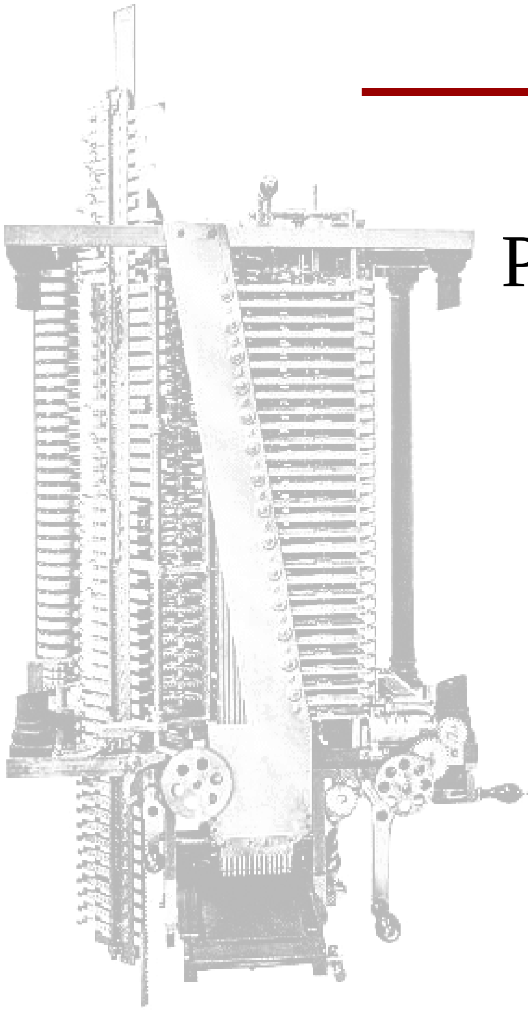
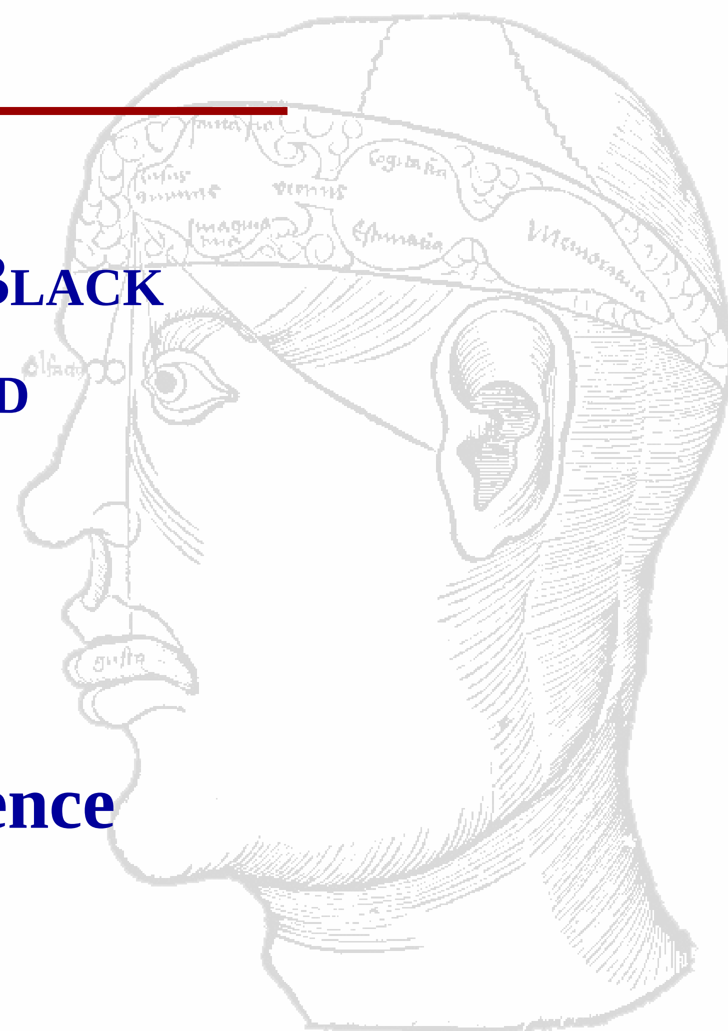
Topics in Brain Computer Interfaces

