“Fast Approximate Energy Minimization via Graph Cuts”

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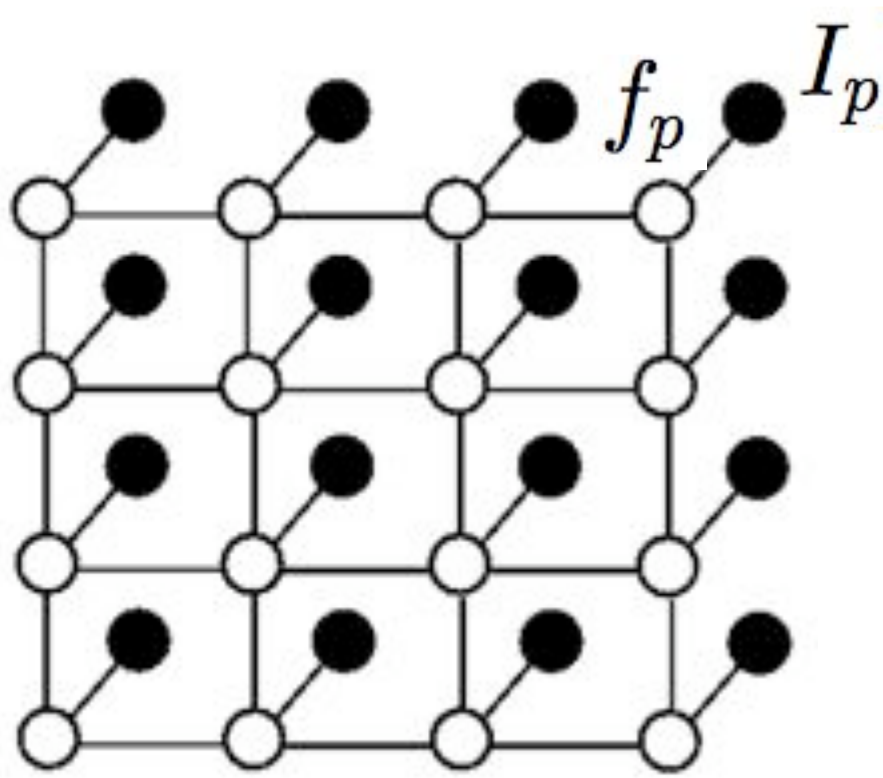
Motivation

- Image Restoration

noisy image

denoised image

MRF for image restoration



f_p : latent intensity

I_p : observed intensity

Image restoration in math…

- Goal:
  labeling $f : \mathcal{P} \rightarrow \mathcal{L}$,
  $\mathcal{P}$: set of all pixels, $\mathcal{L}$: set of possible labels

- Properties of f :
  - piecewise smooth
    - varies smoothly on object surface
    - dramatic change at object boundaries
  - consistent with data

Image restoration as MAP estimation

- Most probable configuration of distribution over all pixel values

- Max of cost function (W&J, eq. (8.5)):

$$\max_{\mu \in L(T)} < \mu, \theta > = \max_{x \in X^m} \left[\sum_{s \in V} \theta_s(x_s) + \sum_{(s,t) \in E} \theta_{st}(x_s, x_t) \right]$$

energy function

Energy minimization challenge

- Global min NP-hard for most interesting energies
- Opt for local min
  - Standard moves, e.g. simulated annealing
    - Change label of 1 pixel each time till no 1-pixel change reduces energy further
- This paper: improved convergence rate of sim. annealing style approaches
  - larger moves, e.g. swaps, expansions
    - optimal move based on graph cuts

Energy function

$$E(f) = E_{smooth}(f) + E_{data}(f)$$

$$E_{smooth}(f) \rightarrow \sum_{\{p,q\}\in\mathcal{N}} V_{p,q}(f_p, f_q)$$

$$E_{data}(f) \rightarrow \sum_{p\in\mathcal{P}} D_p(f_p)$$

$$D_p(f_p) \rightarrow (f_p - I_p)^2$$

f_p : latent intensity

I_p : observed intensity

$\mathcal{N}$: neighborhood

$\mathcal{P}$: set of all pixels

Penalty function properties

semi-metric

$$
\begin{aligned}
V(\alpha,\beta) = 0 \quad &\Leftrightarrow \quad \alpha = \beta, \\
V(\alpha,\beta) \quad &= \quad V(\beta,\alpha) \geq 0, \\
V(\alpha,\beta) \quad &\leq \quad V(\alpha,\gamma) + V(\gamma,\beta),
\end{aligned}
$$

