{m}) = \sum_{i=1}^{n} X_i$, and $\mathrm{E}[S_n(\frac{r}{m})] = n\frac{r}{m}$

Let $Z_i = \mathrm{E}[S_n | \ X_1 = x_1, \ldots, X_i = x_i]$.

We verify that $Z_1, \ldots, Z_n$ is a martingale (which we already know, since it's a Doob martingale.)

Let $r_i = r + \sum_{j=1}^{i} x_j$

$$
\begin{aligned}
Z_i &= \mathrm{E}[S_n | \ X_1 = x_1, \ldots, X_i = x_i] = \sum_{j=1}^{i} x_j + E[S_{n-i}(\frac{r + \sum_{j=1}^{i} x_j}{m+i})] \\
&= \sum_{j=1}^{i} x_j + (n-i)\frac{r + \sum_{j=1}^{i} x_j}{m+i} = \sum_{j=1}^{i} x_j + (n-i)\frac{r_i}{m+i}
\end{aligned}
$$

$r_i = r + \sum_{j=1}^{i} x_j$
$Z_i = \mathrm{E}[S_n | \ X_1 = x_1, \ldots, X_i = x_i] = \sum_{j=1}^{i} x_j + (n-i)\frac{r_i}{m+i}.$

$$\mathrm{E}[Z_{i+1} \mid X_1, \ldots, X_i] = \mathrm{E}[\mathrm{E}[S_n | X_1, X_2, \ldots, X_{i+1}] \mid X_1 = x_1, \ldots, X_i = x_i]$$

$$\begin{aligned}
&= \mathrm{E}\left[\sum_{j=1}^{i} x_j + X_{i+1} + S_{n-i-1}\left(\frac{r_i + X_{i+1}}{m+i+1}\right)\right] \\
&= \sum_{j=1}^{i} x_j + \frac{r_i}{m+i} + (n-i-1)\frac{r_i + \frac{r_i}{m+i}}{m+i+1} \\
&= \sum_{j=1}^{i} x_j + \frac{r_i}{m+i} + (n-i-1)\frac{r_i(1 + \frac{1}{m+i})}{m+i+1} \\
&= \sum_{j=1}^{i} x_j + \frac{r_i}{m+i} + (n-i-1)\frac{r_i}{m+i} = Z_i
\end{aligned}$$

Tail Inequalities: Doob Martingales

Let $X_1, \ldots, X_n$ be sequence of random variables.

Random variable Y:

- $Y = f(X_1, X_2, \ldots, X_n)$ is a function of $X_1, X_2, \ldots, X_n$;
- $\mathrm{E}[|Y|] < \infty$.

Let $Z_i = \mathrm{E}[Y | X_1, \ldots, X_i]$, $i = 0, 1, \ldots, n$.

$Z_0, Z_1, \ldots, Z_n$ is martingale with respect to $X_1, \ldots, X_n$.

If we can use Azuma-Hoeffding inequality:

$$\Pr(|Z_n - Z_0| \geq \lambda) \leq .....$$

then we have,

$$\Pr(|Y - \mathrm{E}[Y]| \geq \lambda) \leq ......$$

We need a bound on $|Z_i - Z_{i-1}|$.

Example: Pattern Matching

$A = (a_1, a_2, \ldots, a_n)$ string of characters, each chosen independently and uniformly at random from Σ, with $m = |\Sigma|$.

pattern: $B = (b_1, \ldots, b_k)$ fixed string, $b_i \in \Sigma$.

$F =$ number occurrences of B in random string S.

$$\mathrm{E}[F] = (n - k + 1)\left(\frac{1}{m}\right)^k \quad .$$

Can we bound the deviation of F from its expectation?

$F =$ number occurrences of B in random string A.

$Z_0 = \mathrm{E}[F]$ and $Z_n = F$.

$Z_i = \mathrm{E}[F|a_1, \ldots, a_i]$, for $i = 1, \ldots, n$.

$Z_0, Z_1, \ldots, Z_n$ is a Doob martingale.

Each character in A can participate in no more than k occurrences of B:

$$|Z_i - Z_{i+1}| \leq k \quad .$$

Azuma-Hoeffding inequality (version 1):

$$\Pr(|F - \mathrm{E}[F]| \geq \lambda) \leq 2e^{-\lambda^2/(2nk^2)} \quad .$$

McDiarmid Bound

In general it is hard to prove a bound on $|Z_i - Z_{i-1}|$. This theorem gives a general condition:

Theorem

Assume that $f(X_1, X_2, \ldots, X_n)$ *satisfies, for all* $1 \le i \le n$,

$$|f(x_1, \ldots, , x_i, \ldots, x_n) - f(x_1, \ldots, , y_i, \ldots, x_n)| \le c_i \quad .$$

and $X_1, \ldots, X_n$ *are independent, then*

$$\Pr(|f(X_1, \ldots, X_n) - \mathrm{E}[f(X_1, \ldots, X_n)]| \ge \lambda) \le 2e^{-2\lambda^2/(\sum_{k=1}^{n} c_k^2)} \quad .$$

[Changing the value of X_i changes the value of the function by at most c_i.]

