LINEAR ALGEBRA

A REVIEW

EIGENVECTORS & EIGENVALUES

evectors

evalues

$$\boxed{Ax = \lambda x}$$

Matrix

evector

evalue

$A \in \mathbb{R}^{n \times n}$ = space of $n \times n$ matrices with real numbers as entries

$\lambda \in \mathbb{C}$ = 1-dim Complex numbers space

$x \in \mathbb{R}^n$ = n dim Real number space

$$Ax = \lambda x$$

↗ Ax leaves x same except multiplied by a scalar λ

The values of λ are computed as roots of the <u>characteristic polynomial</u>

$$p(\lambda) = \det(\lambda I - A) = 0$$

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$$

there are n roots of $p(\lambda)$

some different, some repeated

Note. For a fixed λ we have to solve a linear system of equations

$$Ax = \lambda x$$

Properties of matrix A of interest

$A^T \equiv A$ Transposed

$\boxed{A \text{ is } \boxed{\text{symmetric}}: \ A^T = A}$

- If $A^T = A$ then $\lambda(A)$ are real

$\lambda(A) = \{\lambda_1(A), \dots, \lambda_n(A)\}$ are the evalues of A

- A matrix is positive definite if $x^T A x > 0, \ \forall x \neq 0$ | Notation $A \succ 0$

and is positive semi-definite

if $x^T A x \geq 0, \forall x \neq 0$ | $A \succeq 0$

Lemma

A matrix A is positive definite iff $\lambda(A) > 0$, all evals > 0

• For a matrix B the matrix $A = B^T B$ is positive semi-definite

Indeed

$$x^T A x = x^T B^T B x = (Bx)^T B x = \|Bx\|_F^2 \geq 0$$

dot product $\langle Bx, Bx \rangle = \|Bx\|_F^2$

The Trace of A

$$tr(A) = \sum_i A_{ii} = \sum_i \lambda_i(A)$$

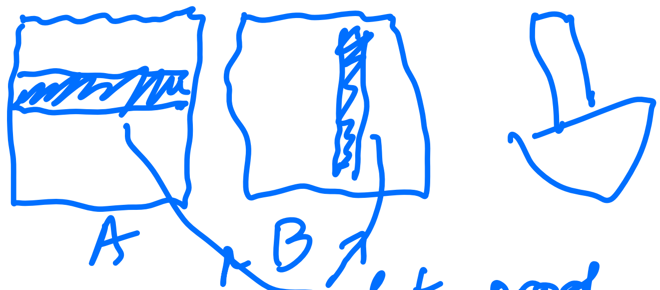
$\boxed{\text{The trace of A equals the sum of the evalues of A.}}$

The Frobenious Norm

$$\|A\|_F^2 = \sum_{i,j} A_{ij}^2 = tr(A^T A)$$

- The dot product of A and B

$$(A \cdot B)_{ij} = \sum_k A_{ik} B_{kj}$$



dot prod. of row i and column j

Let us prove $\boxed{\|A\|_F^2 = \mathrm{tr}(A^T A)}$

indeed,

$$(A^T A)_{ij} = \sum_k A_{ki} A_{kj}$$

$$\mathrm{tr}(A^T A) = \sum_i (A^T A)_{ii} =$$

$$= \sum_i \sum_k A_{ki} A_{ki} =$$

$$= \sum_{i,k} A_{ki}^2 = \|A\|_F^2$$

We have

$$\boxed{tr(AB) = tr(BA)}$$

So

$$\underbrace{x^T}_{1\times n} \underbrace{A}_{n\times n} \underbrace{x}_{1\times n} = \text{a number} = tr(x^T A x)$$

$$= tr(A x x^T).$$

Norms for Matrices

Frobenious:

$$\|A\|_F^2 = \sum_{i,j} A_{ij}^2 = \langle A, A \rangle$$

where $\langle \cdot , \cdot \rangle$ is the DOT PRODUCT

L1 norm:

$$\|A\|_1 = \sum_{i,j} |A_{ij}|$$

$$tr(A^T B) = \sum_{i,j} A_{ij} B_{ij} =$$

$$= \langle A, B \rangle$$

Dot Product

DOT PRODUCT:

$$\langle A, B \rangle = tr(A^T B)$$

$$\langle A, A \rangle = \|A\|_F^2$$

- dot product between two vectors is computing the angle between the vectors
- Dot Product for matrices exactly as for vectors

$$\langle A, B \rangle = tr(A^T B) = \sum_{i,j} A_{ij} B_{ij}$$

MATRIX DECOMPOSITION

1) EIGENVALUE DECOMPOSITION

$Ax = \lambda x$ ← evector

↑ evalue

λ's are the roots of the polynomial

$$p(\lambda) = \det(\lambda I - A)$$

there are n roots of an $n \times n$ matrix A

$A \in \mathbb{R}^{n \times n}$

$\{\lambda_i\}_{i=1}^{n}$ evalues of A

$\{v_i\}_{i=1}^{n}$ evectors of A

$$A v_i = \lambda_i v_i \,, \quad 1 \le i \le n$$

$$A [v_1, \dots, v_n] = [\lambda_1 v_1, \dots, \lambda_n v_n]$$

$$= \underbrace{[v_1, \dots, v_n]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

Therefore

$$AV = V\Lambda$$

$\boxed{\text{If}}$ V is invertible then

$$\boxed{A = V \Lambda V^{-1}}$$

!!!

the eigenvalue decomposition

If the eigenvectors are linearly independent

then V is invertible

These matrices: $\left\{ \begin{array}{l} V \text{ is an } n\times n \text{ Complex matrix in } \mathbb{C}^{n\times n} \\ \\ \Lambda \text{ —— " —— in } \mathbb{C}^{n\times n} \\ \text{diagonal matrix} \end{array} \right.$

A real matrix $A \in \mathbb{R}^{n\times n}$ could have V and Λ complex $\underbrace{\quad}_{\text{number}}$ matrices

Very important special case

A is SYMMETRIC

THEOREM

If $A = A^T$ (symmetric)

then

① $\lambda_i \in \mathbb{R}$, $\forall i$, and

② the v_i can be selected to be ORTHOGONAL

i.e.

$v_i^T v_j = 0$, $\forall i \neq j$

Def v and w two vectors are <u>orthogonal</u> (or perpendicular) if

$$v^T w = 0$$

Recall: evectors are always defined up to a scalar.

$$Ax = \lambda x$$

if c is any scalar $\neq 0$

$$Axc = \lambda xc$$

$$A(xc) = \lambda(xc)$$

xc is also evector

Because of this, we can always assume

$$\|v_i\|_F^2 = 1 \quad \text{normalized to } 1$$

Ex. any x has the property $\exists c$ such that

$$\|ax\|_F^2 = 1$$

We write $\underbrace{,\ \|v_i\|}_{\text{above}}$ for simplicity without the F and "2".

An example

let $x = (2,3)$ a vector

$$\|x\|_F^2 = 2^2 + 3^2$$

take $c = \dfrac{1}{\sqrt{2^2+3^2}}$

then $cx = \left(\dfrac{2}{\sqrt{2^2+3^2}}, \dfrac{3}{\sqrt{2^2+3^2}} \right)$

$$\|cx\|_E^2 = \left(\frac{2}{\sqrt{2^2+3^2}}\right)^2 + \left(\frac{3}{\sqrt{2^2+3^2}}\right)^2$$

$$= \frac{2^2+3^2}{2^2+3^2} = 1$$

So we can assume without loss of generality that the vectors are normalized $\|v_i\| = 1, \forall i$.

Consider

$$V^T V = \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \cdots & v_1^T v_n \\ \vdots & & & \\ v_n^T v_1 & v_n^T v_2 & \cdots & v_n^T v_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix} = I$$

I is the identity matrix

Def The Orthogonal Group of dimension n is denoted

$$\mathcal{O}(n) = \{R \in \mathbb{R}^{n\times n} \mid R^T R = I\}$$

= all real matrices whose columns are orthogonal to each other

If A is symmetric the matrix of its evectors is orthogonal !!!