for any labels $\alpha, \beta, \gamma \in \mathcal{L}$

metric

Graph-based energy minimization

1. Swap algorithm
2. Expansion algorithm

1. Swap algorithm

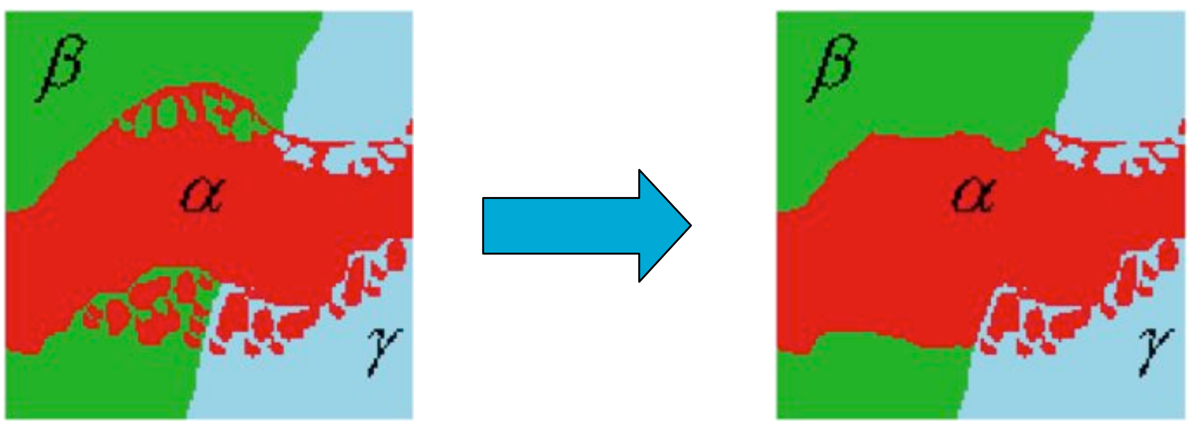
α-β swap move

- labeling $f \Leftrightarrow$ partition $\mathbf{P} = \{\mathcal{P}_l \mid l \in \mathcal{L}\}$

$$\mathcal{P}_l = \{p \in \mathcal{P} \mid f_p = l\}$$

- $\alpha\text{-}\beta\text{-}swap : \mathbf{P} \rightarrow \mathbf{P}'$,

$$\mathcal{P}_l = \mathcal{P}'_l \text{ for any label } l \neq \alpha, \beta$$



before

after α-β swap

Swap algorithm (1/2)

- Energy minimization based on α-β swaps
- Challenge: exponential number of swap moves!
  - exp time even for checking for min!
- Linear time when standard moves are allowed

  ⇒ ***extend standard move concept***:

  standard move → optimal α-β swap move

Swap algorithm (2/2)

1. Start with an arbitrary labeling f
2. Set success := 0
3. For each pair of labels $\{\alpha, \beta\} \subset \mathcal{L}$
   - 3.1. Find $\hat{f} = \arg\min E(f')$ among f' within one α-β swap of f
   - 3.2. If $E(\hat{f}) < E(f)$, set f := $\hat{f}$ and success := 1
4. If success = 1 goto 2
5. Return f