CS295-7



Professor: **MICHAEL BLACK**

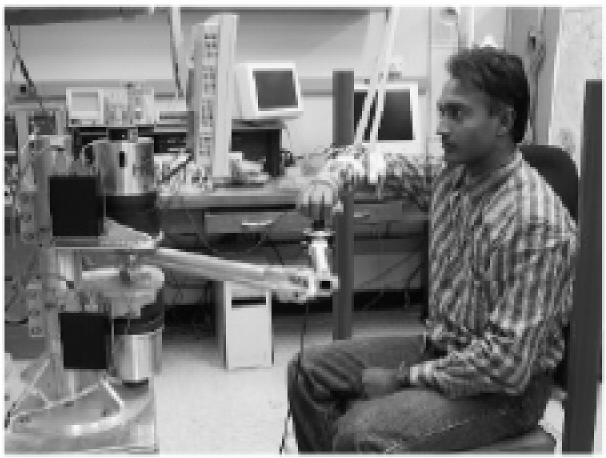
TA: **FRANK WOOD**

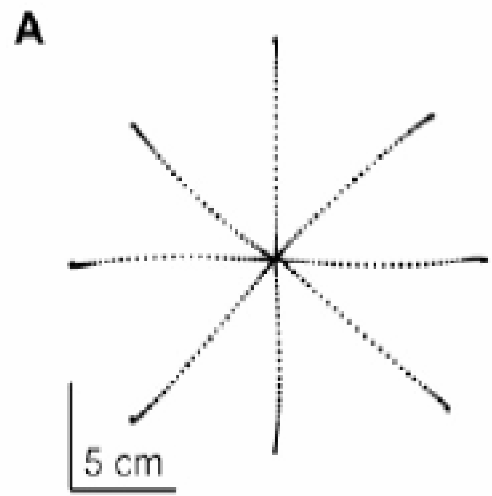


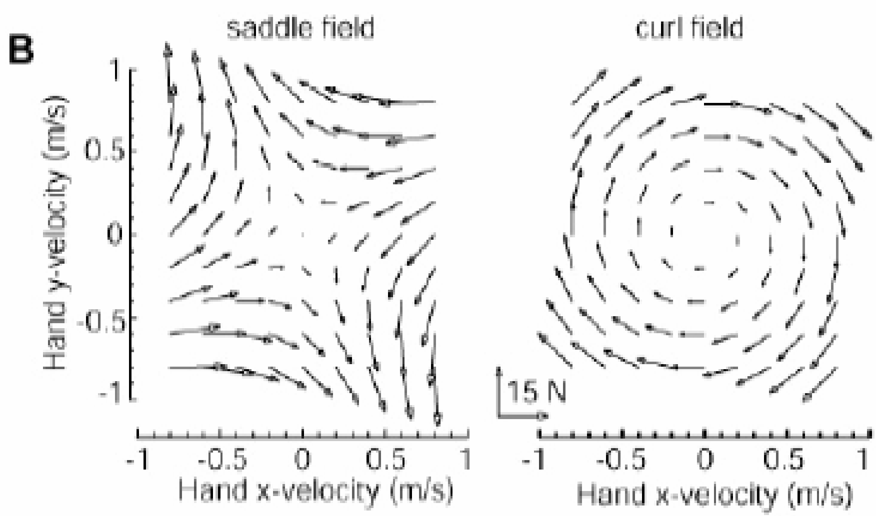
Spring 2005

Probabilistic Inference

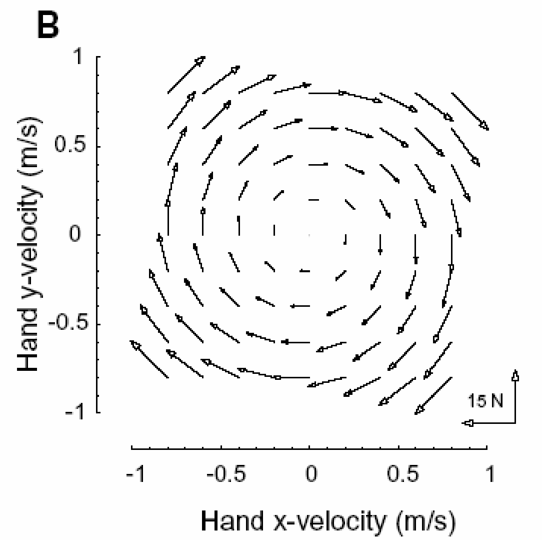
Reza Shadmehr

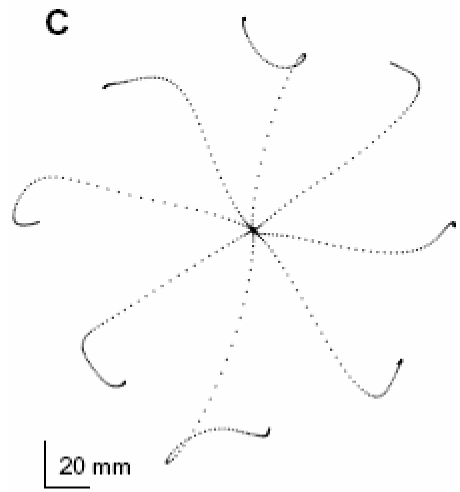


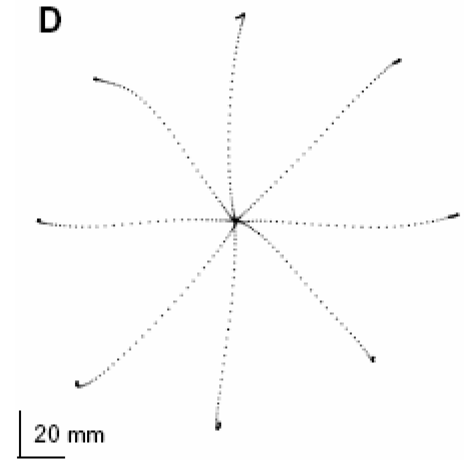




Reza Shadmehr

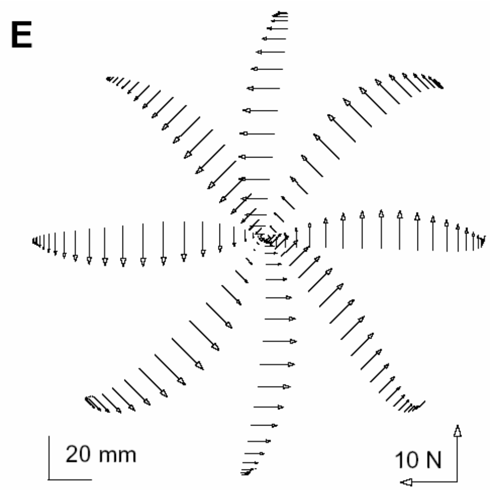


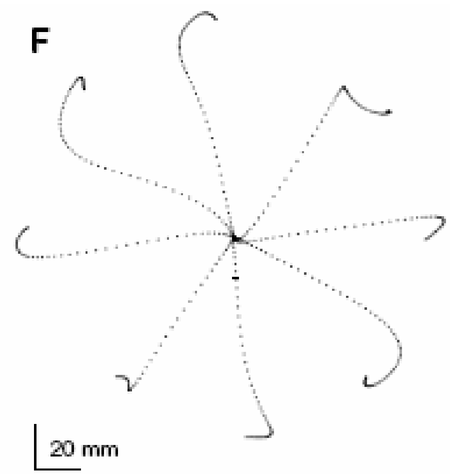




After 300 trails

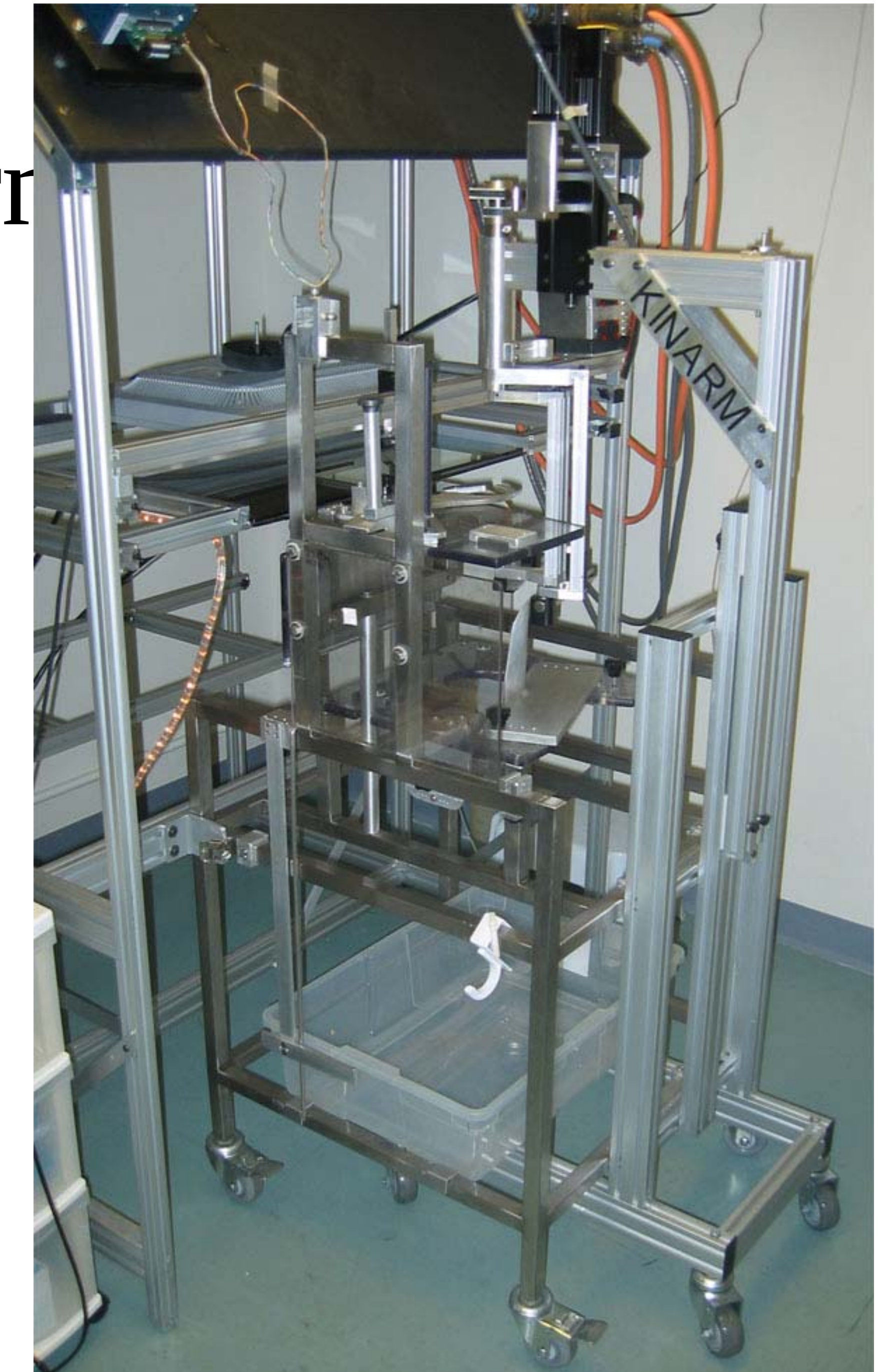
Forces used
to counter
field.





