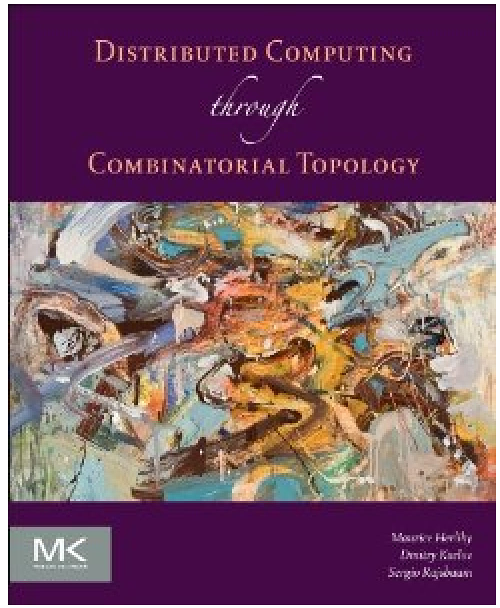
Elements of Combinatorial Topology



Companion slides for

Distributed Computing
Through Combinatorial Topology

Maurice Herlihy & Dmitry Kozlov & Sergio Rajsbaum

Road Map

- Simplicial Complexes
- Standard Constructions
- Carrier Maps
- Connectivity
- Subdivisions
- Simplicial & Continuous Approximations

Road Map

Simplicial Complexes

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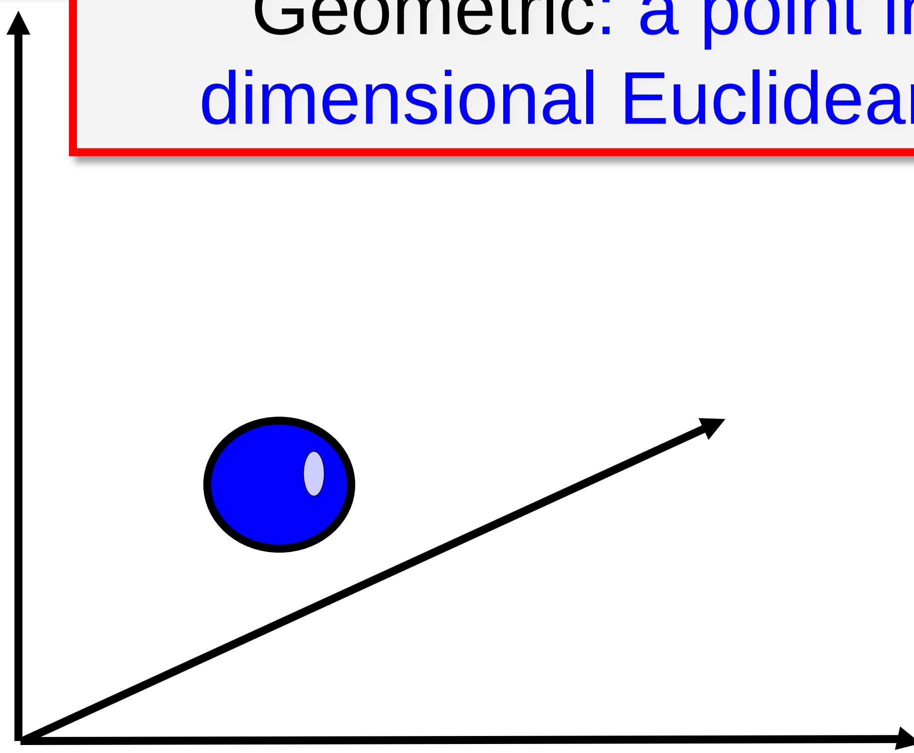
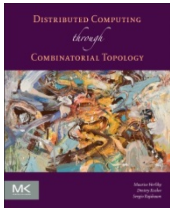
A Vertex

Combinatorial: an element of a set

Geometric: a point in high-dimensional Euclidean Space

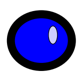




Simplexes

Combinatorial: a set of vertexes

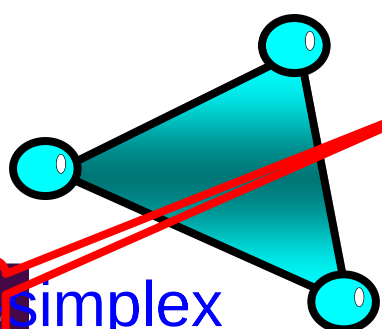
Geometric: convex hull of points in general position

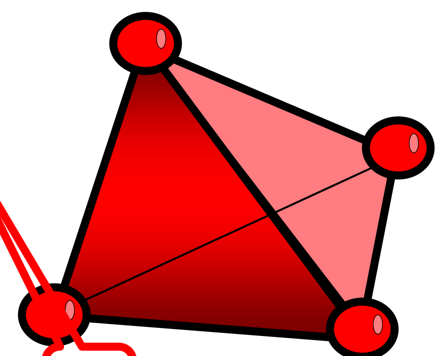


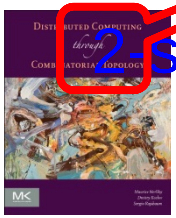
0-simplex

1-simplex

dimension







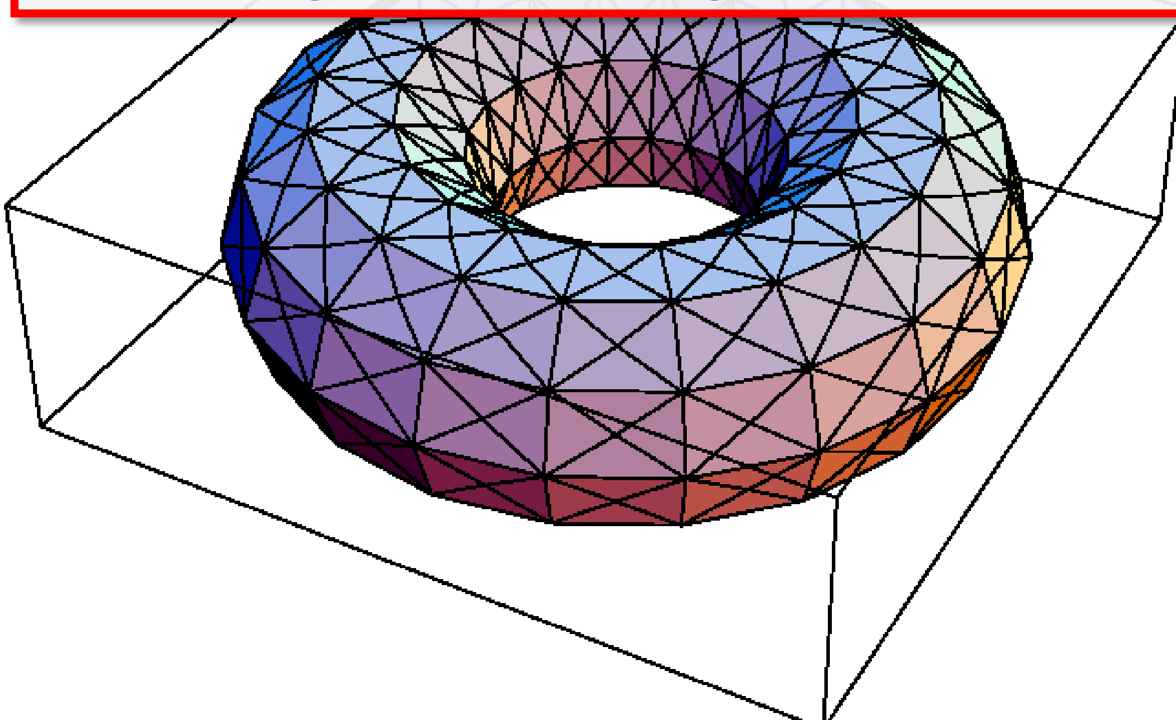
2-simplex

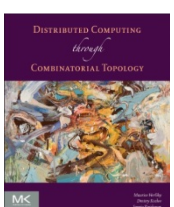
3-simplex

Simplicial Complex

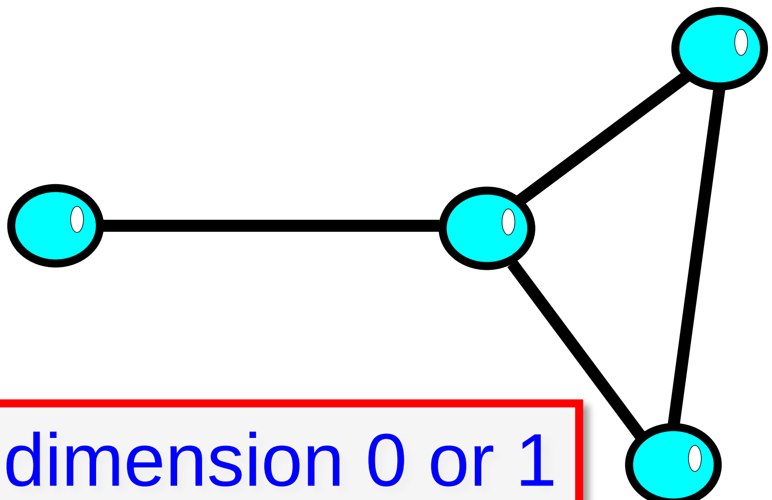
Combinatorial: a set of simplexes close under inclusion

