CSCI1950-J: Take-Home Exam 1

Out: February 17, 2011 (revised February 22 to fix Problem 2)
Due: February 24, 2011 (in class)

This is a strictly non-collaborative assignment. You may only discuss the questions and answers with the course staff. You are permitted to use the textbook, but no external resources.

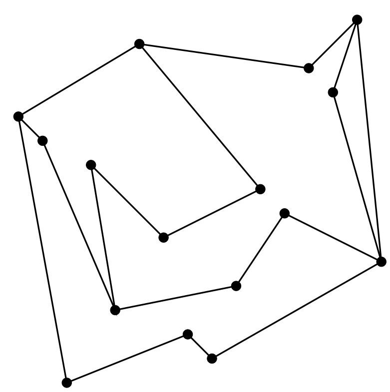
The writeup for this homework will be collected in class on the due date. All work should be typed, preferably in LaTeX.

Readings: Textbook §1.2 (except §1.2.3.2), §1.3 (except §1.3.3), §1.4 (except pp. 33–35), §2.1, §2.2 (except 60–62), §2.3

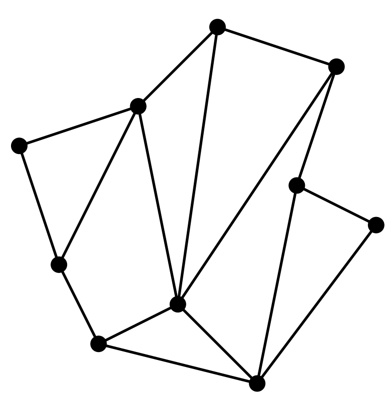
Problem 1 (20 points)

Note: all figures will be made available on the course website as both .fig and .pdf files.

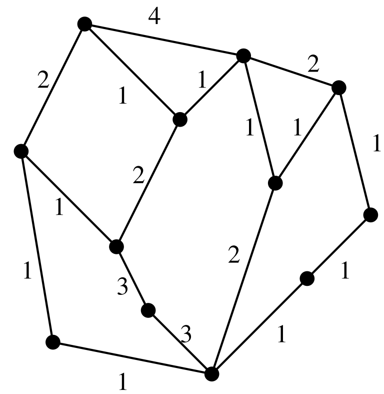
(a) Show the result of a regularization sweep (as used in the chain method) on the given planar straight-line graph. Distinguish edges added in the downward sweep from those added in the upward sweep.



(b) Show the labels assigned by the chain method to the given regularized planar straight-line graph (i) after the top to bottom pass (ii) after both passes.



(c) Show the chains used implicitly by the chain method on the given labeled planar straight-line graph.



Problem 2 (20 points)

A variant of the chain method for point location achieves query time $O(\log n)$ through a technique known as **fractional cascading**. In a simpler setting for fractional cascading, we have k sorted lists $L^{(1)}, \ldots, L^{(k)}$, each of length n/k. For a search query q, we need to determine for each $i \in \{1, \ldots, k\}$ the least index j_i such that $L^{(i)}_{j_i} \geq q$. (If q is greater than all elements of $L^{(i)}$, then $j_i = k + 1$.)

For example, consider the lists

$$L^{(1)} = \texttt{(1, 6, 8, 11)}$$
$$L^{(2)} = \texttt{(3, 9, 10, 12)}$$
$$L^{(3)} = \texttt{(2, 4, 5, 7)}.$$

When $q = 6.5$, the search algorithm should return $j_1 = 3$ and $j_2 = 2$ and $j_3 = 4$.

A naive approach is to perform k binary searches, for a total cost of $k \log(n/k)$. A fractional cascading approach requires k additional lists $M^{(1)}, \ldots, M^{(k)}$ constructed as follows. Start by setting $M^{(k)} = L^{(k)}$. To construct $M^{(i)}$, merge $L^{(i)}$ with every second element of $M^{(i+1)}$. Construct also two lists $A^{(i)}$ and $B^{(i)}$, where $A^{(i)}_j$ is the result of searching for $M^{(i)}_j$ in $L^{(i)}$, and $B^{(i)}_j$ is the result of searching for $M^{(i)}_j$ in $M^{(i+1)}$.

$$M^{(3)} = \texttt{(2, 4, 5, 7)}$$
$$M^{(2)} = \texttt{(3, 4, 7, 9, 10, 12)}$$
$$A^{(2)} = \texttt{(1, 2, 2, 2, 3, 4)}$$
$$B^{(2)} = \texttt{(2, 2, 4, 5, 5, 5)}$$
$$M^{(1)} = \texttt{(1, 4, 6, 8, 9, 11, 12)}$$
$$A^{(1)} = \texttt{(1, 2, 2, 3, 4, 4, 5)}$$
$$B^{(1)} = \texttt{(1, 2, 3, 4, 4, 6, 6)}.$$

(a) Compute the M, A, B lists for

$$L^{(1)} = \texttt{(2, 6, 8, 10)}$$
$$L^{(2)} = \texttt{(1, 4, 7, 12)}$$
$$L^{(3)} = \texttt{(3, 5, 9, 11)}.$$

(b) Show that each list M_j contains less than $2n/k$ elements. Conclude that fractional cascading requires only linear space.

(c) Suppose that j is the result of searching for q in $M^{(i)}$. Give pseudocode to search in constant time for q in (i) $L^{(i)}$ and (ii) $M^{(i+1)}$.

(d) Given a query q and a $\sqrt{n} \times \sqrt{n}$ matrix C having rows whose elements increase left to right and columns whose elements decrease top to bottom, give an algorithm to determine in time $O(\sqrt{n})$, with no preprocessing, whether q appears in C.

Problem 3 (20 points)

(a) Show that in every planar straight-line graph, there exists a vertex of degree at most 5. (Hint: at least how many edges would there be in total if every vertex had degree at least 6? Use Euler's formula.)

(b) A k-**coloring** of a graph is a map from vertices to $\{1, \ldots, k\}$ such that neighboring vertices map to different values. Design and analyze a linear-time algorithm that computes a 6-coloring of a planar straight-line graph.[1] (Hint: save a vertex of degree at most 5 for last.)

Problem 4 (20 points)

(a) Design and analyze a linear-time algorithm that given a planar straight-line graph G and a set of "interesting" vertices S, identifies a subset of at least $|S|/6$ interesting vertices between any two of which there is no edge. You may assume the existence of the algorithm requested in 3(b).

(b) Recall from our analysis of Kirkpatrick's triangulation refinement method for point location that α is the maximum fraction of vertices not on the infinite face that can survive one refinement step. Determine the value of α when the points removed are identified using the previous subroutine, where the set S consists of the vertices of degree at most $K = 12$ not on the infinite face.

Problem 5 (20 points)

In this problem, you will analyze the multistage direct access technique for range searching described in Textbook §2.3.3.

(a) Consider an l-stage search structure, which has l increasingly fine gauges. Obtain a method that uses $O(N^{c_2})$ words of storage and has query time $O(c_1 \log N)$, where c_2 is minimal. Express c_1 and c_2 as functions of l. (Hint: work out the case $l = 3$.)

(b) Using (a), prove Theorem 2.10 (Textbook p. 83).

[1] It is known how to compute a 5-coloring in linear time. Every PSLG admits a 4-coloring, but the best known algorithm is quadratic.