Finding optimal α-β swap move

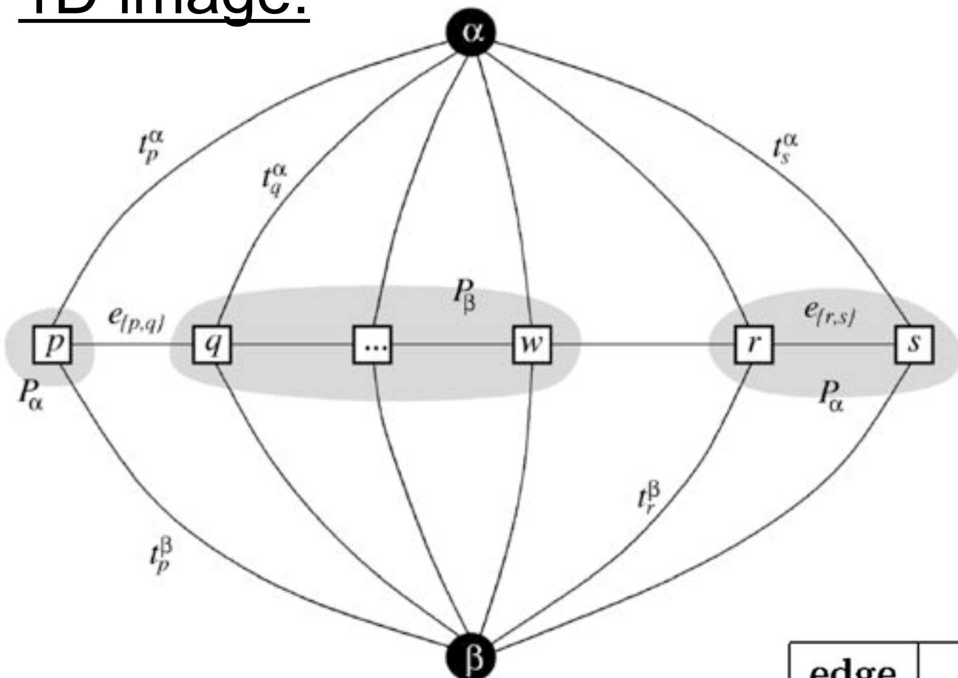
- Min cut on swap graph $\mathcal{G}_{\alpha\beta} = \langle \mathcal{V}_{\alpha\beta}, \mathcal{E}_{\alpha\beta} \rangle$
  a. What is a (min) graph cut?
  b. How is the swap graph formulated?
  c. Why optimal swap corresponds to a min cut on the swap graph?

a. Graph cut

- $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$: weighted graph, 2 terminals
- $cut\ \mathcal{C} \subset \mathcal{E}$: minimal set of edges s.t. terminals separated in $\mathcal{G}(\mathcal{C}) = \langle \mathcal{V}, \mathcal{E} - \mathcal{C} \rangle$
- ***min cut problem***: find cheapest cut that separates terminals
- min cut calculation based on max-flow
  - – nearly linear running time

b. Swap graph $\mathcal{G}_{\alpha\beta}$ formulation

<u>1D image:</u>



$$\mathcal{P}_{\alpha\beta} = \mathcal{P}_\alpha \cup \mathcal{P}_\beta$$
$$\mathcal{P}_\alpha = \{p, r, s\}$$
$$\mathcal{P}_\beta = \{q, \dots, w\}$$

edge	weight	for
t_p^α	$D_p(\alpha) + \sum_{q \in \mathcal{N}_p, q \notin \mathcal{P}_{\alpha\beta}} V(\alpha, f_q)$	$p \in \mathcal{P}_{\alpha\beta}$
t_p^β	$D_p(\beta) + \sum_{q \in \mathcal{N}_p, q \notin \mathcal{P}_{\alpha\beta}} V(\beta, f_q)$	$p \in \mathcal{P}_{\alpha\beta}$
$e_{\{p,q\}}$	$V(\alpha, \beta)$	$\{p,q\} \in \mathcal{N}$, $p,q \in \mathcal{P}_{\alpha\beta}$

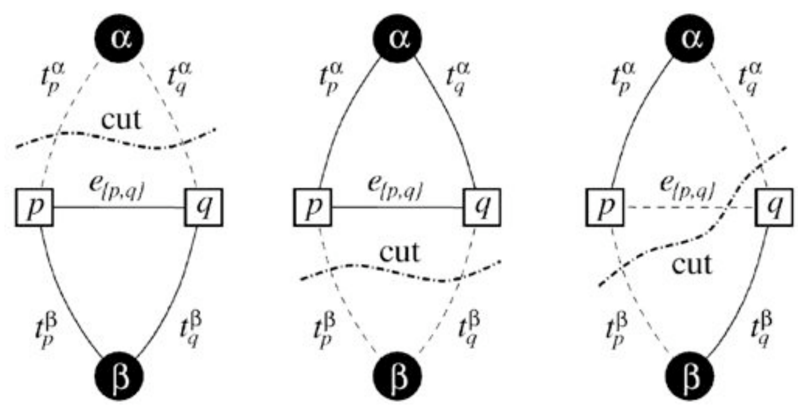
t-link (arrows to rows t_p^α and t_p^β)

n-link (arrow to row $e_{\{p,q\}}$)

c. Why optimal swap = min cut? (1/3)