Geometric: simplexes "glued together" along faces …

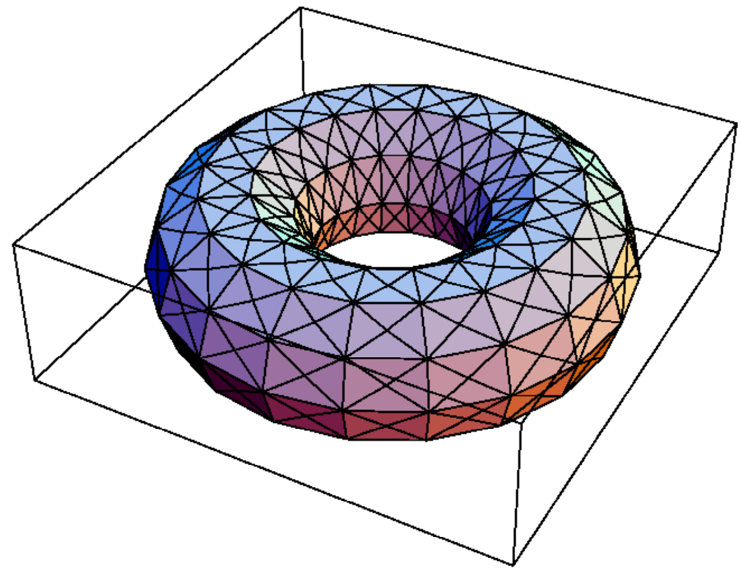




Graphs vs Complexes



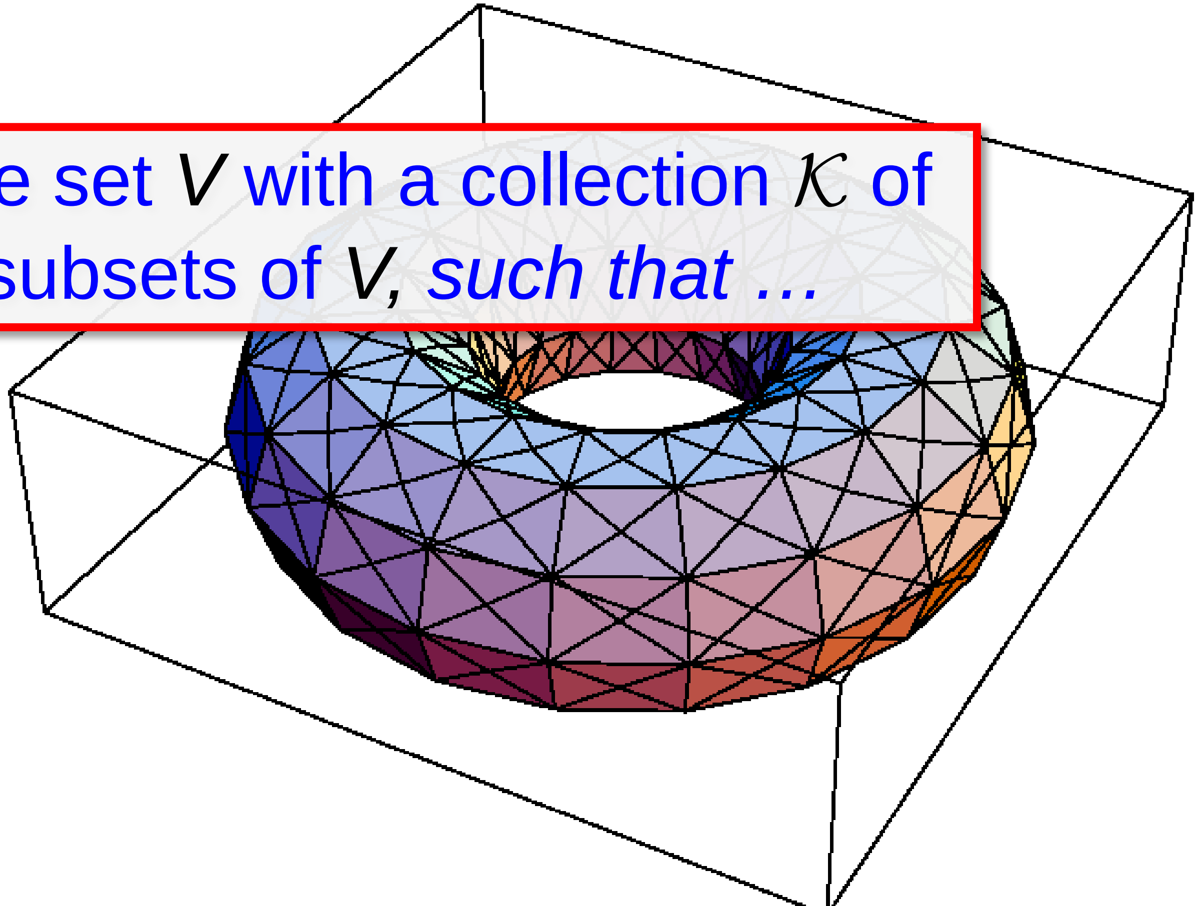
dimension 0 or 1



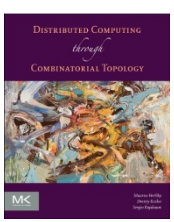
arbitrary dimension

complexes are a natural *generalization* of graphs

Abstract Simplicial Complex



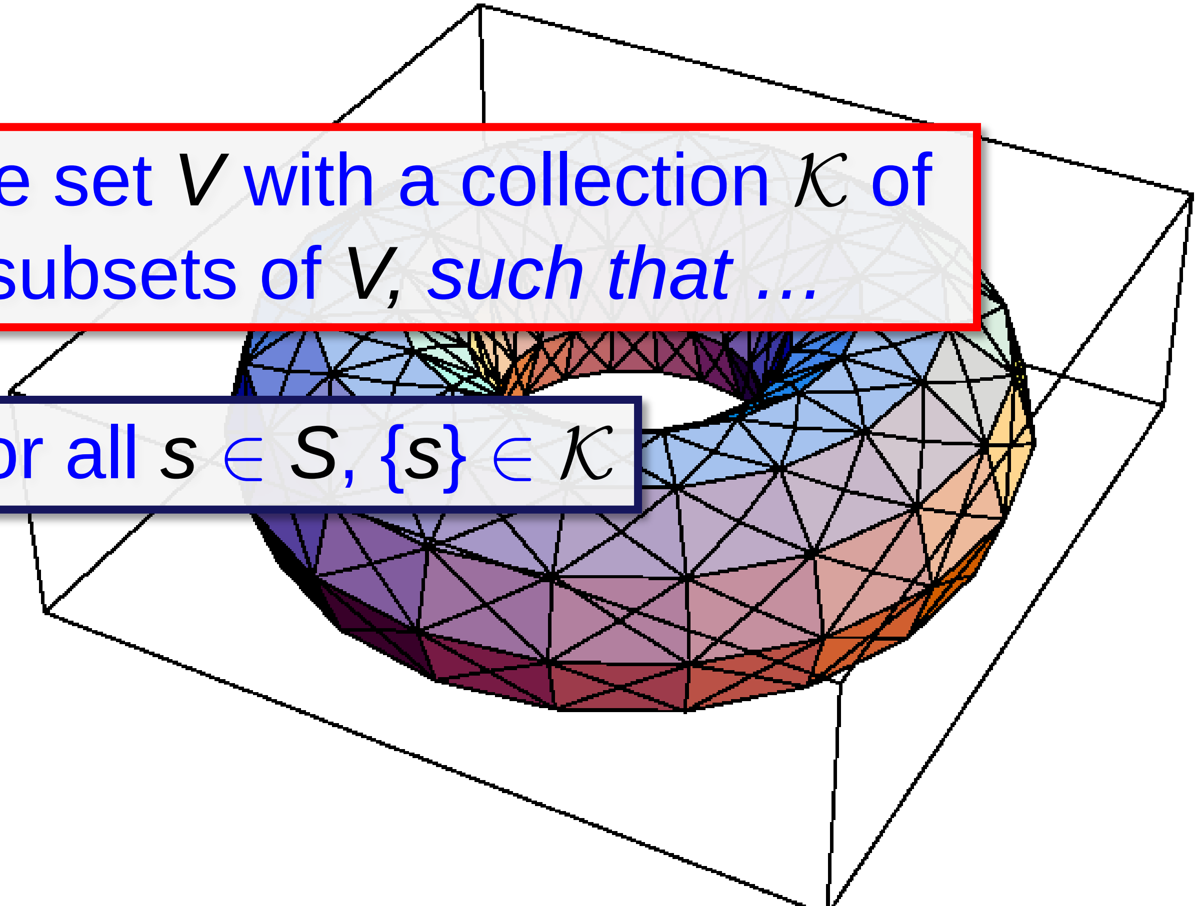
finite set V with a collection $\mathcal{K}$ of subsets of V, *such that …*

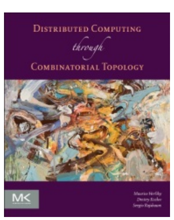


Abstract Simplicial Complex

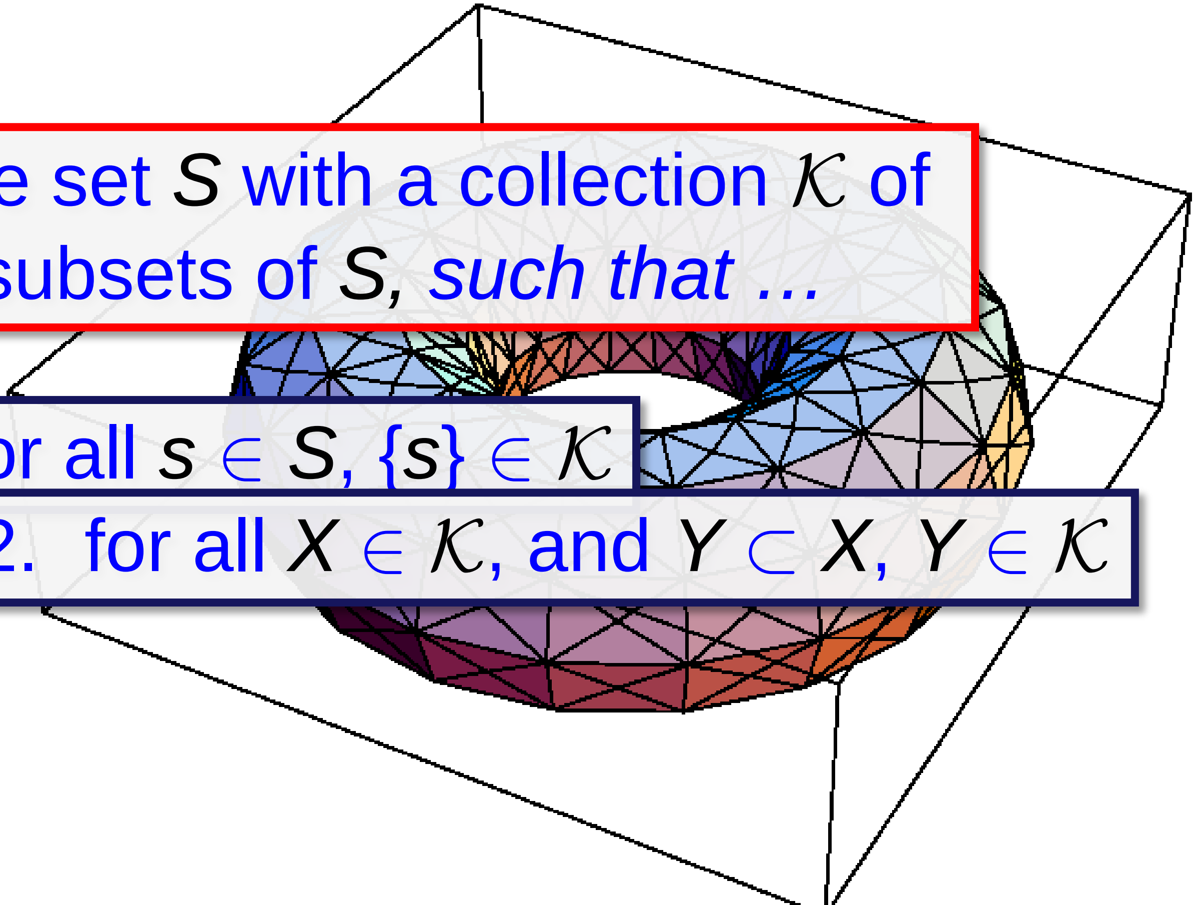
finite set V with a collection $\mathcal{K}$ of subsets of V, *such that …*