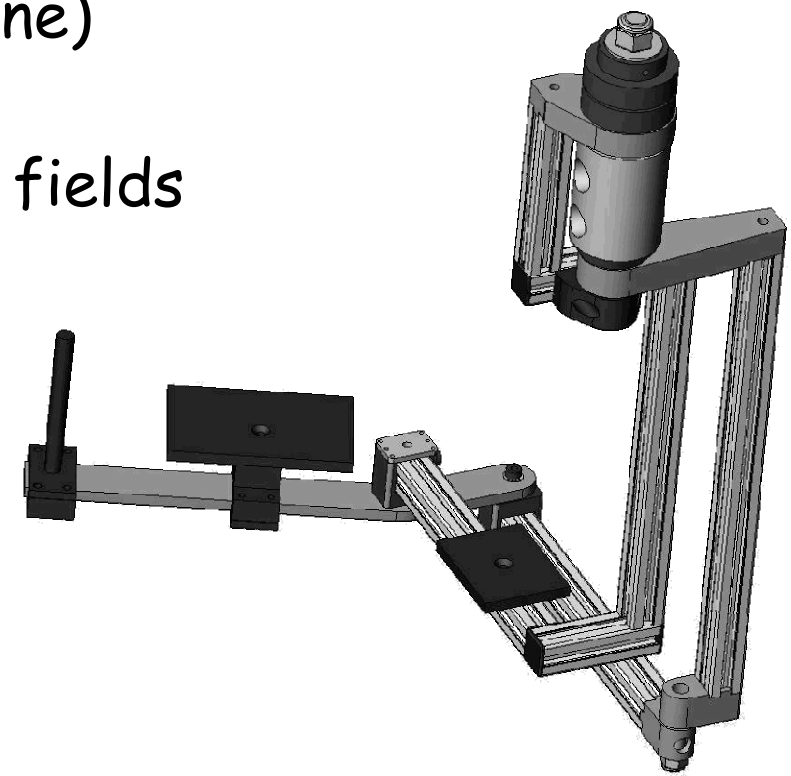
“Catch” trails


KINARM


KINARM

Kinarm project

- Connect Kinarm with TG2 game software (partially done).
- Calibrate Kinarm (partially done)
- Develop a high-level way of programming different force fields (not done).
- Develop a basic set of fields.

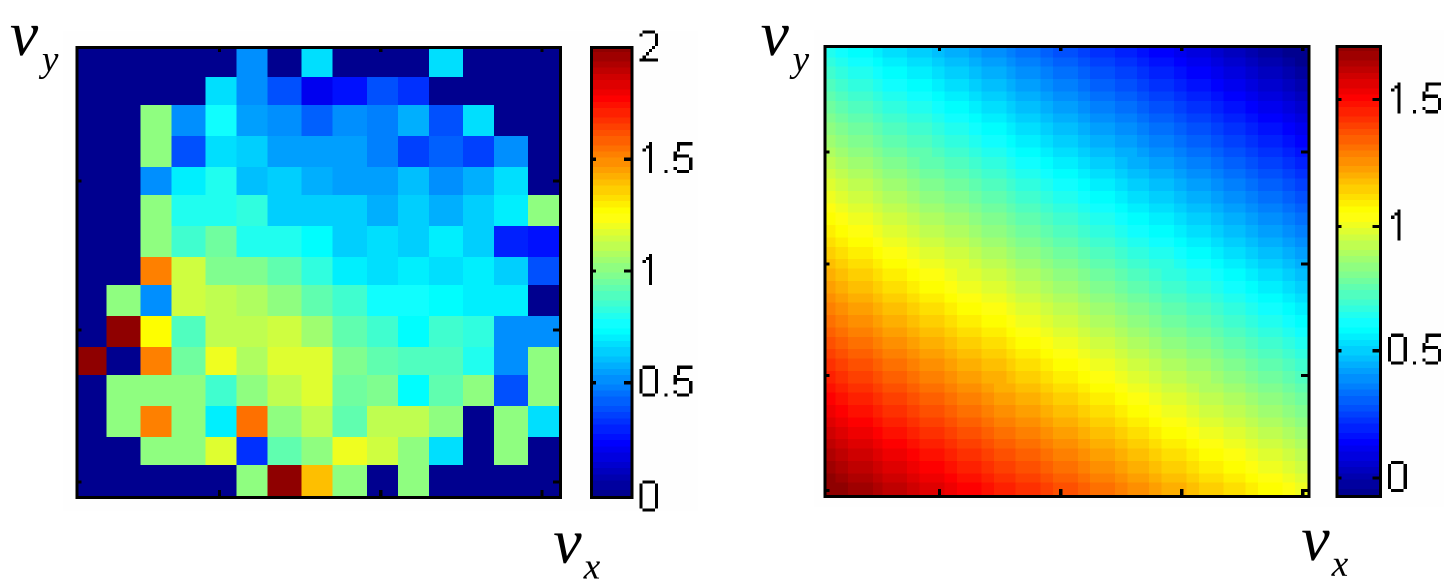


Encoding Review

Moran & Schwartz ('99):

$$z_k = h_0 + speed\,(h_x \cos(\theta_k) + h_y \sin(\theta_k))$$

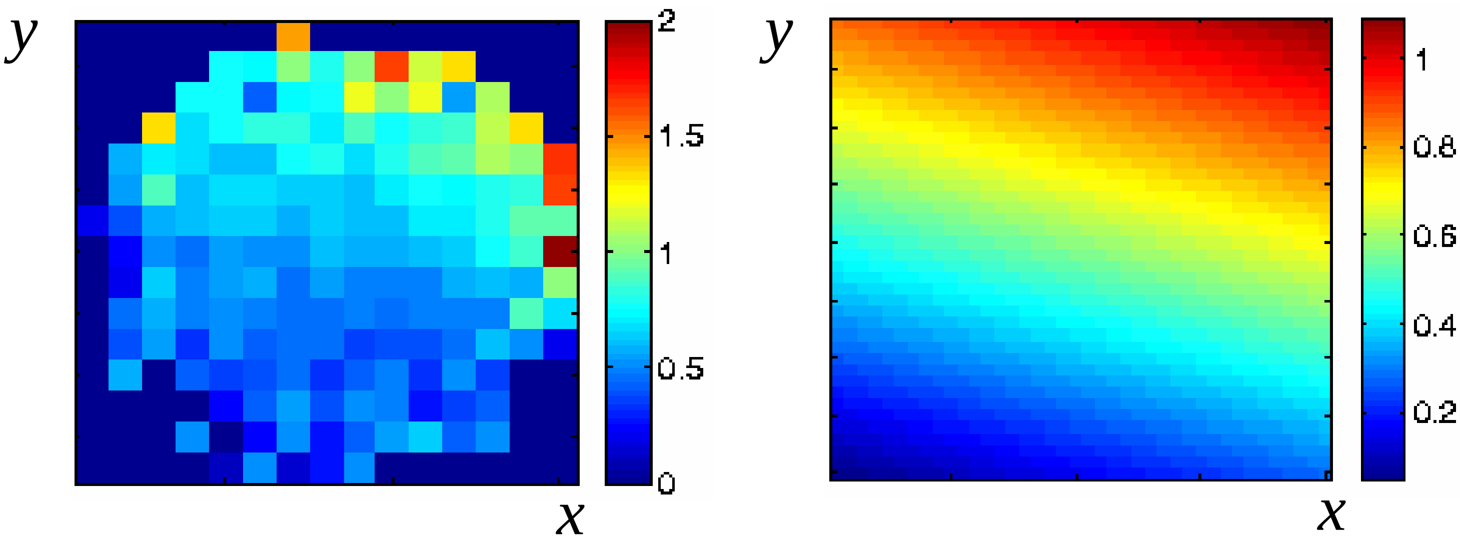
$$= h_0 + h_x v_{x_k} + h_y v_{y_k} \quad \textbf{(\textit{Linear} in velocity)}$$



Encoding Review

Kettner et al ('88):

$$z_k = b_0 + b_x x_k + b_y y_k$$ (*Linear* in position)



