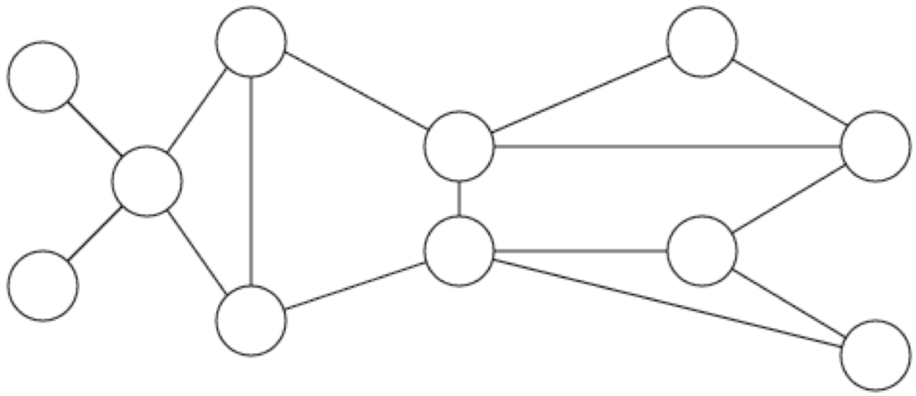
Learning and Inference in Probabilistic Graphical Models

Gaussian Belief Propagation

March 22, 2010

Pairwise Markov Random Fields



$$p(x \mid y) = \frac{1}{Z} \prod_{(s,t) \in \mathcal{E}} \psi_{st}(x_s, x_t) \prod_{s \in \mathcal{V}} \psi_s(x_s, y)$$

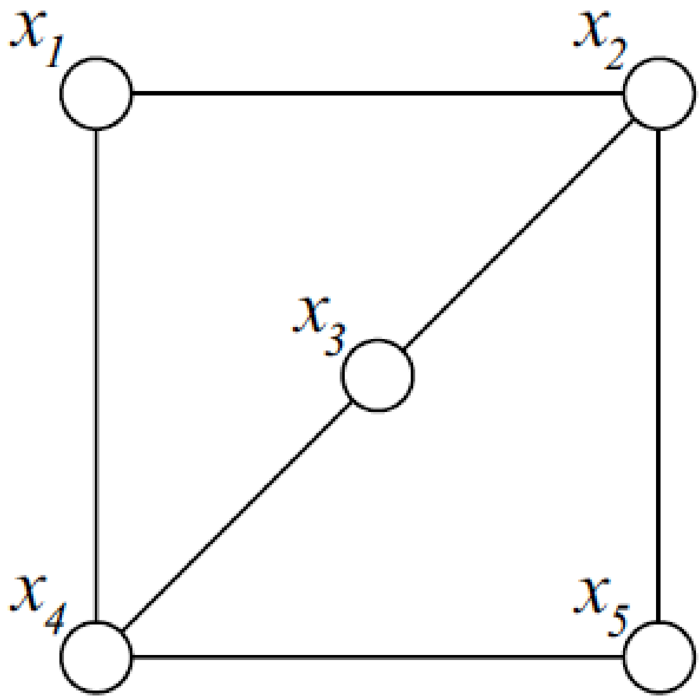
$\mathcal{V}$ ⟶ set of N nodes $\{1, 2, \ldots, N\}$

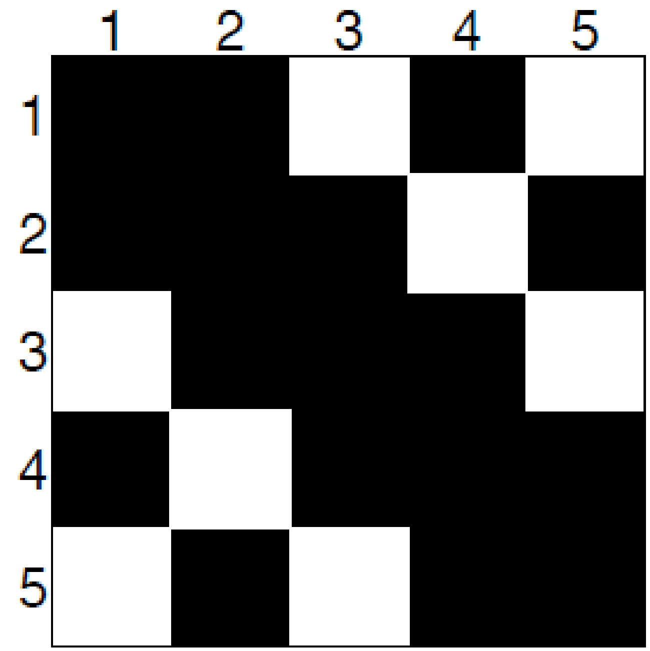
$\mathcal{E}$ ⟶ set of edges (s, t) connecting nodes $s, t \in \mathcal{V}$

Z ⟶ normalization constant (partition function)

- Product of arbitrary positive *clique potential* functions
- Guaranteed Markov with respect to corresponding graph