1. for all $s \in S$, $\{s\} \in \mathcal{K}$



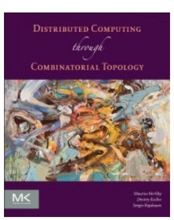


Abstract Simplicial Complex



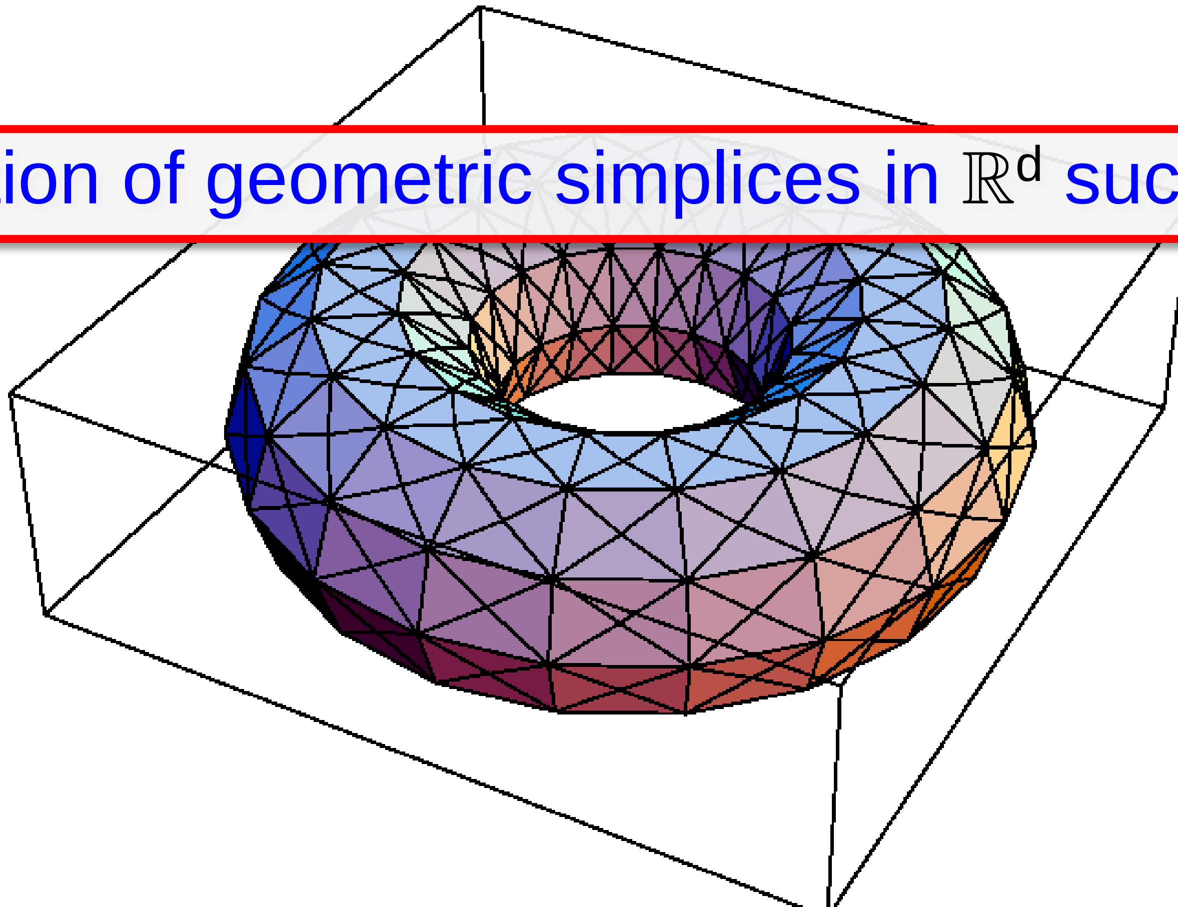
finite set S with a collection $\mathcal{K}$ of subsets of S, *such that …*

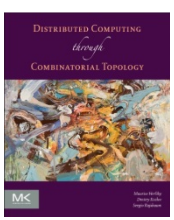
1. for all $s \in S$, $\{s\} \in \mathcal{K}$
2. for all $X \in \mathcal{K}$, and $Y \subset X$, $Y \in \mathcal{K}$