Flament et al ('88): Firing rate is also related to hand acceleration

Notation

Firing rates of *n* cells at time *k*

$$\vec{z}_k = \begin{bmatrix} z_{1,k} \\ z_{2,k} \\ \vdots \\ z_{n,k} \end{bmatrix}$$

Hand *kinematics* at time *k*

$$\vec{x}_k = \begin{bmatrix} x_k \\ y_k \\ v_{x,k} \\ v_{y,k} \\ a_{x,k} \\ a_{y,k} \end{bmatrix}$$

$$\vec{Z}_k = (\vec{z}_k, \vec{z}_{k-1}, \ldots, \vec{z}_1)$$

$$\vec{X}_k = (\vec{x}_k, \vec{x}_{k-1}, \ldots, \vec{x}_1)$$

Decoding Methods

Direct decoding methods:

$$\vec{x}_k = f(\vec{z}_k, \vec{z}_{k-1}, \ldots)$$

Simple linear regression method

$$x_k = \vec{f}_1^{\,T} \vec{Z}_{k:k-d}$$

$$y_k = \vec{f}_2^{\,T} \vec{Z}_{k:k-d}$$

Decoding Methods

Direct decoding methods:

$$\vec{x}_k = f(\vec{z}_k, \vec{z}_{k-1}, \ldots)$$

In contrast to generative encoding models:

$$\vec{z}_k = f(\vec{x}_k)$$

Need a sound way to exploit generative models for decoding.

Uncertainty

Typically there will be uncertainty in our models

$$\vec{z}_k = f(\vec{x}_k) + \text{noise}$$

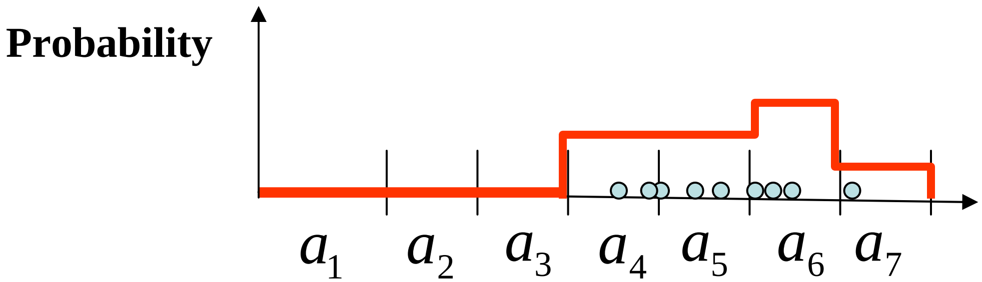
This defines the likelihood of the observations given the state:

$$p(\vec{z}_k \mid \vec{x}_k)$$

But we want to estimate something about the state x given noisy measurements z

$$p(\vec{x}_t \mid \vec{Z}_{t-j}) = p(\vec{x}_t \mid \vec{z}_{t-j}, \ldots, \vec{z}_1)$$

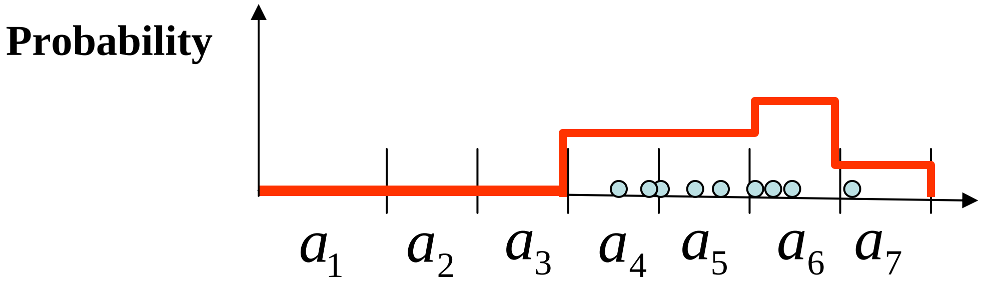
Probability Review



Let X be a *random variable* that can take on one of the discrete values

$$X \in [a_1, \ldots, a_7]$$

Basic facts

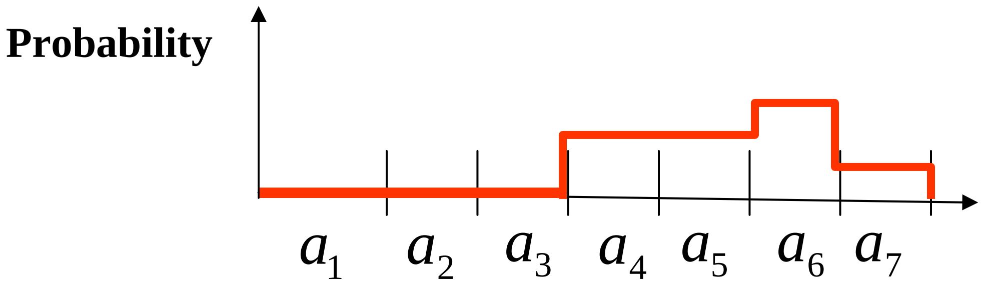


$$p(X = a_i) \text{ or just } p(a_i)$$

$$0 \leq p(X = a_i) \leq 1$$

$$\sum_{i=1}^{7} p(a_i) = 1$$

Basic facts



Expected value or expectation of a random variable

$$\mu = E[x] = \textbf{?}$$

$$\sigma^2 = \mathrm{var}[x] = E[(x - E(x))^2] = \sum_x (x - \mu)^2 p(x)$$

Joint Probability

$$p(X_1 = a_{1,i}, X_2 = a_{2,j}) = p(a_{1,i}, a_{2,j})$$

$$\sum_{a_{1,i}} \sum_{a_{2,j}} p(a_{1,i}, a_{2,j}) = 1$$

Statistical independence

$$p(x, y) = p(x)\, p(y)$$

- knowing *y* tells you nothing about *x*

Conditional Probability

Dependence - Knowing the value of one random variable tells us something about the other.

$$p(A \mid B) = \frac{p(A, B)}{p(B)}$$

$$p(A \mid B)\, p(B) = p(A, B)$$

Statistical Independence

$$p(A \mid B) = \textbf{?}$$

If A and B are statistically independent?

Statistical Independence

$$p(A \mid B) = \frac{p(A,B)}{p(B)} = \frac{p(A)\,p(B)}{p(B)} = p(A)$$

A and B are *statistically independent* if and only if

$$p(A,B) = p(A \mid B)\,p(B) = p(A)\,p(B)$$

$$p(A \mid B) = p(A)$$

$$p(B \mid A) = p(B)$$

Conditional Independence

A is independent of B, conditioned on C

$$p(A,B,C) = p(A,B \mid C)\,p(C)$$
$$= p(A \mid C)\,p(B \mid C)\,p(C)$$

If I know *C*, then knowing *B* doesn't give me any more information about *A*.

This does not mean that *A* and *B* are statistically independent

More generally

$$p(A, B \mid C) = p(A \mid B, C)\, p(B \mid C)$$

Marginalizing over a random variable

$$p(A \mid C) = \sum_B p(A, B \mid C) = \sum_B p(A \mid B, C) p(B \mid C)$$

Bayes' Theorem

$$p(A, B) = p(A \mid B)\, p(B) = p(B \mid A)\, p(A)$$

Revd. Thomas Bayes, 1701-1761

Bayesian Inference

Likelihood (evidence)

Prior (*a priori* – before the evidence)

Posterior

$$p(\text{kinematics} \mid \text{firing}) = \frac{p(\text{firing} \mid \text{kinematics})\, p(\text{kinematics})}{p(\text{firing})}$$

a posteriori probability (after the evidence)

normalization constant (independent of mouth)

We ***infer*** hand kinematics from uncertain evidence and our prior knowledge of how hands move.

GENERATIVE MODEL

Encoding:

linear, non-linear?