Any cut C includes only 1 t-link per pixel

- 0 t-links in C → path between terminals
- 2 t-links in C → cut subset of C



Cut C defines natural labeling

$$f_p^{\mathcal{C}} = \begin{cases} \alpha & \text{if } t_p^{\alpha} \in \mathcal{C} \text{ for } p \in \mathcal{P}_{\alpha\beta} \\ \beta & \text{if } t_p^{\beta} \in \mathcal{C} \text{ for } p \in \mathcal{P}_{\alpha\beta} \\ f_p & \text{for } p \in \mathcal{P},\ p \notin \mathcal{P}_{\alpha\beta}. \end{cases}$$

c. Why optimal swap = min cut? (2/3)

Lemma 4.1. *A labeling f^{C} corresponding to a cut $\mathcal{C}$ on $\mathcal{G}_{\alpha\beta}$ is one α-β swap away from the initial labeling f.*

Only pixels with labels a,b change (their new labels are again a or b)

There is a one to one correspondence between cuts $\mathcal{C}$ on $\mathcal{G}_{\alpha\beta}$ and labelings that are one α-β swap from f

severed t-links uniquely determine the labels assigned to pixels p and the n-links that must be cut

c. Why optimal swap = min cut? (3/3)

The cost of a cut $\mathcal{C}$ *on* $\mathcal{G}_{\alpha\beta}$ *is* $|\mathcal{C}| = E(f^{\mathcal{C}})$ *plus a constant.*

Proof: on the board

Corollary 4.5. *The lowest energy labeling within a single* α-β *swap move from* f *is* $\hat{f} = f^{\mathcal{C}}$, *where* $\mathcal{C}$ *is the minimum cut on* $\mathcal{G}_{\alpha\beta}$.

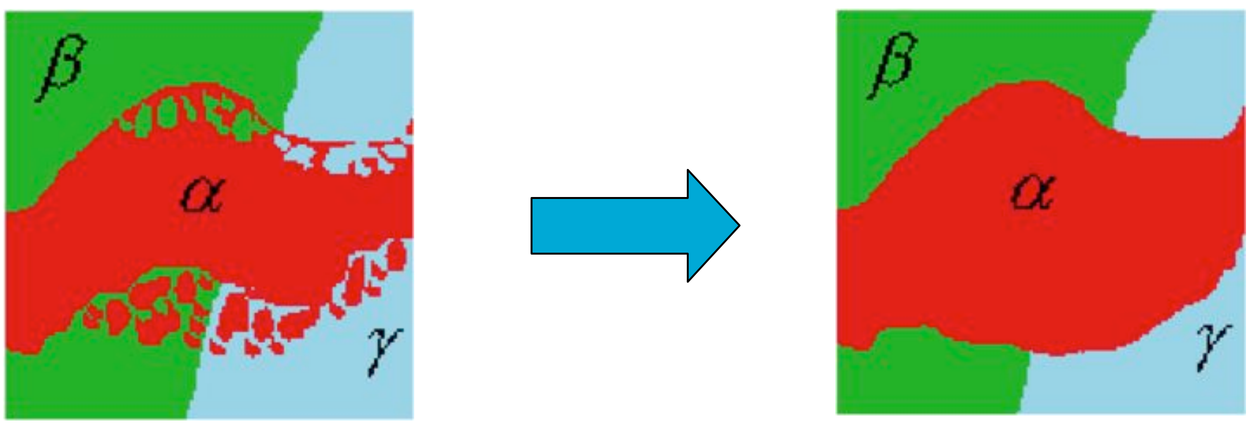
2. Expansion algorithm

Expansion algorithm

1. Start with an arbitrary labeling f
2. Set success := 0
3. For each label $\alpha \in \mathcal{L}$
   - 3.1. Find $\hat{f} = \arg\min E(f')$ among f' within one α-expansion of f
   - 3.2. If $E(\hat{f}) < E(f)$, set f := $\hat{f}$ and success := 1
4. If success = 1 goto 2
5. Return f