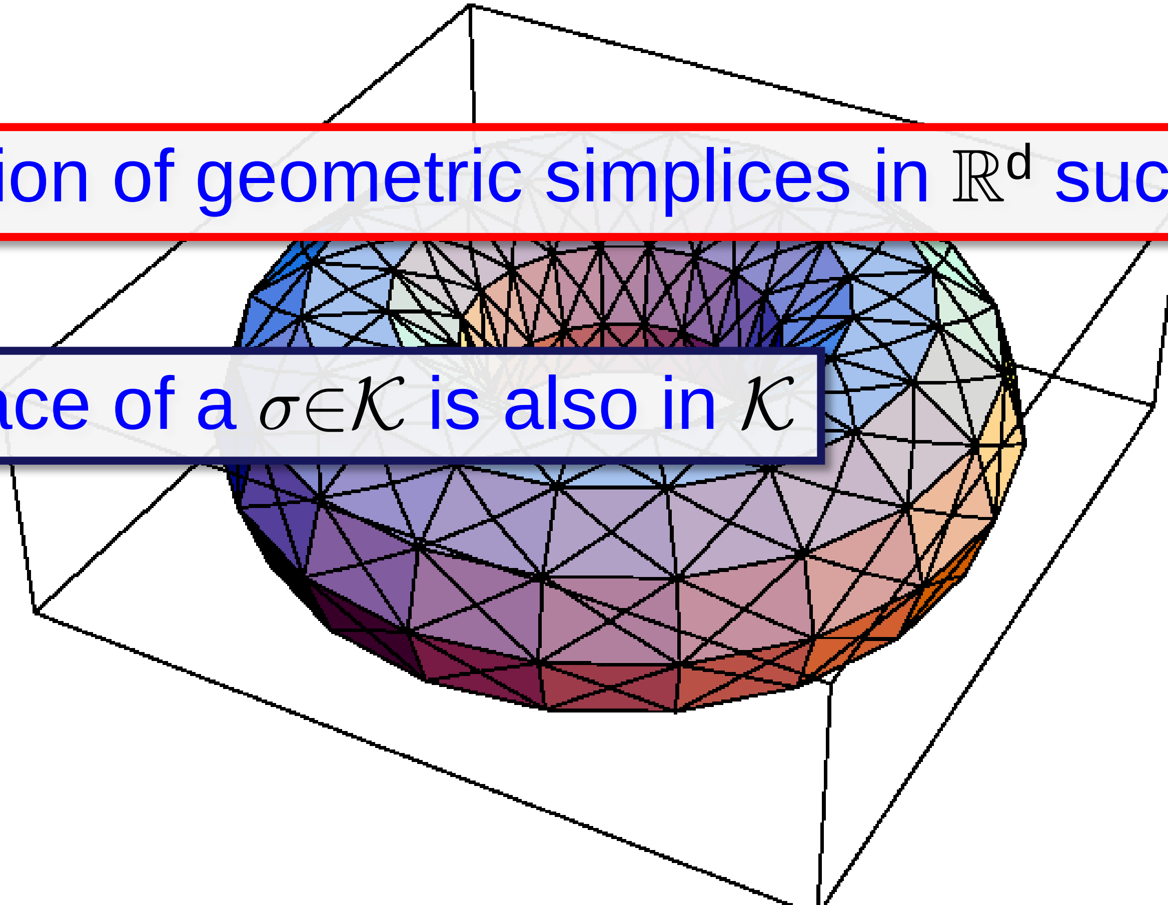
Geometric Simplicial Complex

A collection of geometric simplices in $\mathbb{R}^d$ such that



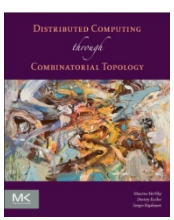


Geometric Simplicial Complex

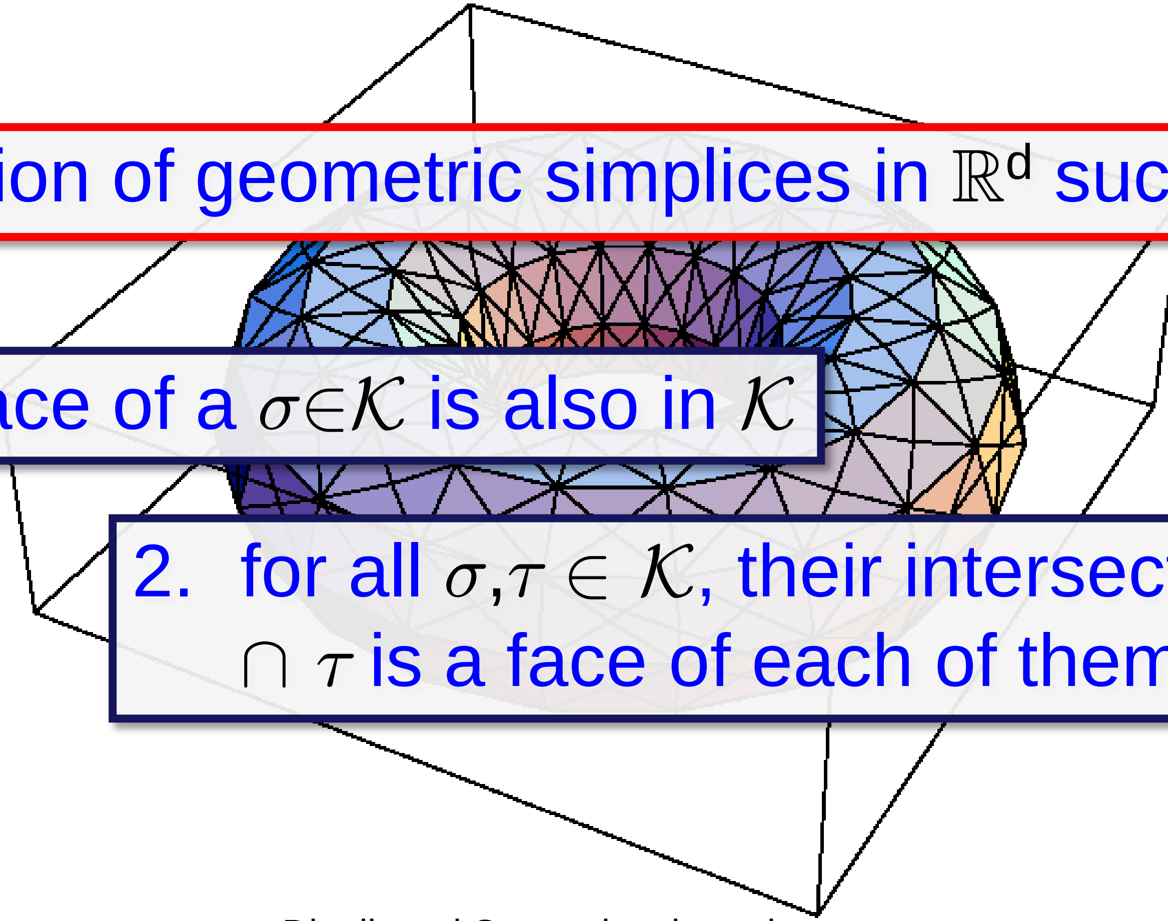


A collection of geometric simplices in $\mathbb{R}^d$ such that

1. any face of a $\sigma \in \mathcal{K}$ is also in $\mathcal{K}$

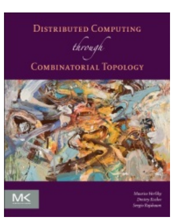


Geometric Simplicial Complex



A collection of geometric simplices in $\mathbb{R}^d$ such that

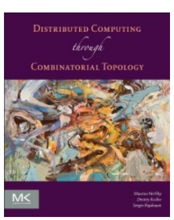
1. any face of a $\sigma \in \mathcal{K}$ is also in $\mathcal{K}$
2. for all $\sigma, \tau \in \mathcal{K}$, their intersection $\sigma \cap \tau$ is a face of each of them.



Abstract vs Geometric Complexes

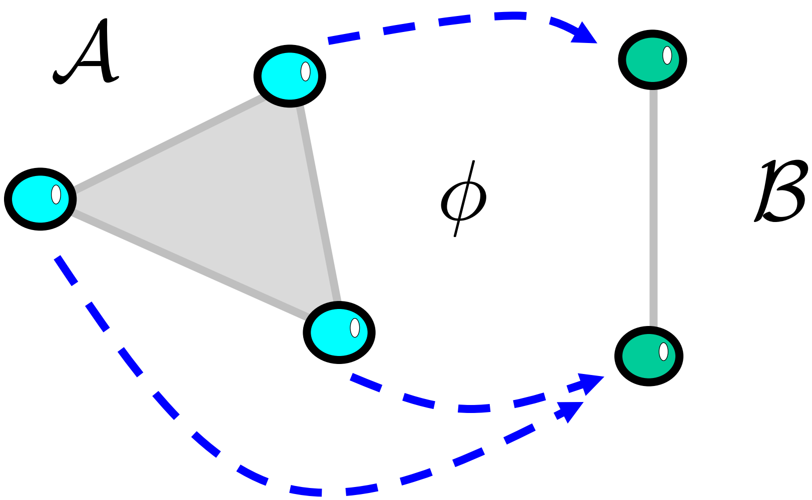
Abstract: $\mathcal{A}$

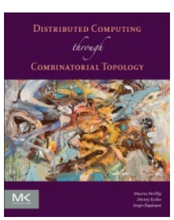
Geometric: $|\mathcal{A}|$



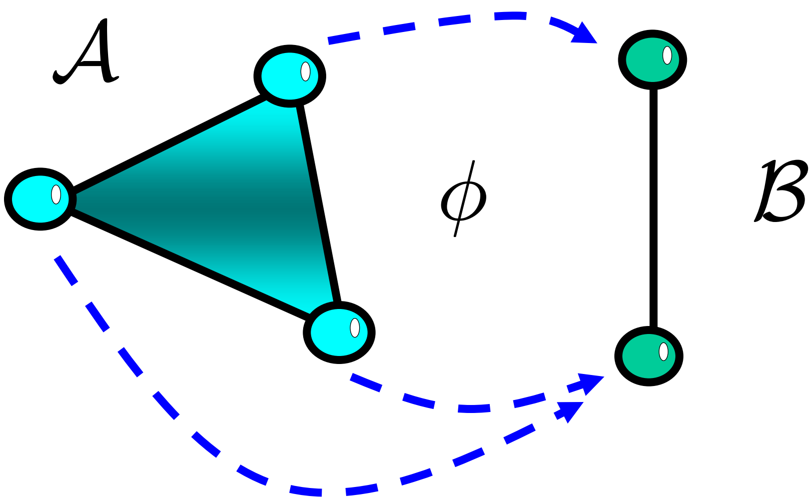
Simplicial Maps

Vertex-to-vertex map …





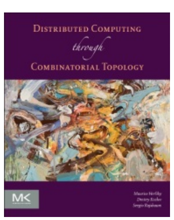
Simplicial Map



Vertex-to-vertex map …

that sends simplexes to simplexes

$$\boxed{\phi \colon \mathcal{A} \to \mathcal{B}}$$



Road Map

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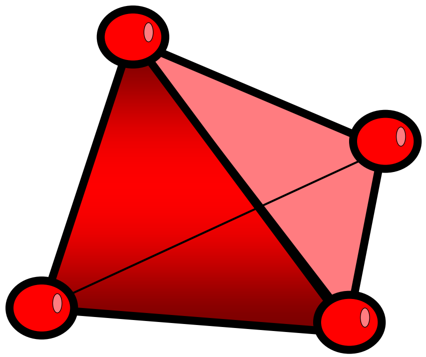
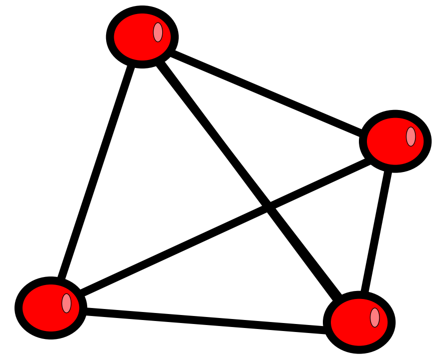
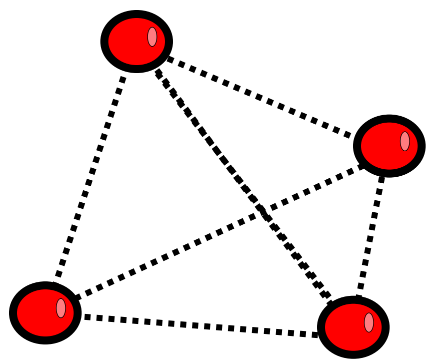
Skeleton