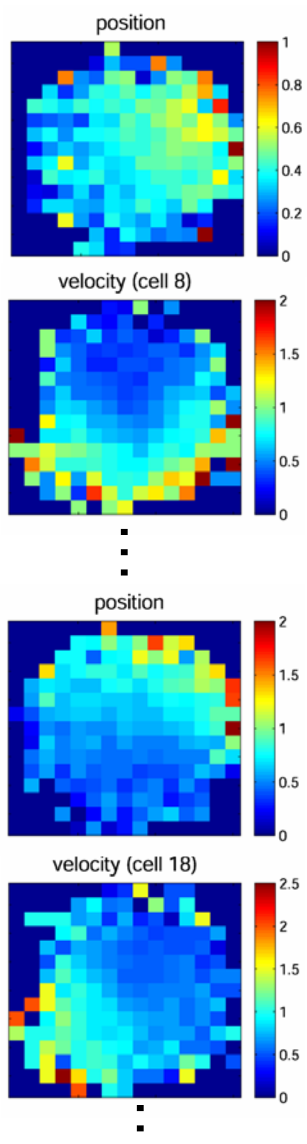
$$\vec{z}_k = f_1(\vec{x}_k) + \vec{q}_k$$

noise (e.g. Normal or Poisson)

$$\vec{x}_k = f_2(\vec{x}_{k-1}) + \vec{w}_k$$

neural firing rate of N=42 cells in M=70ms

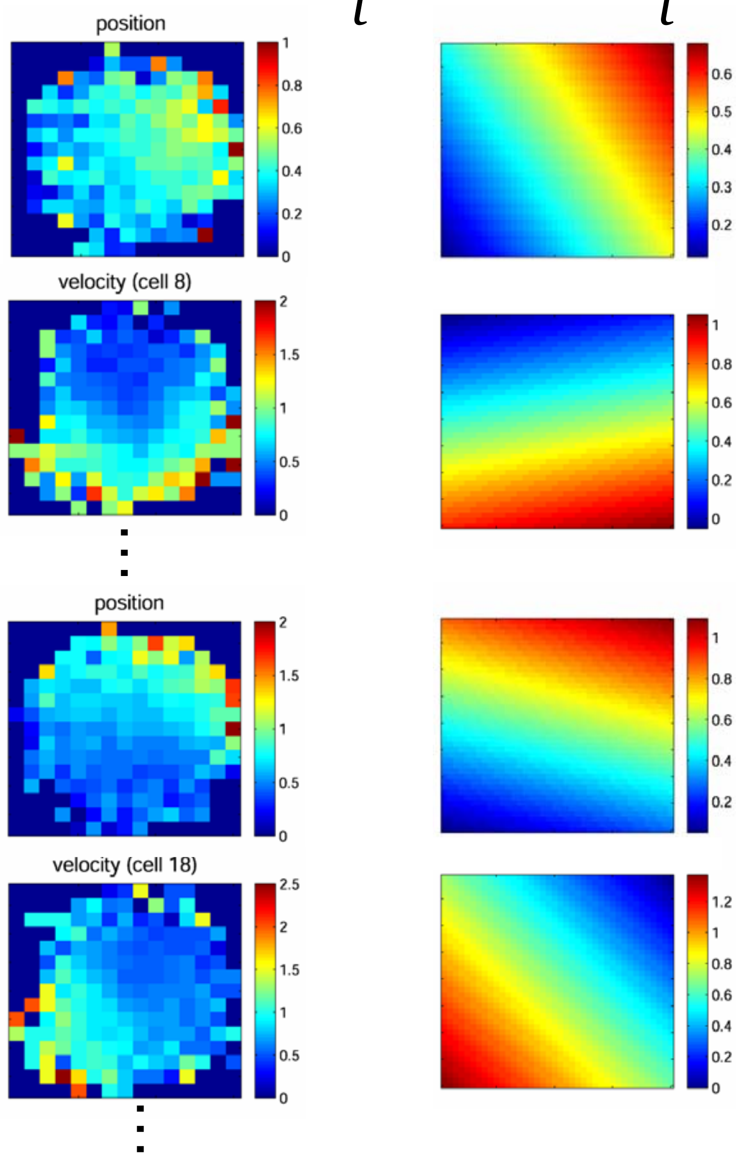
behavior (e.g. hand position, velocity, acceleration)



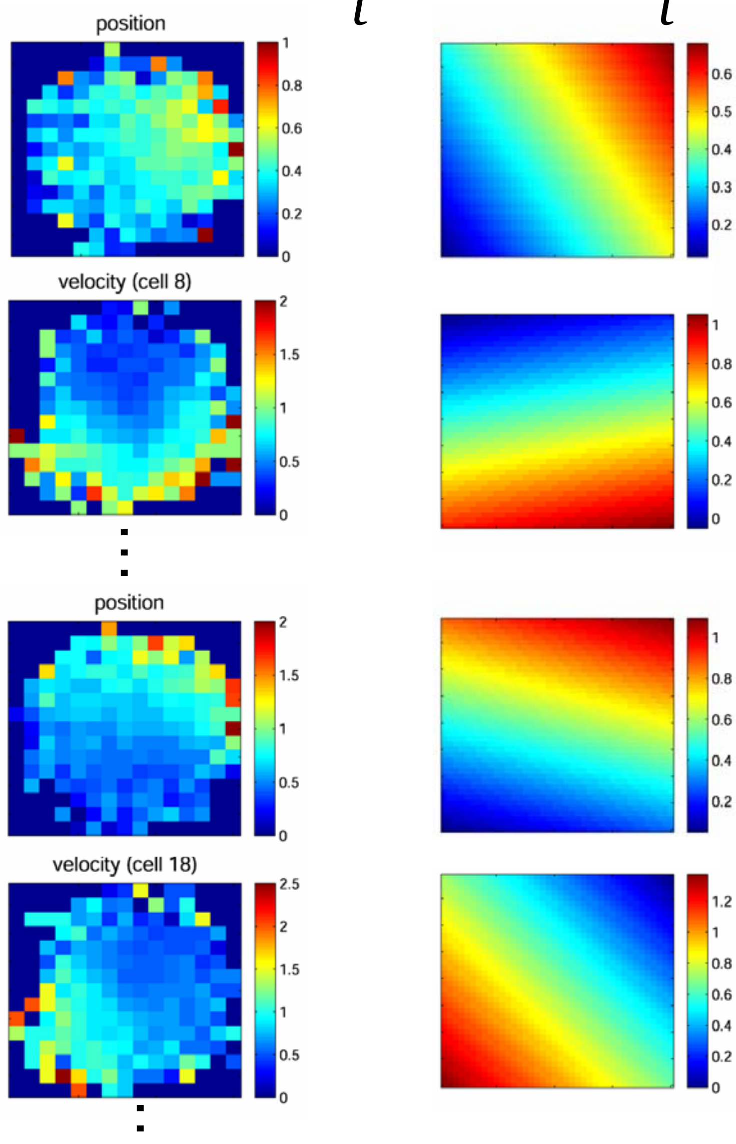
"cell 8"

"cell 18"

$$\vec{z}_t = H \ \vec{x}_t + noise$$



$$\vec{z}_t = H\ \vec{x}_t + noise$$



$$\begin{bmatrix} z_{1,t} \\ \vdots \\ z_{n,t} \end{bmatrix} = \begin{bmatrix} h_{1,x} & h_{1,y} & \cdots & h_{1,a_y} \\ h_{2,x} & h_{2,y} & \cdots & h_{2,a_y} \\ \vdots & & & \\ h_{n,x} & h_{n,y} & \cdots & h_{n,a_y} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ \vdots \\ a_{y,t} \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}$$