Gaussian MRFs





$$p(x) = \frac{1}{Z} \prod_{(s,t) \in \mathcal{E}} \psi_{s,t}(x_s, x_t)$$

$$\sum_{t \in N(s)} J_{s(t)} = J_{s,s}$$

$$\psi_{s,t}(x_s, x_t) = \exp\left\{ -\frac{1}{2} \begin{bmatrix} x_s^T & x_t^T \end{bmatrix} \begin{bmatrix} J_{s(t)} & J_{s,t} \\ J_{t,s} & J_{t(s)} \end{bmatrix} \begin{bmatrix} x_s \\ x_t \end{bmatrix} \right\}$$

Gaussian Potentials

$$p(x) = \frac{1}{Z}\exp\left\{-\frac{1}{2}x^T P^{-1} x\right\} = \frac{1}{Z}\prod_{s=1}^{N}\prod_{t=1}^{N}\exp\left\{-\frac{1}{2}x_s^T J_{s,t} x_t\right\} =$$

$$\frac{1}{Z}\prod_{(s,t)\in\mathcal{E}}\exp\left\{-\frac{1}{2}\begin{bmatrix} x_s^T & x_t^T \end{bmatrix}\begin{bmatrix} J_{s(t)} & J_{s,t} \\ J_{t,s} & J_{t(s)} \end{bmatrix}\begin{bmatrix} x_s \\ x_t \end{bmatrix}\right\} = \frac{1}{Z}\prod_{(s,t)\in\mathcal{E}}\psi_{s,t}(x_s, x_t)$$

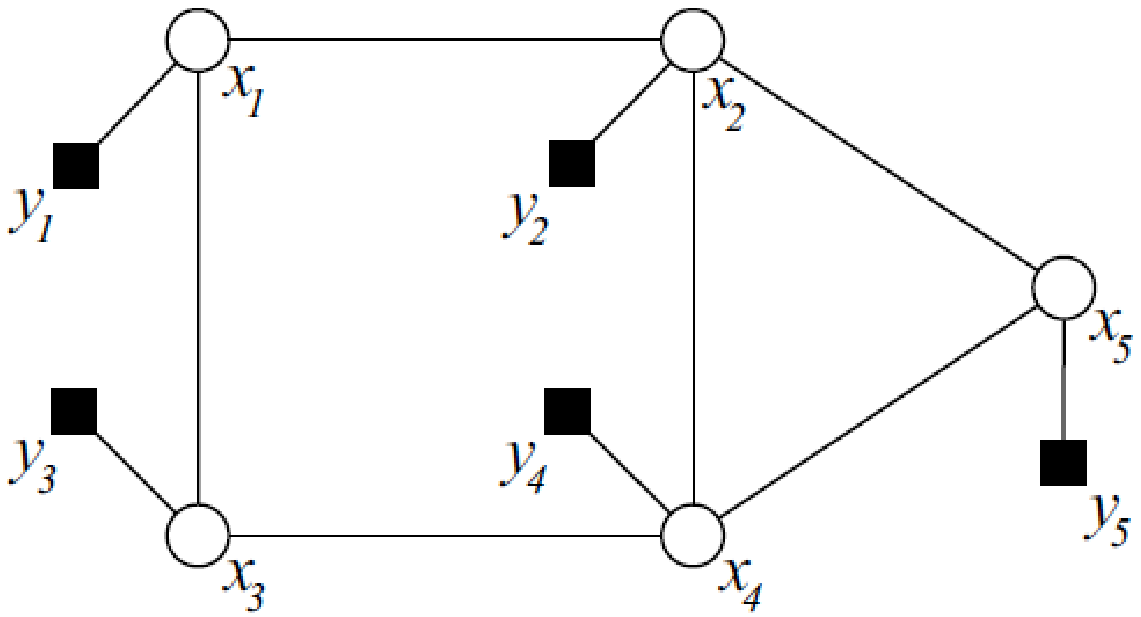
$$Z = \left((2\pi)^N \det P\right)^{1/2}$$

$$p(x) = \frac{1}{Z}\prod_{(s,t)\in\mathcal{E}}\psi_{s,t}(x_s, x_t)$$

$$\sum_{t\in N(s)} J_{s(t)} = J_{s,s}$$

$$\psi_{s,t}(x_s, x_t) = \exp\left\{-\frac{1}{2}\begin{bmatrix} x_s^T & x_t^T \end{bmatrix}\begin{bmatrix} J_{s(t)} & J_{s,t} \\ J_{t,s} & J_{t(s)} \end{bmatrix}\begin{bmatrix} x_s \\ x_t \end{bmatrix}\right\}$$

Linear Gaussian Observations



$$y_s = C_s x_s + v_s$$

$$v_s \sim \mathcal{N}(0, R_s)$$

$$p(y \mid x) = \prod_{s=1}^{N} p(y_s \mid x_s)$$

$$C = \text{diag}(C_1, C_2, \dots, C_N)$$

$$R = \text{diag}(R_1, R_2, \dots, R_N)$$

$$p(x_s \mid y) \sim \mathcal{N}\left(\widehat{x}_s, \widehat{P}_s\right)$$

Gaussian Belief Propagation

Correctness of Belief Propagation in Gaussian Graphical Models of Arbitrary Topology