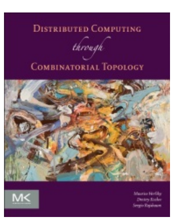
$\mathcal{C}$



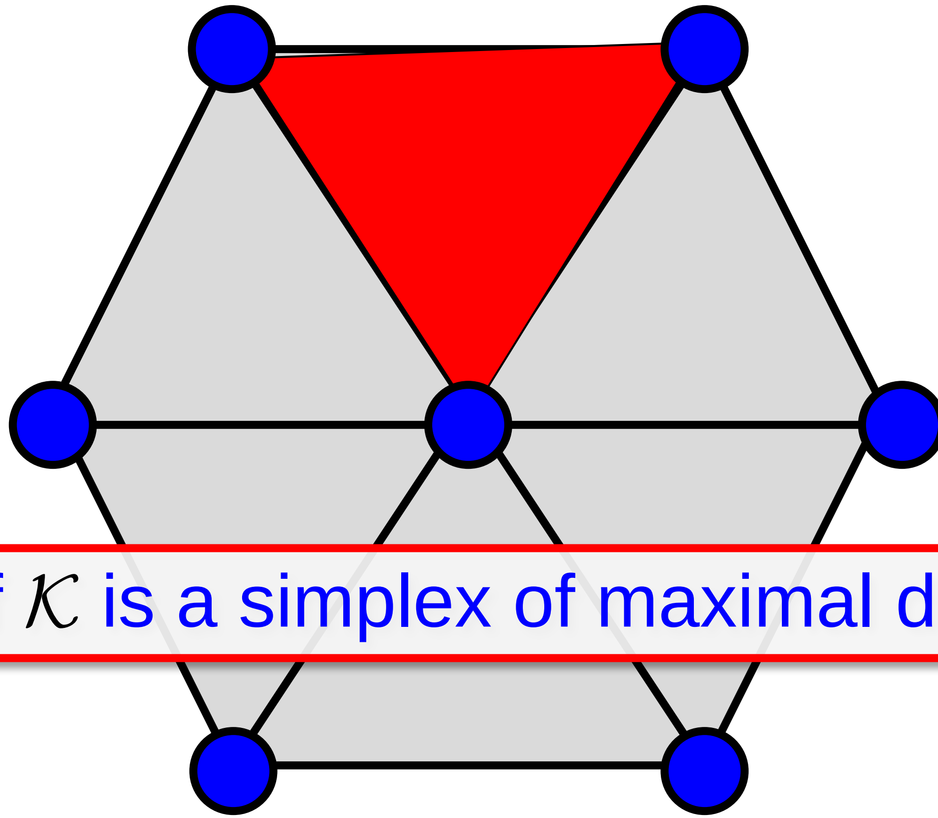
$\text{skel}^1\, \mathcal{C}$



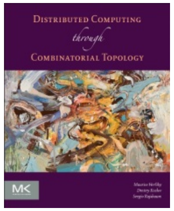
$\text{skel}^0\, \mathcal{C}$



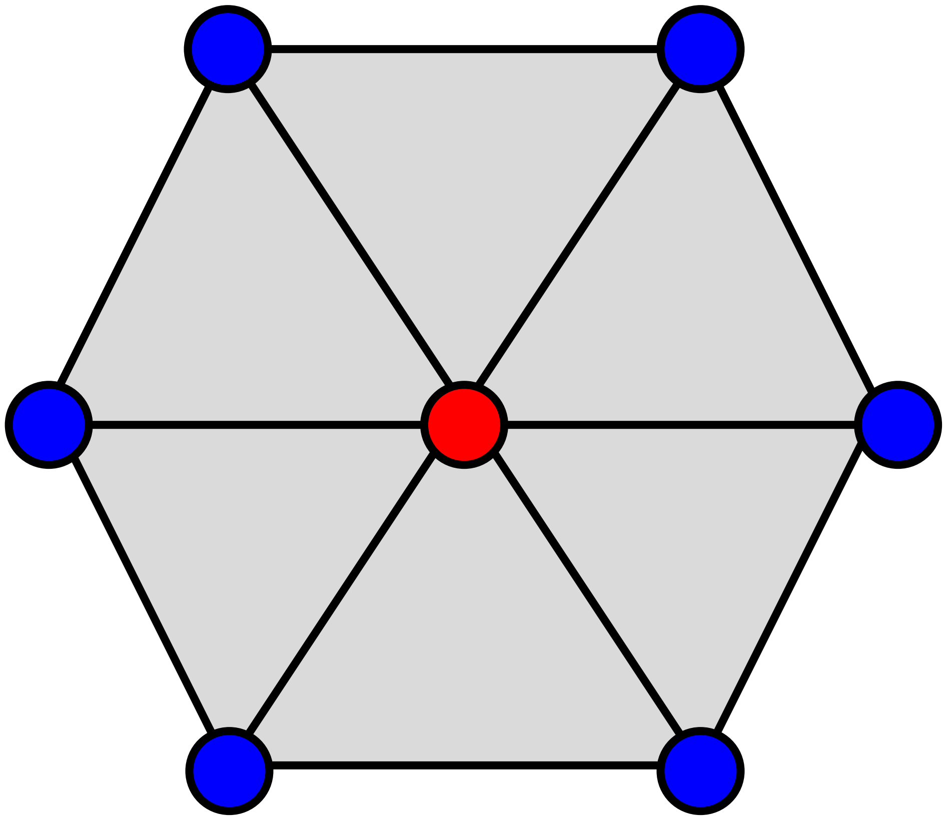
Facet



A facet of $\mathcal{K}$ is a simplex of maximal dimension



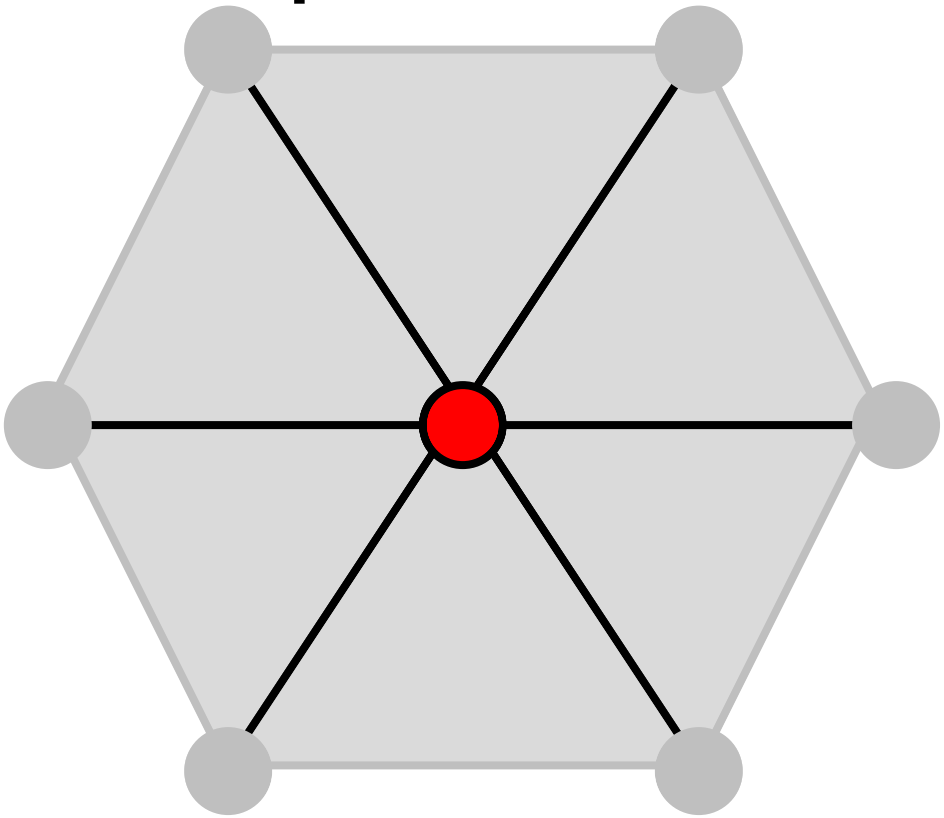
Star



Star(σ,$\mathcal{K}$) is the complex of facets of $\mathcal{K}$ containing σ

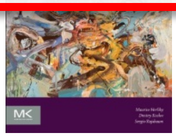
Complex

Open Star

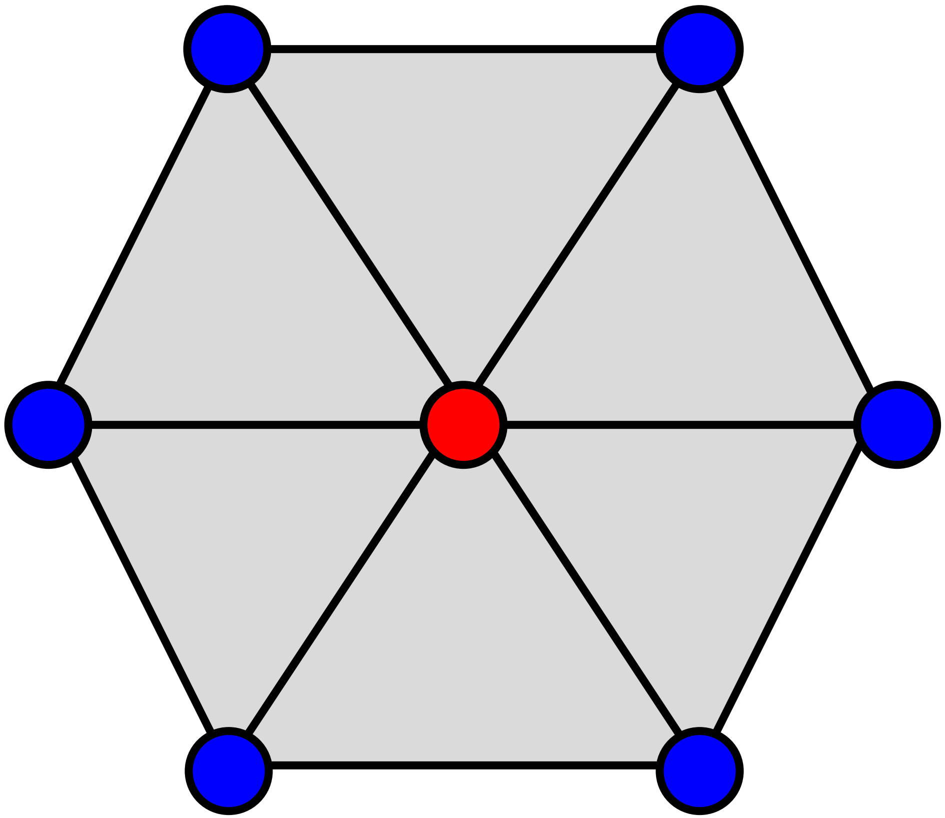


Star$^{o}(\sigma,\mathcal{K})$ union of interiors of simplexes containing σ

Point Set



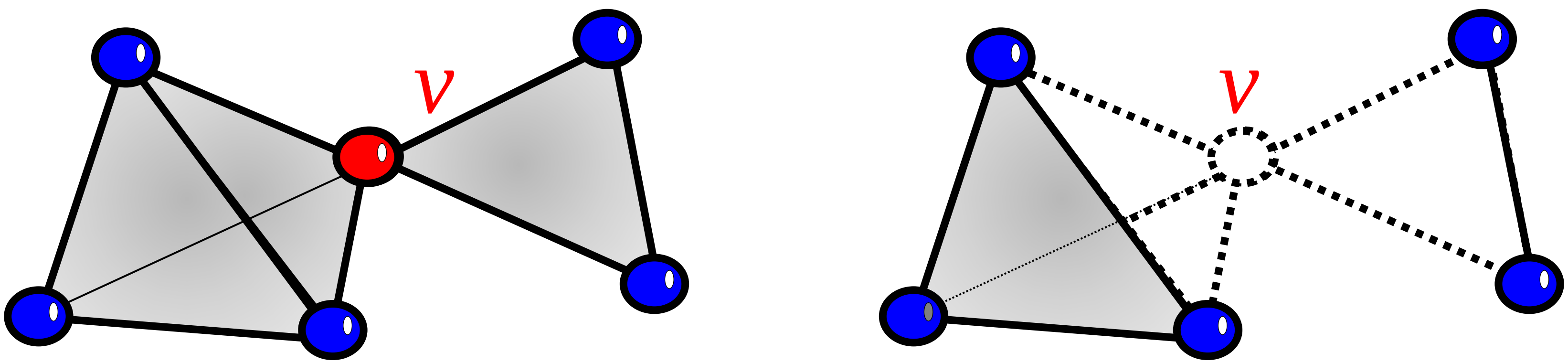
Link



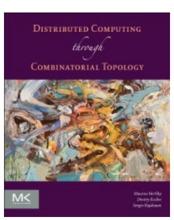
Link(σ,$\mathcal{K}$) is the complex of simplices of Star(σ,$\mathcal{K}$) not containing σ