GENERATIVE MODEL

Observation Equation:

$$\vec{z}_{k-j} = H \vec{x}_k + \vec{q}_k$$

firing rate vector (zero mean, sqrt)

$$\begin{pmatrix} z^1_{k-j} \\ z^2_{k-j} \\ \vdots \\ z^{42}_{k-j} \end{pmatrix}$$

system state vector (zero mean)

$$\begin{pmatrix} x_k \\ y_k \\ v_{x_k} \\ v_{y_k} \\ a_{x_k} \\ a_{y_k} \end{pmatrix}$$

42 X 6 matrix

Assumption

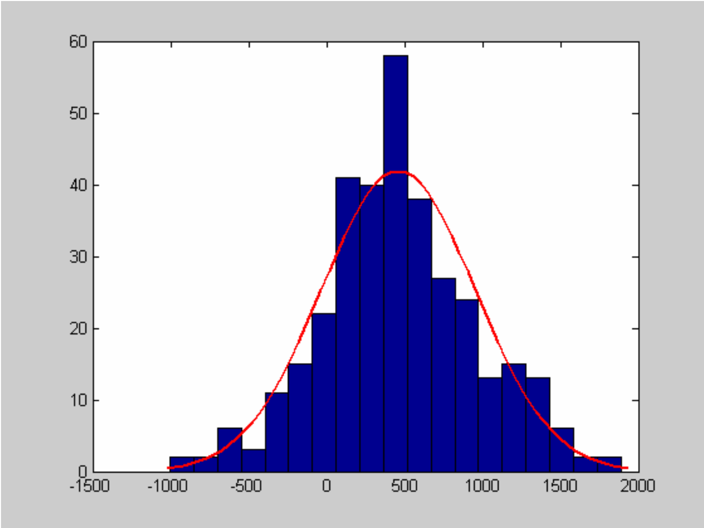
Gaussian distribution:

$$\vec{z}_k \sim N(H\vec{x}_k, Q)$$

$$\vec{z}_k - H\vec{x}_k = \vec{q}_k \sim N(0, Q)$$

Recall:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2}(x-\mu)^2 / \sigma^2)$$



Gaussian

For a single cell:

$$p(z_{i,t} \mid \vec{x}_t) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2}(z_{i,t} - (h_{i,x}x_t + h_{i,y}y_t + \cdots + h_{i,a_y}a_{y,t}))^2 / \sigma^2)$$

What about multiple cells?

If the firing rates are conditionally independent:

$$p(z_{1,t}, z_{2,t}, \ldots, z_{n,t} \mid \vec{x}_t) = \prod_{i=1}^{n} p(z_{i,t} \mid \vec{x}_t)$$

If we know x_t, then the firing rates of the other cells tell us nothing more about $z_{i,t}$

Covariance

first moment

$$\sigma_x^2 = var[x] = E[(x-\mu_x)] = \sum_x \sum_y (x-\mu_x)^2 p(x,y)$$

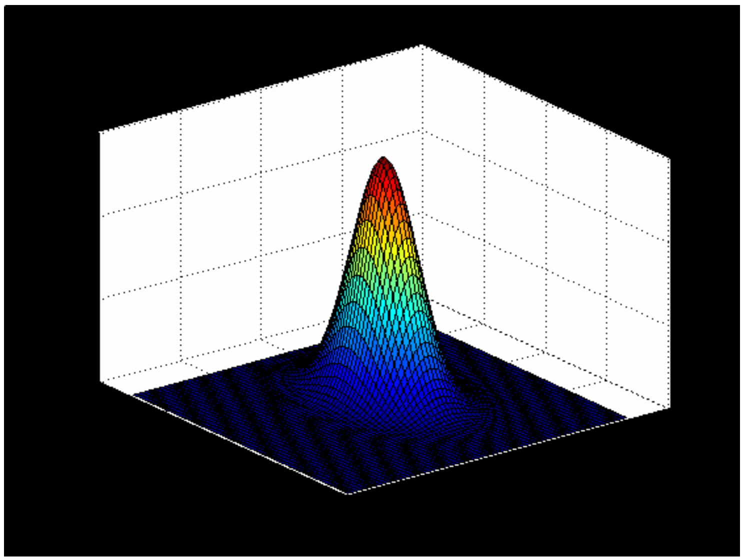
second moment

$$\sigma_{xy} = E[(x-\mu_x)(y-\mu_y)] = \sum_x \sum_y (x-\mu_x)(y-\mu_y) p(x,y)$$

$$\vec{\mu} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$C = E[(\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^T] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$$

Covariance

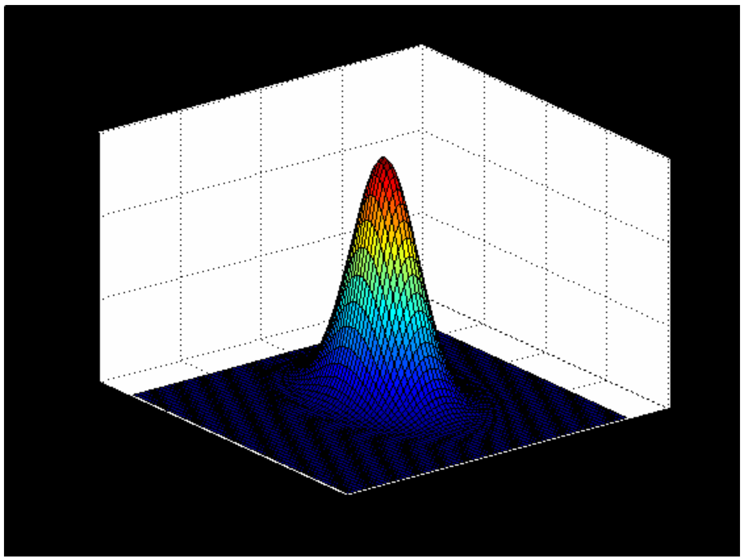


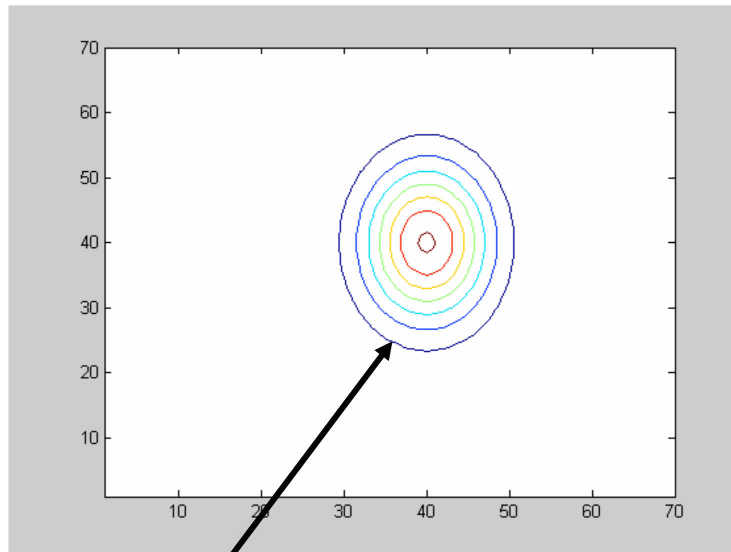
Multivariate Gaussian
(Normal)

Mahalanobis distance Δ^2

$$p(\vec{x}) = \frac{1}{(2\pi)^{D/2} |C|^{1/2}} \exp\left(-\frac{1}{2} \underbrace{(\vec{x}-\vec{\mu})^T C^{-1} (\vec{x}-\vec{\mu})} \right)$$

Covariance Ellipse





hyperellipsoids of constant Mahalanobis distance Δ^2

Some Facts

$$C = \mathrm{E}[(\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^T] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$$