A $n\times n$

$V \in \mathcal{O}(n)$

From $V^T V = I$ it follows

$$V^{-1} = V^T$$

So

$$\boxed{\begin{aligned} A &= V \Lambda V^{-1} = \\ A &= V \Lambda V^T \end{aligned}}$$

Symmetric matrices have an Eigenvalue Decomposition into real values matrices V, Λ

2) SINGULAR VALUE DECOMPOSITION (SVD)

The previous Decomposition, the Eigenvalue Decomposition has some shortcomings:

- Not always exist (remember: $\boxed{\text{If}}$ V was invertible it was possible)
- Matrix A must be a square $n \times n$ matrix

- Decomposition was into possible complex matrices

This decomposition SVD applies to arbitrary matrix $A \in \mathbb{R}^{m \times n}$

Reminder: The evalues of $A \in \mathbb{R}^{m \times n}$ can be positive or negative in general

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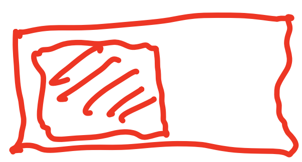
The SVD THEOREM

For any matrix $A \in \mathbb{R}^{m \times n}$

there exist matrices U, V and Σ as follows:

$$U \in \mathcal{O}(m)$$

$$V \in \mathcal{O}(n)$$



rank

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & 0 & \\ & & \sigma_r & & \\ & 0 & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \in \mathbb{R}^{m \times n}$$

diagonal

Such that:

1. $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

2. $r = \text{rank}(A)$

3. $$\boxed{A = U \Sigma V^T}$$
$$\underset{m \times m}{U} \quad \underset{m \times n}{\Sigma} \quad \underset{n \times n}{V^T}$$

$U = [u_1, \ldots, u_m]$ are the left-singular vectors of A

$V = [v_1, \ldots, v_n]$ are the right-singular vector of A

$\sigma_1, \ldots, \sigma_r$ are the singular values of A

So: any matrix can be written as a product of an orthogonal × diagonal × orthogonal matrix product

$\boxed{\text{The COMPACT SVD}}$

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_r & u_{r+1} & \cdots & u_m \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_r & & & \\ & 0 & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T$$

$$= \begin{bmatrix} \sigma_1 u_1, & \sigma_2 u_2, & \cdots & \sigma_r u_r, & 0, & \cdots & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} =$$

$$= \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

$$= \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix}}_{U}\ \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}}_{\Sigma}\ \underbrace{\begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}^T}_{V^T}$$

$m \times r \qquad r \times r \qquad r \times n$

$$\boxed{A = U \Sigma V^T}$$

the compact SVD.

$$AA^T = U \Sigma V^T V \Sigma U^T$$

$$= U \Sigma^2 U^T$$

$V^T V = I$

evectors(AA^T) = left-singular vectors(A)

evectors(A^TA) = right-singular vectors(A)

$$A^TA = V\Sigma U^T U \Sigma V^T$$
$$= V\Sigma^2 V^T$$

$U^TU = I$

evalues(A^TA) = (singular values(A))^2

$$tr(A) = \sum_i \lambda_i(A)$$

$$\mathrm{tr}(A^T A) = \sum_i \lambda_i (A^T A)$$
$$= \sum_i \sigma_i^2 = \|A\|_F^2$$

$$\langle A, A \rangle = \|A\|_F^2$$

$$\boxed{\begin{array}{c}\text{The Nullspace}(A) \\ \equiv \text{Kernel}(A)\end{array}}$$

$$\mathrm{Ker}(A) = \{x \mid Ax = 0\}$$

You can compute it directly from SVD which are the vectors mapped to 0?

$$A v_{r+1} = \sum_{i=1}^{r} \sigma_i u_i v_i^T v_{r+1} = 0$$

$$A v_{r+2} = 0$$

$$\vdots$$

$$A v_n = 0$$

$$\{ \underbrace{v_{r+1}, \ldots, v_n}_{n-r} \} \in \ker(A)$$

They are all orthogonal so they are linearly independent

$$\boxed{\dim \operatorname{Ker}(A) = n - r}$$

if $V = [\underbrace{V_1}_{r} \quad \underbrace{V_2}_{n-r}]$

$$V_2 = \text{Nullspace}(A) = \operatorname{Ker}(A).$$