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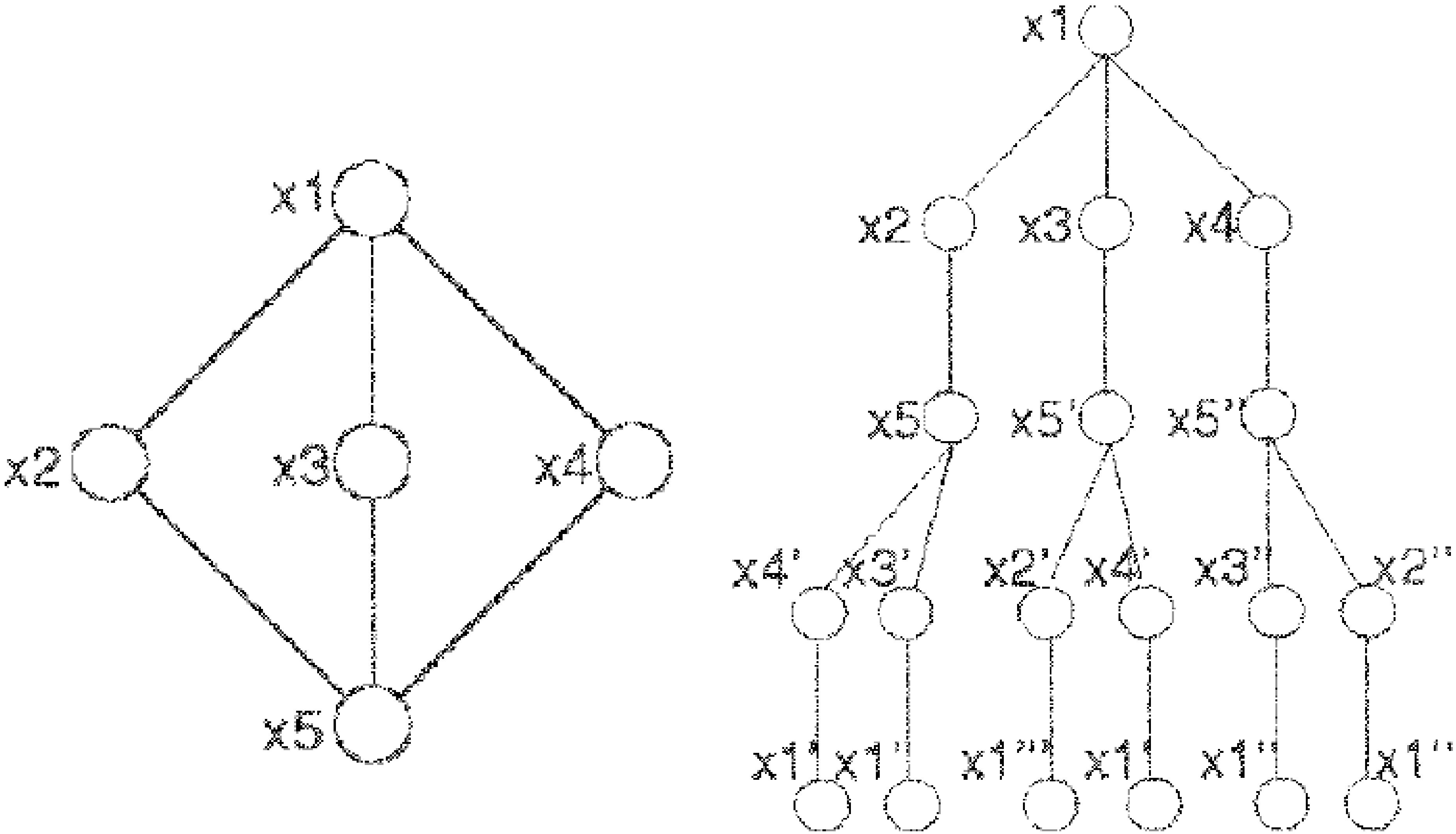
Computation Trees



Analysis of Gaussian BP

Claim 1:

$$\tilde{\mu}(1) = \mu(1) + \tilde{C}_{x_1|y} r, \tag{3.1}$$

where r is a vector that is zero everywhere but the last L components (corresponding to the leaf nodes).

Claim 2:

$$\tilde{\sigma}^2(1) = \sigma^2(1) + \tilde{C}_{x_1|y} r_1 - \tilde{C}_{x_1|y} r_2, \tag{3.2}$$

where r_1 is a vector that is zero everywhere but the last L components and r_2 are equal to 1 for all components corresponding to nonroot nodes in the unwrapped tree that reference x_1. All other components of r_2 are zero.

Claim 3: If the conditional correlation between the root node and the leaf nodes decreases rapidly enough, then (1) belief propagation converges, (2) the belief propagation means are exact, and (3) the belief propagation variances are equal to the correct variances minus the summed conditional correlations between $\tilde{x}_1$ and all $\tilde{x}_j$ that are replicas of x_1.

Surface Reconstruction