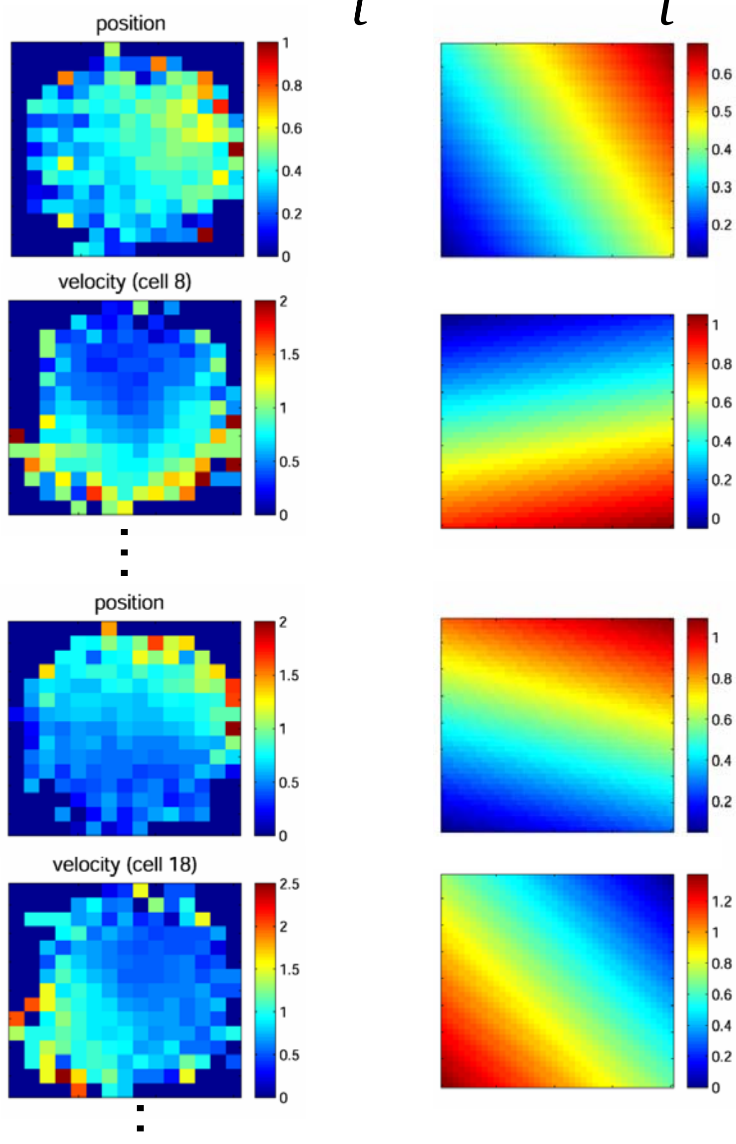
If *x* and *y* are statistically independent then σ_{xy}=0.

If σ_{xy}=0, then x and y are uncorrelated.

Uncorrelated does not imply statistically independent.
Uncorrelated and Gaussian does.

PCA de-correlates the directions but unless the data is Gaussian, the coefficients are not statistically independent.

$$\vec{z}_t = H\ \vec{x}_t + noise$$



Approximation: Linear Gaussian (generative) model

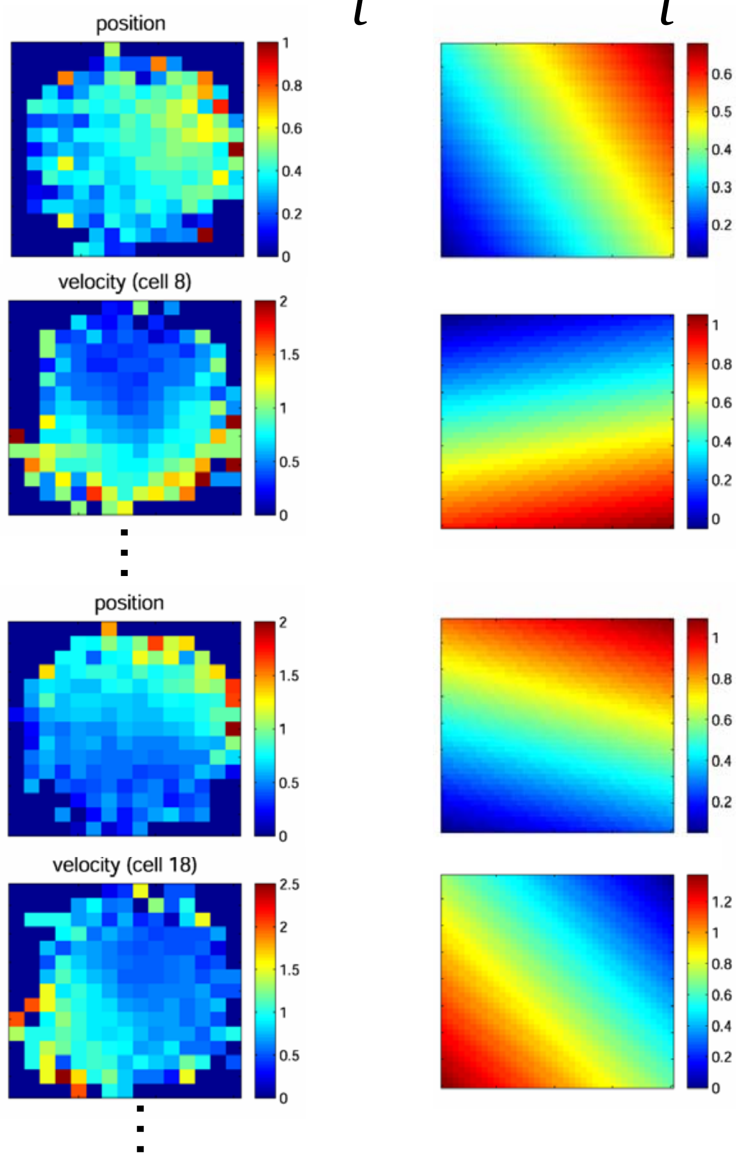
observation model

$$\vec{z}_{t-j} \sim \mathcal{N}(H\ \vec{x}_t,\ Q_t)$$

Full covariance Q matrix models correlations between cells.

H models how firing rates relate to full kinematic model (position, velocity, and acceleration).

$$\vec{z}_t = H\ \vec{x}_t + noise$$



Approximation: Linear Gaussian (generative) model

likelihood

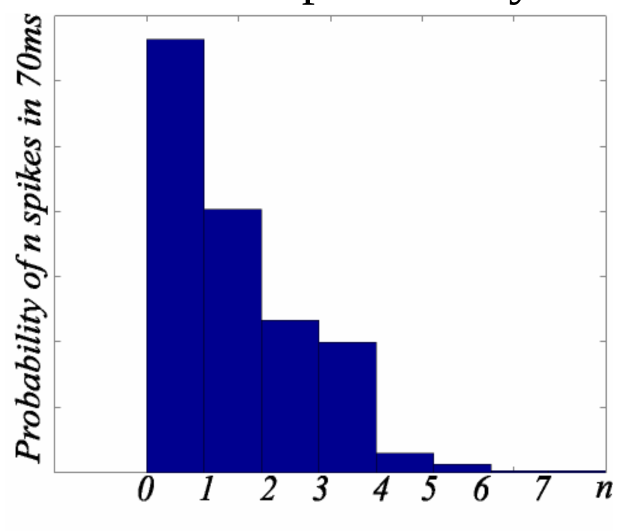
$$p(\vec{z}_t \mid \vec{x}_t) =$$

$$\frac{1}{D}\exp\left(\frac{-1}{2}(\vec{z}_t - H\vec{x}_t)^{\mathrm{T}} Q_t^{-1} (\vec{z}_t - H\vec{x}_t)\right)$$

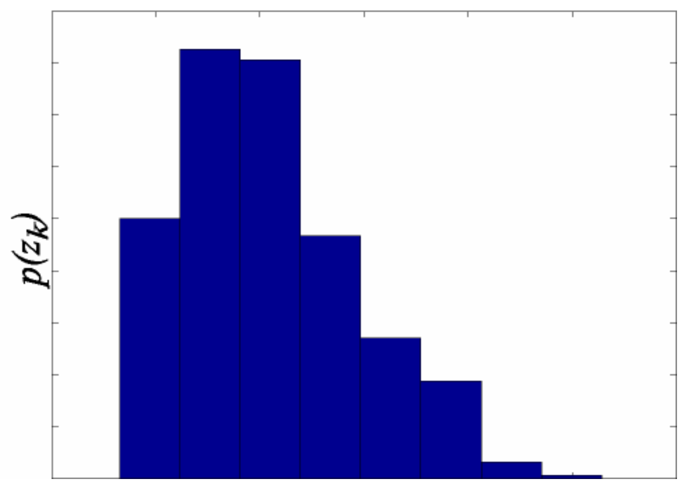
TRANSFORMED RATE

Firing rates are not normally distributed.

