CSCI1950-J: Take-Home Exam 4 (100 points)

Out: Tuesday, April 26, 2011
Due: Tuesday, May 3, 2011 (in class)

Note: in lieu of Tuesday office hours, I (David) will hold office hours on Friday, April 29 from 6:00 pm to 8:00 in a location to be announced on the website. I am also available by appointment.

This is a strictly non-collaborative assignment. You may only discuss the questions and answers with the course staff. You are permitted to use the textbook, but no external resources.

The writeup for this homework will be collected in class on the due date. All work should be typed, preferably in LaTeX.

Problem 1 (25 points)

Define the *polar dual* of a set $K \subseteq \mathbb{R}^2$ to be the following intersection of half-planes.

$$K^\dagger \triangleq \bigcap_{(a,b)\in K} \{(x,y) : (x,y) \in \mathbb{R}^2,\ ax + by \leq 1\}.$$

For each point p, the boundary of the half-plane corresponding to p is the polar of p with respect to the unit circle.

(a) Let $K \subseteq L \subseteq \mathbb{R}^2$ be sets. Prove that $L^\dagger \subseteq K^\dagger$.

(b) Let $K, L \subseteq \mathbb{R}^2$ be nonempty convex sets. The convex closure of their union is the set

$$M \triangleq \{\lambda p + (1-\lambda)q : \lambda \in [0,1],\ p \in K,\ q \in L\}.$$

Prove that $M^\dagger = K^\dagger \cap L^\dagger$.

(c) Describe the polar dual of the line $\{(x,y) : (x,y) \in \mathbb{R}^2,\ x + y = 0\}$.

(d) Describe the polar dual of the filled triangle with vertices $(1,0), (-4/5, 3/5), (-3/5, -4/5)$.

(e) Describe the polar dual of the filled ellipse $\{(x,y) : (x,y) \in \mathbb{R}^2,\ x^2 + 2y^2 \leq 1\}$.

(f) Let $K, L \subseteq \mathbb{R}^2$ be sets. Prove that $K \subseteq L^\dagger$ if and only if $L \subseteq K^\dagger$.

Problem 2 (25 points)

Given are a point p and pairwise disjoint disks $D_1, \ldots, D_n$, none of which contain p. Design and analyze (correctness and running time) an algorithm to determine which disks can be seen from p, that is, determine the set of disks containing a point q such that the interior of the segment $\overline{pq}$ does not intersect the union of the disks. Get the best asymptotic running time you can.

Problem 3 (25 points)

Given are $3n$ points $a_1, b_1, c_1, a_2, b_2, c_2, \ldots, a_n, b_n, c_n$. Design and analyze (correctness and running time) an algorithm to determine whether there exists a line ℓ such that

- for all i, the segment $\overline{a_i b_i}$ intersects ℓ
- for all i, the segment $\overline{b_i c_i}$ does not intersect ℓ.

Get the best asymptotic running time you can.

Problem 4 (25 points)

Consider the following two problems.

ENCLOSURE-OF-SQUARES Given filled isothetic squares $S_1, \ldots, S_n \subseteq \mathbb{R}^2$, report all ordered pairs (i, j) such that $S_i \subseteq S_j$.

3-D DOMINANCE Given points $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n) \in \mathbb{R}^3$, report all ordered pairs (i, j) such that $x_i \leq x_j$ and $y_i \leq y_j$ and $z_i \leq z_j$.

Show that ENCLOSURE-OF-SQUARES and 3-D DOMINANCE are linear-time transformable to one another.