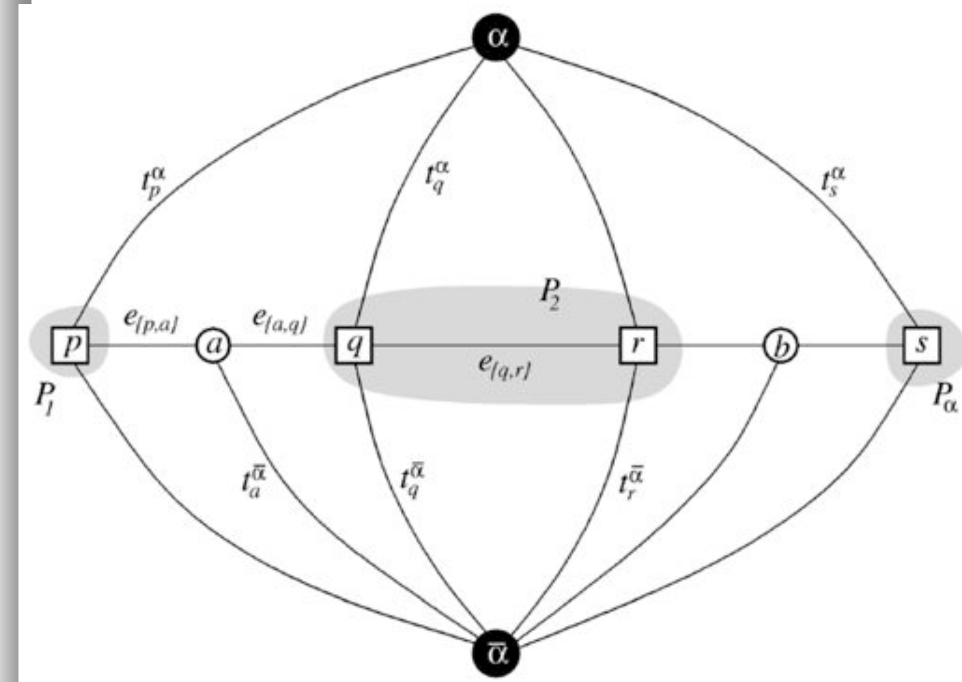
α-expansion

- labeling f $\Leftrightarrow$ partition $\mathbf{P} = \{\mathcal{P}_l \mid l \in \mathcal{L}\}$
  $\mathcal{P}_l = \{p \in \mathcal{P} \mid f_p = l\}$
- α*-expansion* : $\mathbf{P} \rightarrow \mathbf{P}'$,
  $\mathcal{P}_\alpha \subset \mathcal{P}'_\alpha$ and $\mathcal{P}'_l \subset \mathcal{P}_l$ for any label $l \neq \alpha$



Finding optimal expansion move

- Min cut on graph $\mathcal{G}_{\alpha\beta} = \langle \mathcal{V}_{\alpha\beta}, \mathcal{E}_{\alpha\beta} \rangle$



edge	weight	for
$t_p^{\bar{\alpha}}$	∞	$p \in \mathcal{P}_\alpha$
$t_p^{\bar{\alpha}}$	$D_p(f_p)$	$p \notin \mathcal{P}_\alpha$
t_p^α	$D_p(\alpha)$	$p \in \mathcal{P}$
$e_{\{p,a\}}$	$V(f_p, \alpha)$	$\{p,q\} \in \mathcal{N},\ f_p \neq f_q$
$e_{\{a,q\}}$	$V(\alpha, f_q)$	
$t_a^{\bar{\alpha}}$	$V(f_p, f_q)$	
$e_{\{p,q\}}$	$V(f_p, \alpha)$	$\{p,q\} \in \mathcal{N},\ f_p = f_q$

Cut properties

- Any cut C includes only 1 t-link per pixel
- Natural labeling from cut C

$$f_p^{\mathcal{C}} = \begin{cases} \alpha & \text{if} \quad t_p^{\alpha} \in \mathcal{C} \\ \\ f_p & \text{if} \quad t_p^{\bar{\alpha}} \in \mathcal{C} \end{cases} \qquad \forall p \in \mathcal{P}$$

- **Lemma 5.1.** *A labeling $f^{\mathcal{C}}$ corresponding to a cut $\mathcal{C}$ on $\mathcal{G}_\alpha$ is one α-expansion away from the initial labeling f.*