Complex

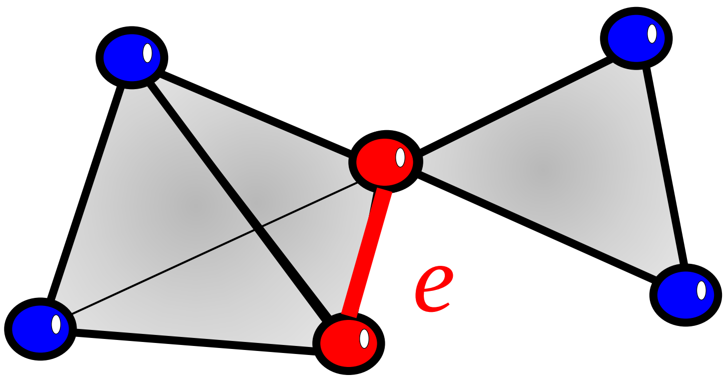
More Links



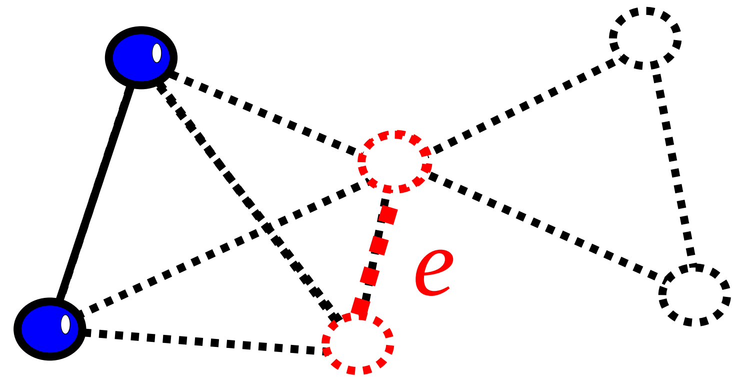
$\mathcal{C}$ Link(v,$\mathcal{C}$)



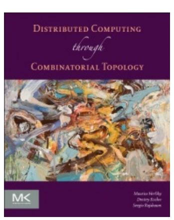
More Links



$\mathcal{C}$



Link(e,$\mathcal{C}$)



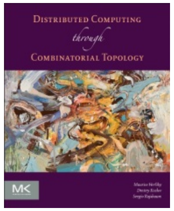
Join

Let $\mathcal{A}$ and $\mathcal{B}$ be complexes with disjoint sets of vertices

their *join* $\mathcal{A}*\mathcal{B}$ is the complex

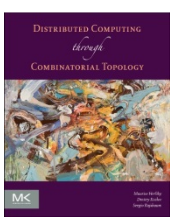
with vertices $V(\mathcal{A}) \cup V(\mathcal{B})$

and simplices $\alpha \cup \beta$, where $\alpha \in \mathcal{A}$, and $\beta \in \mathcal{B}$.



Join

$\mathcal{A}$ $\mathcal{B}$ $\mathcal{A} * \mathcal{B}$



Road Map

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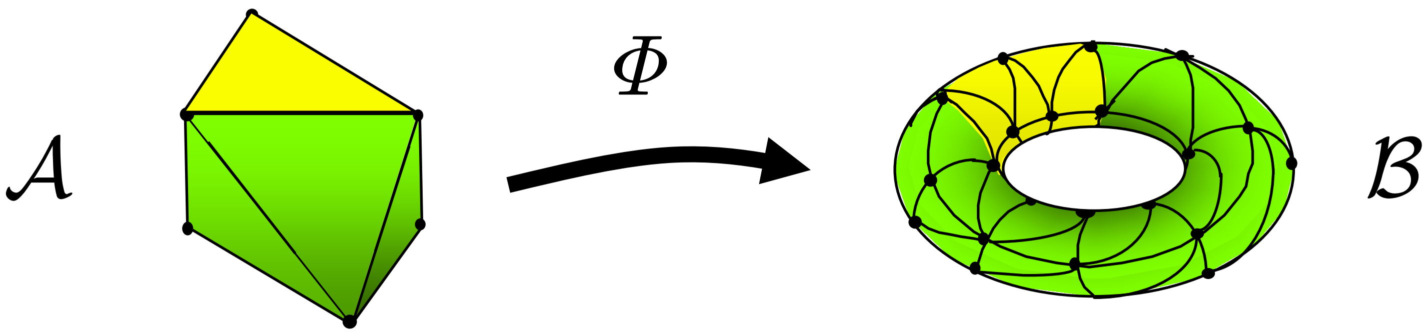
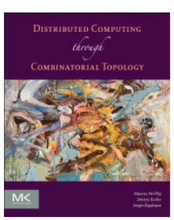
Carrier Map



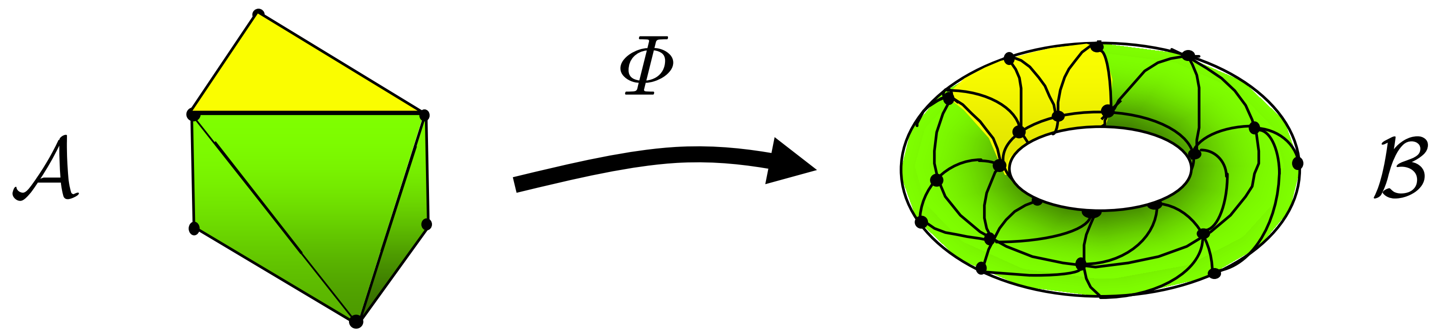
Maps simplex of $\mathcal{A}$

to subcomplex of $\mathcal{B}$

$$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$$



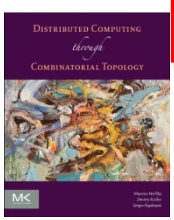
Carrier Maps are Monotonic



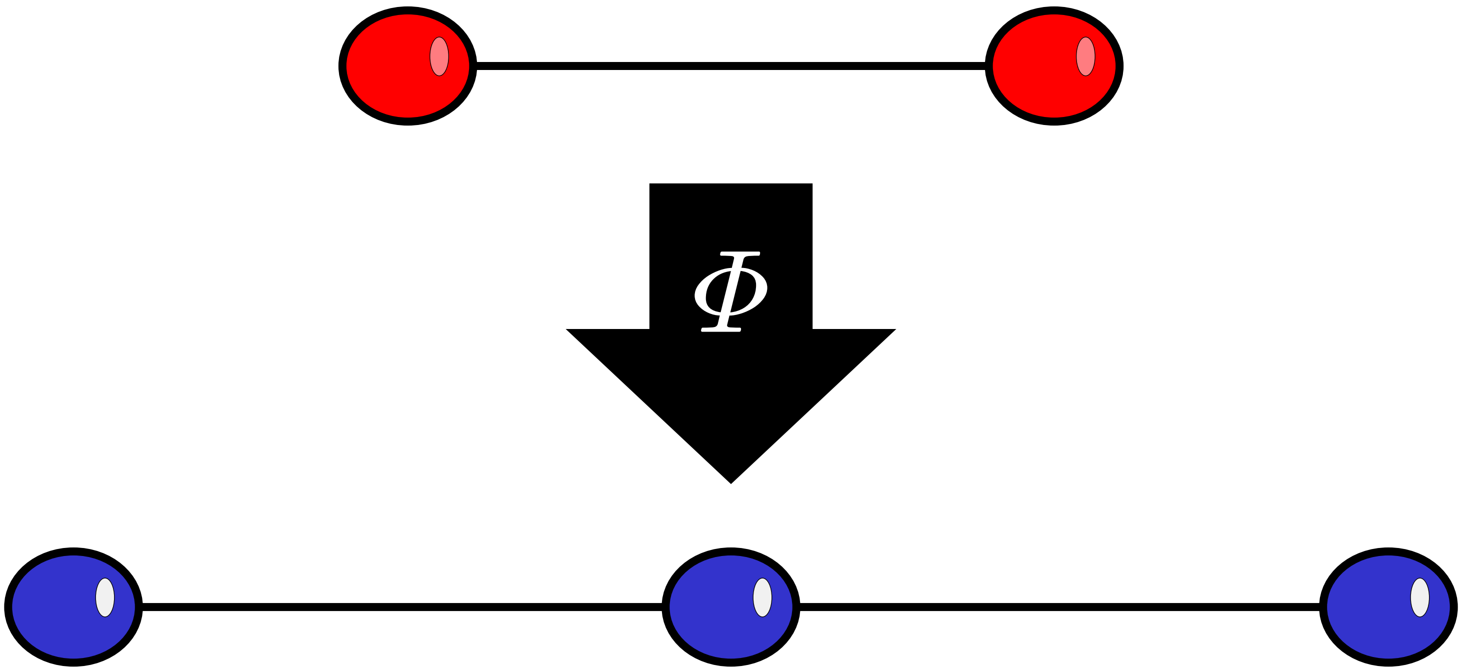
If $\tau \subseteq \sigma$ then $\Phi(\tau) \subseteq \Phi(\sigma)$

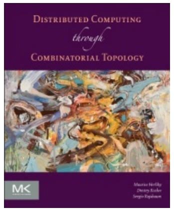
or

for $\sigma, \tau \in \mathcal{A}$, $\Phi(\sigma \cap \tau) \subseteq \Phi(\sigma) \cap \Phi(\tau)$