Poisson probability



$$\tilde{z}_k = \sqrt{z_k + 1} - mean(\sqrt{z_k + 1})$$



$$\vec{\tilde{z}}_{k-j} = H \vec{x}_k + \vec{q}_k$$

We'll come back to Poisson models…

GENERATIVE MODEL

Observation Equation:

firing rate vector (zero mean, sqrt)

$$\begin{pmatrix} z^1_{k-j} \\ z^2_{k-j} \\ \vdots \\ z^{42}_{k-j} \end{pmatrix}$$

$$\vec{z}_{k-j} = H \vec{x}_k + \vec{q}_k$$

system state vector (zero mean)

$$\begin{pmatrix} x_k \\ y_k \\ v_{x_k} \\ v_{y_k} \\ a_{x_k} \\ a_{y_k} \end{pmatrix}$$

42 X 6 matrix

42 X 42 matrix

$$\vec{q}_k \sim N(0, Q) \quad k = 0,1,2,\cdots$$

System Equation:

$$\vec{x}_{k+1} = A \vec{x}_k + \vec{w}_k$$

6 X 6 matrix

6 X 6 matrix

$$\vec{w}_k \sim N(0, W) \quad k = 0,1,2,\cdots$$

Model Fitting

How do we fit *H* and *A*?

Linear regression:

$$H = \arg\min_{H} \sum_{k} \| \vec{z}_k - H\vec{x}_k \|^2$$

$$A = \arg\min_{A} \sum_{k} \| \vec{x}_{k+1} - A\vec{x}_k \|^2$$

TRAINING

What about the covariance matrices?

$$
\begin{aligned}
Q &= \mathbf{cov}(\{\vec{z}_k - H\vec{x}_k\}_k) \\
&= (\mathbf{z} - H\mathbf{x})(\mathbf{z} - H\mathbf{x})^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
W &= \mathbf{cov}(\{\vec{x}_{k+1} - A\vec{x}_k\}_k) \\
&= (\mathbf{x}_{k+1} - A\mathbf{x}_k)(\mathbf{x}_{k+1} - A\mathbf{x}_k)^{\mathrm{T}}
\end{aligned}
$$

Centralize the training data, such that

$$\mathrm{E}(\{\vec{z}_k\}) = 0, \quad \mathrm{E}(\{\vec{x}_k\}) = 0$$

Matlab

$$H = \underset{H}{\operatorname{argmin}} \sum_k \| \vec{z}_k - H\vec{x}_k \|^2$$

```
z = rates';
x = kinematics';
H = z*x'*inv(x*x');
Q = (z-H*x)*(z-H*x)'/size(z,2);
```

Bayesian Inference

Likelihood (evidence)

Prior (*a priori* – before the evidence)

Posterior

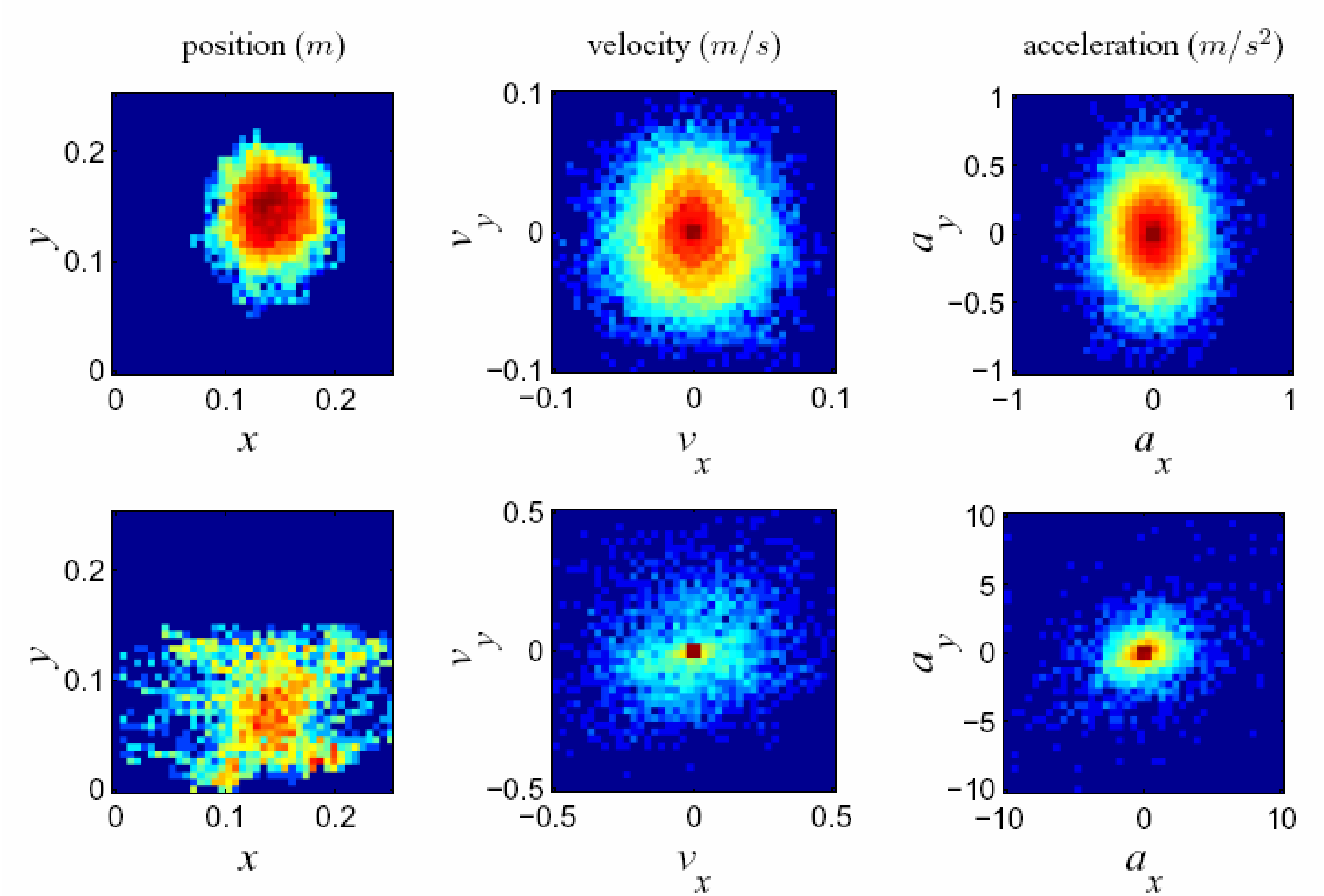
$$p(\text{kinematics} \mid \text{firing}) = \frac{p(\text{firing} \mid \text{kinematics})\, p(\text{kinematics})}{p(\text{firing})}$$

a posteriori probability (after the evidence)

normalization constant (independent of mouth)

We ***infer*** hand kinematics from uncertain evidence and our prior knowledge of how hands move.

Two Tasks


position (m)
velocity (m/s)
acceleration (m/s^2)
0.2
0.1
0
0
0.1
0.2
x
y
0.1
0
−0.1
−0.1
0.1
v_x
v_y
1
0.5
−0.5
−1
a_x
a_y
0.5
−0.5
10
5
−5
−10