Cut properties

- **Property 5.2.** *If $\{p,q\} \in \mathcal{N}$ and $f_p \neq f_q$, then a minimum cut $\mathcal{C}$ on $\mathcal{G}_\alpha$ satisfies:*

$$(a) \quad \text{If} \quad t_p^\alpha, t_q^\alpha \in \mathcal{C} \quad \text{then} \quad \mathcal{C} \cap \mathcal{E}_{\{p,q\}} = \emptyset.$$

$$(b) \quad \text{If} \quad t_p^{\bar{\alpha}}, t_q^{\bar{\alpha}} \in \mathcal{C} \quad \text{then} \quad \mathcal{C} \cap \mathcal{E}_{\{p,q\}} = t_a^{\bar{\alpha}}.$$

$$(c) \quad \text{If} \quad t_p^{\bar{\alpha}}, t_q^\alpha \in \mathcal{C} \quad \text{then} \quad \mathcal{C} \cap \mathcal{E}_{\{p,q\}} = e_{\{p,a\}}.$$

$$(d) \quad \text{If} \quad t_p^\alpha, t_q^{\bar{\alpha}} \in \mathcal{C} \quad \text{then} \quad \mathcal{C} \cap \mathcal{E}_{\{p,q\}} = e_{\{a,q\}}.$$

- **Lemma 5.3.** *If $\{p,q\} \in \mathcal{N}$ and $f_p \neq f_q$, then the minimum cut $\mathcal{C}$ on $\mathcal{G}_\alpha$ satisfies*

$$|\mathcal{C} \cap \mathcal{E}_{\{p,q\}}| = V\left(f_p^{\mathcal{C}}, f_q^{\mathcal{C}}\right).$$

- **Theorem 5.4.** *Let $\mathcal{G}_\alpha$ be constructed as above given f and α. Then, there is a one to one correspondence between elementary cuts on $\mathcal{G}_\alpha$ and labelings within one α-expansion of f. Moreover, for any elementary cut $\mathcal{C}$, we have $|\mathcal{C}| = E(f^{\mathcal{C}})$.*

Optimality properties

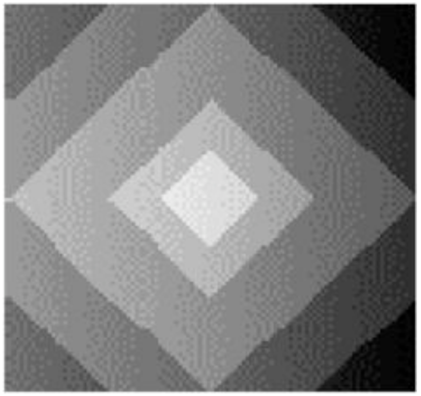
- Swap algorithm
  - local min can be arbitrarily far from global min
  - tighter bound for semimetric V
- Expansion algorithm
  - local min within known factor from global min

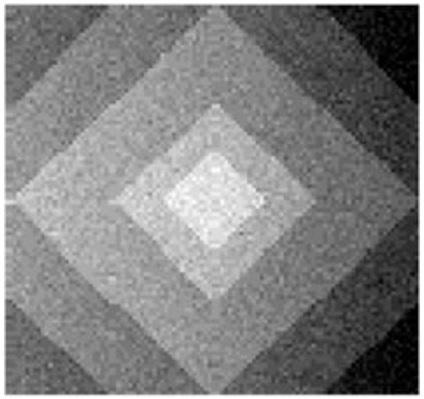
Expansion move optimal

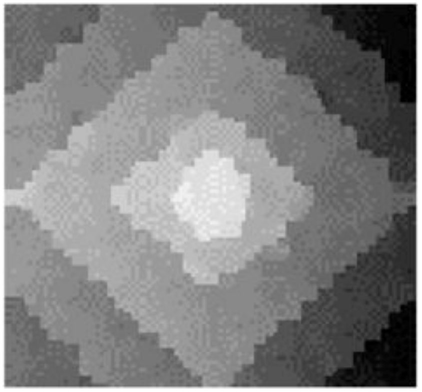
Theorem 6.1. *Let $\hat{f}$ be a local minimum when the expansion moves are allowed and f^* be the globally optimal solution. Then, $E(\hat{f}) \leq 2cE(f^*)$.*

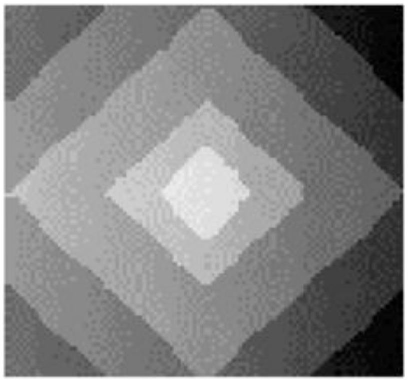
Proof: on the board

Image restoration results


original image


noisy image


standard moves denoised image


expansion moves denoised image