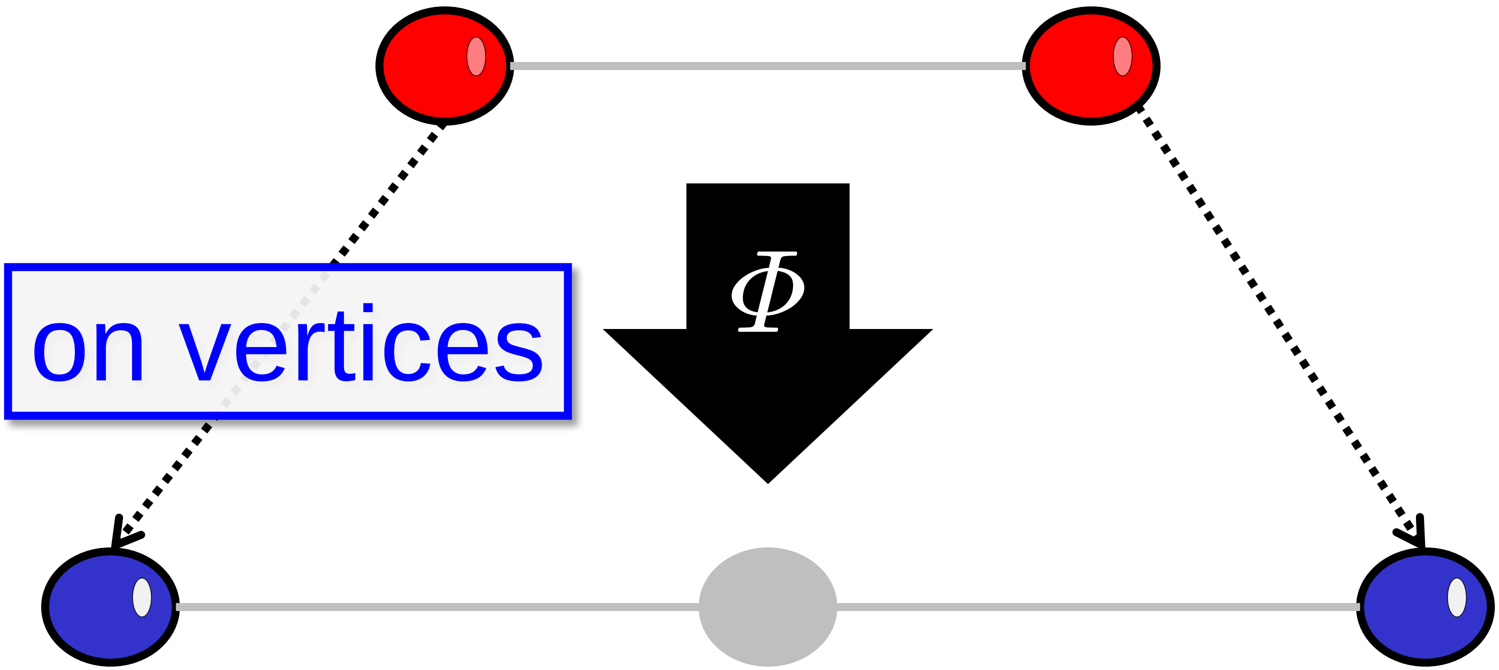


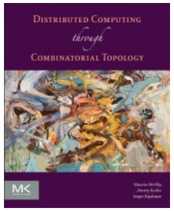
Example


Φ


Distributed Computing
through
Combinatorial Topology
MK

Example


on vertices
Φ


Distributed Computing
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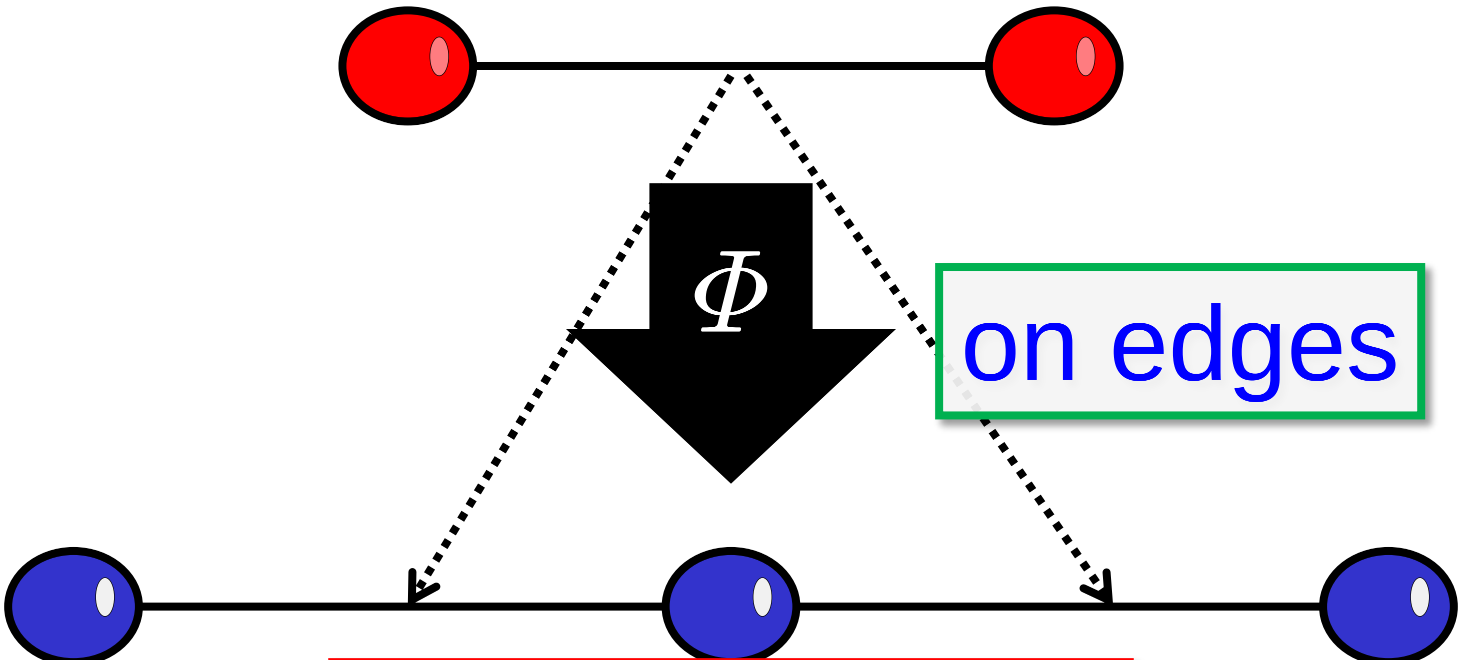
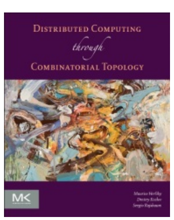
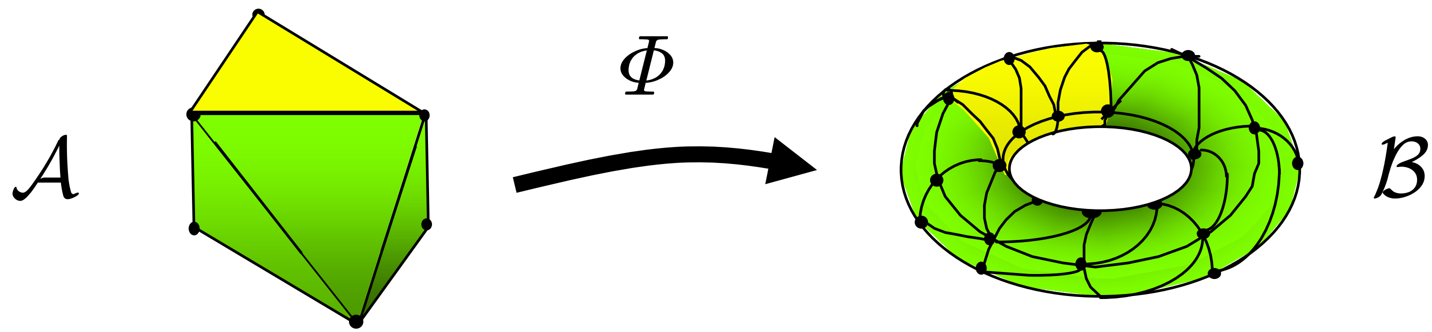
Example



There is no simplicial map carried by Φ

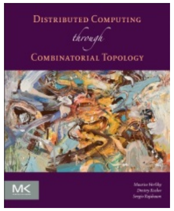


Strict Carrier Maps

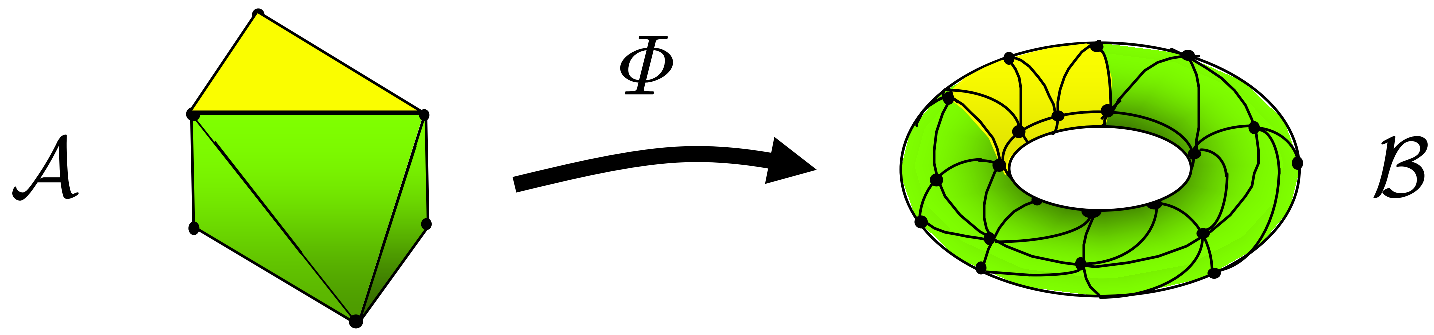


for all $\sigma,\tau \in \mathcal{A}$, $\Phi(\sigma\cap\tau) = \Phi(\sigma)\cap\Phi(\tau)$