Proof

Define a Doob martingale $Z_0, Z_1, \ldots, Z_n$:

- $Z_0 = \mathrm{E}[f(X_1, \ldots, X_n)] = \mathrm{E}[f(\bar{X})]$
- $Z_i = \mathrm{E}[f(X_0, \ldots, X_n) \mid X_1, \ldots, X_i] = \mathrm{E}[f(X_i, \ldots, X_n) \mid X^i]$
- $Z_n = f(X_1, \ldots, X_n) = f(\bar{X})$

We want to prove that this martingale satisfies the conditions of

Theorem (Azuma-Hoeffding Inequality)

Let $Z_0, Z_1, \ldots,$ be a martingale with respect to $X_0, X_1, X_2, \ldots,$ such that

$$B_k \leq Z_k - Z_{k-1} \leq B_k + c_k \ ,$$

for some constants c_k and for some random variables B_k that may be functions of $X_0, X_1, \ldots, X_{k-1}$. Then, for all $t \geq 0$ and any $\lambda > 0$,

$$\Pr(|Z_t - Z_0| \geq \lambda) \leq 2e^{-2\lambda^2/(\sum_{k=1}^{t} c_k^2)} \ .$$

Lemma

If $X_1, \ldots, X_n$ *are independent and*

$$|f(x_1, \ldots, , x_i, \ldots, x_n) - f(x_1, \ldots, , y_i, \ldots, x_n)| \leq c_i \quad .$$

then for some random variable B_k*,*

$$B_k \leq Z_k - Z_{k-1} \leq B_k + c_k \quad ,$$

$$Z_k - Z_{k-1} \quad = \quad \mathrm{E}[f(\bar{X}) \mid X^k] - \mathrm{E}[f(\bar{X}) \mid X^{k-1}] \quad .$$

Hence $Z_k - Z_{k-1}$ is bounded above by

$$\sup_x \mathrm{E}[f(\bar{X}) \mid X^{k-1}, X_k = x] - \mathrm{E}[f(\bar{X}) \mid X^{k-1}]$$

and bounded below by

$$\inf_y \mathrm{E}[f(\bar{X}) \mid X^{k-1}, X_k = y] - \mathrm{E}[f(\bar{X}) \mid X^{k-1}] \quad .$$

$$Z_k - Z_{k-1} \quad = \quad \sup_{x,y} \mathrm{E}[f(\bar{X}, X_k = x) - f(\bar{X}, X_k = y) \mid X^{k-1}].$$

Because the X_i are independent, the values for $X_{k+1}, \ldots, X_n$ do not depend on the values of $X_1, \ldots, X_k$.

$$\sup_{x,y} \mathrm{E}[f(\bar{X}, x) - f(\bar{X}, y) \mid X_1 = x_1, \ldots, X_{k-1} = x_{k-1}]$$

$$= \sup_{x,y} \sum_{x_{k+1},\ldots,x_n} \Pr\big((X_{k+1} = x_{k+1}) \cap \ldots \cap (X_n = x_n)\big) \cdot$$

$$\big(f(x_{[1,k-1]}, x, x_{[k+1,n]} - f(x_{[1,k-1]}, y, x_{[k+1,n]})\big)$$

But

$$\big(f(x_{[1,k-1]}, x, x_{[k+1,n]} - f(x_{[1,k-1]}, y, x_{[k+1,n]})\big) \leq c_k$$

and therefore

$$\mathrm{E}[f(\bar{X}, x) - f(\bar{X}, y) \mid X^{k-1}] \leq c_k$$

Application: Balls and Bins

We are throwing m balls independently and uniformly at random into n bins.
Let $X_i =$ the bin that the ith ball falls into.
Let F be the number of empty bins after the m balls are thrown.

$$E[F] = n\left(1 - \frac{1}{n}\right)^m \quad ,$$

The sequence $Z_i = E[F \mid X_1, \ldots, X_i]$ is a Doob martingale.
$F = f(X_1, X_2, \ldots, X_m)$ satisfies the Lipschitz condition with bound 1, and the X_i's are independent. We therefore obtain

$$\Pr(|F - E[F]| \geq \epsilon) \leq 2e^{-2\epsilon^2/m}$$

Example: Polya's Urn

- Start with m balls, r red, $m - r$ blue.
- Repeat n times:
  1. Pick a ball uniformly at random, check its color and return it to the urn.
  2. If red, add a new red ball, else add a new blue ball.

Let $X_i = 1$ if we add a red ball at step i, else $X_i = 0$
$S_n\left(\frac{r}{m}\right) = \sum_{i=1}^{n} X_i = f(X_1, \ldots, X_n)$ satisfies the Lipschitz condition with bound 1, and the X_i's are independent.

$\mathrm{E}[S_n(\frac{r}{m})] = n\frac{r}{m}$.

$Z_i = \mathrm{E}[S_n | \; X_1 = x_1, \ldots, X_i = x_i]$ is a Doob martingale.

$$\Pr(|S_n - n\frac{r}{m}| \geq \epsilon) \leq 2\mathrm{e}^{-2\epsilon^2/m}$$

Application: Chromatic Number

Given a random graph G in $G_{n,p}$, the *chromatic number* $\chi(G)$ is the minimum number of colors required to color all vertices of the graph so that no adjacent vertices have the same color.
We use the vertex exposure martingale defined below:
Let G_i be the random subgraph of G induced by the set of vertices $1, \ldots, i$, let $Z_0 = \mathrm{E}[\chi(G)]$, and let

$$Z_i = \mathrm{E}[\chi(G) \mid G_1, \ldots, G_i] \ .$$

Since a vertex uses no more than one new color, again we have that the gap between Z_i and Z_{i-1} is at most 1.
We conclude

$$\Pr(|\chi(G) - \mathrm{E}[\chi(G)]| \geq \lambda\sqrt{n}) \leq 2\mathrm{e}^{-2\lambda^2} \ .$$

This result holds even without knowing $\mathrm{E}[\chi(G)]$.