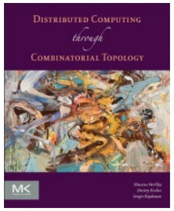
replace $\subseteq$ with $=$



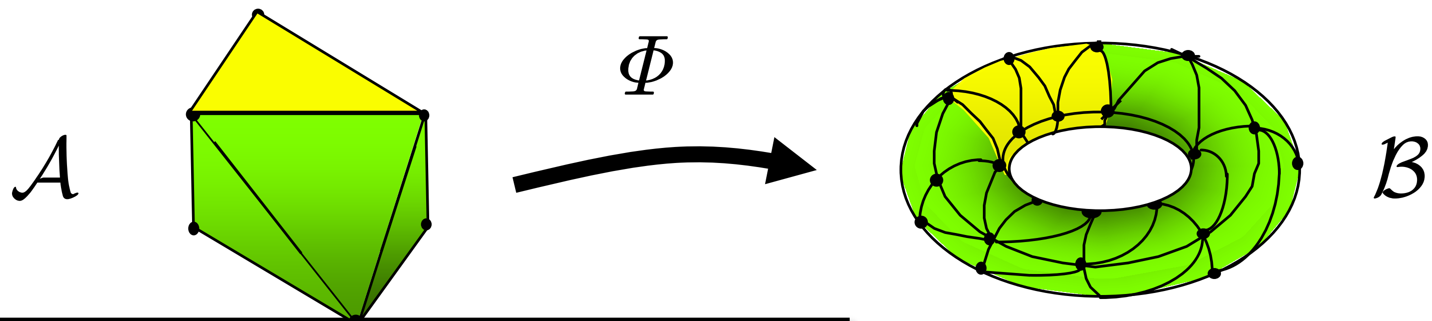
Rigid Carrier Maps



for $\sigma \in \mathcal{A}$, $\Phi(\sigma)$ is pure of dimension dim σ



Carrier of a Simplex



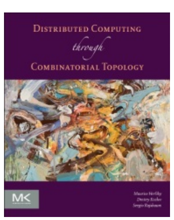
given *strict* $\Phi: \mathcal{A} \to 2^{\mathcal{B}}$

for each $\tau \in \mathcal{B}$,

$\exists$ unique smallest $\sigma \in \mathcal{A}$ such that $\tau \in \Phi(\sigma)$.

$\sigma = \mathrm{Car}(\tau, \Phi(\sigma))$

sometimes omitted



Carrier Map Carried By Carrier Map

Given carrier maps

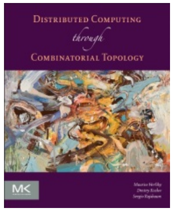
$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$

$\Psi: \mathcal{A} \to 2^{\mathcal{B}}$

Φ is *carried by* Ψ if

for all $\sigma \in \mathcal{A}$, $\Phi(\sigma) \subseteq \Psi(\sigma)$

written: $\Phi \subseteq \Psi$



Simplicial Map Carried By Carrier Map

Given carrier and simplicial maps

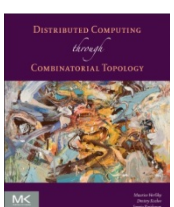
$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$

$\varphi: \mathcal{A} \to \mathcal{B}$

φ is *carried by* Φ if

for all $\sigma \in \mathcal{A}$, $\varphi(\sigma) \subseteq \Phi(\sigma)$

written: $\varphi \subseteq \Phi$



Continuous Map Carried By Carrier Map

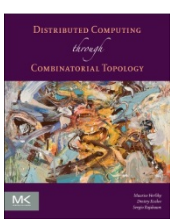
Given carrier and continuous maps

$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$

f: $|\mathcal{A}| \to |\mathcal{B}|$

f is *carried by* Φ if

for all $\sigma \in \mathcal{A}$, $f(\sigma) \subseteq |\Phi(\sigma)|$



Compositions

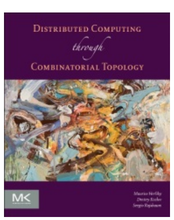
Given carrier maps

$$\Phi\colon \mathcal{A} \to 2^{\mathcal{B}}$$

$$\Psi\colon \mathcal{B} \to 2^{\mathcal{C}}$$

their *composition* is

$$(\Psi \circ \Phi)(\sigma) := \bigcup_{\tau \in \Phi(\sigma)} \Psi(\tau)$$



Theorem

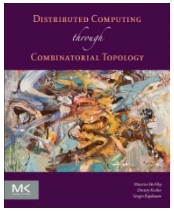
If Φ, Ψ are both

strict

so is $\Phi \circ \Psi$

rigid

so is $\Phi \circ \Psi$



Compositions

Given carrier and simplicial maps

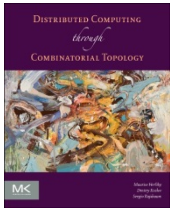
$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$ $\varphi: \mathcal{C} \to \mathcal{A}$

their *composition* is the *carrier map*

$(\Phi \circ \varphi): \mathcal{C} \to 2^{\mathcal{B}}$

defined by

$(\Phi \circ \varphi)(\sigma) := \Phi(\varphi(\sigma))$



Compositions

Given carrier and simplicial maps

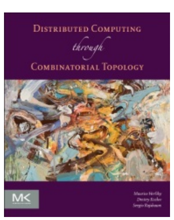
$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$ $\varphi: \mathcal{B} \to \mathcal{C}$

their *composition* is the *carrier map*

$(\varphi \circ \Phi): \mathcal{A} \to 2^{\mathcal{C}}$

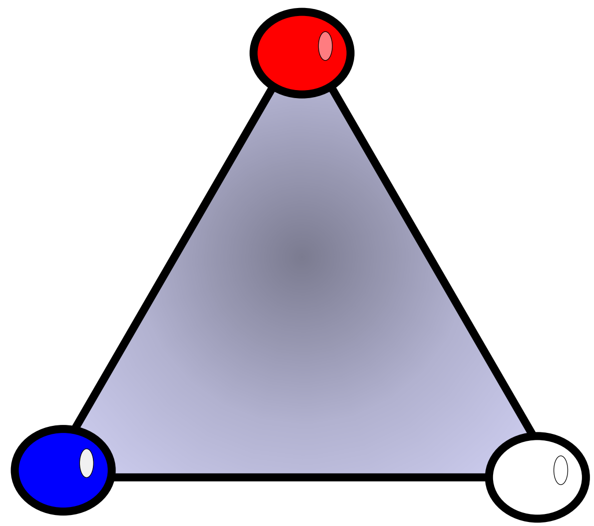
defined by

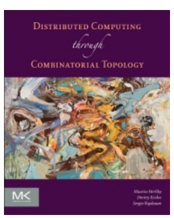
$$(\Phi \circ \varphi)(\sigma) := \cup_{\tau \in \Phi(\sigma)} \varphi(\tau)$$



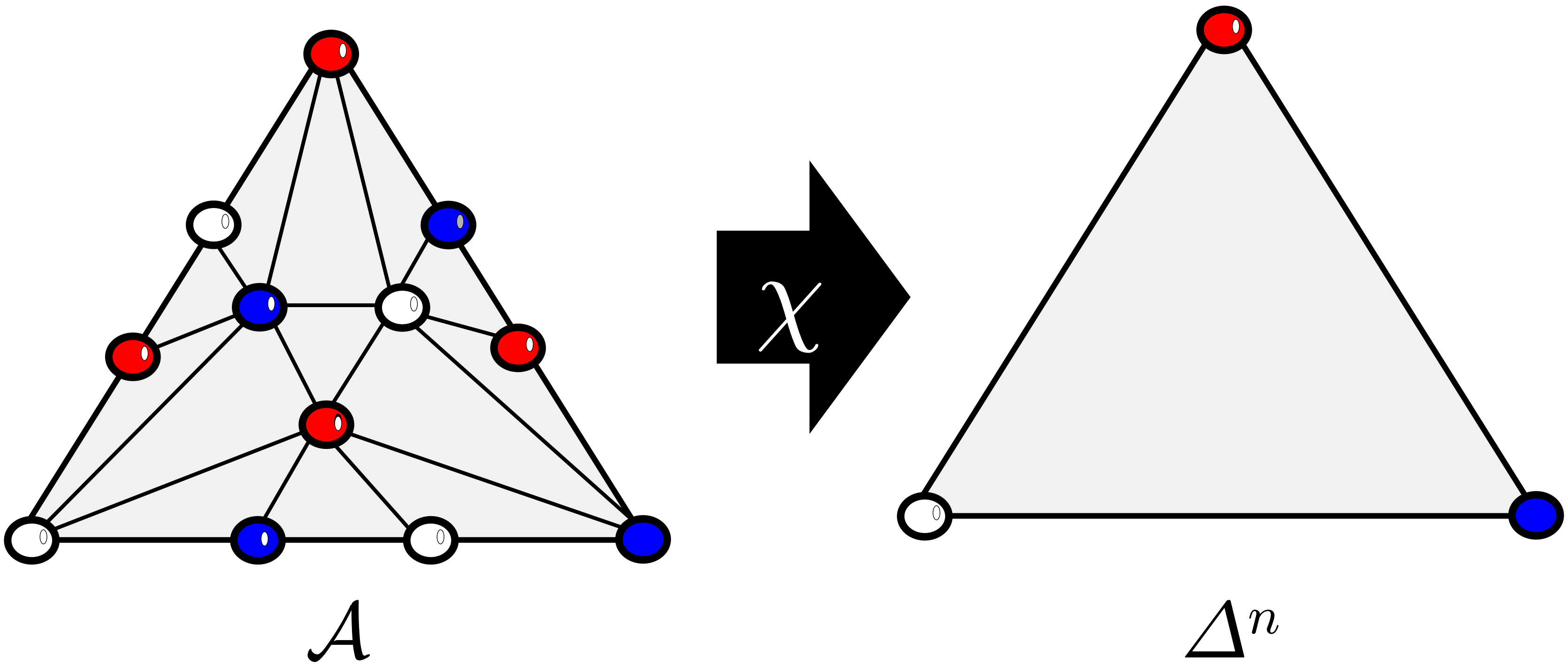
Colorings

$$\Delta^n :=$$

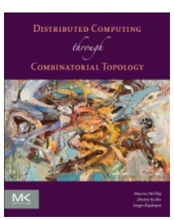




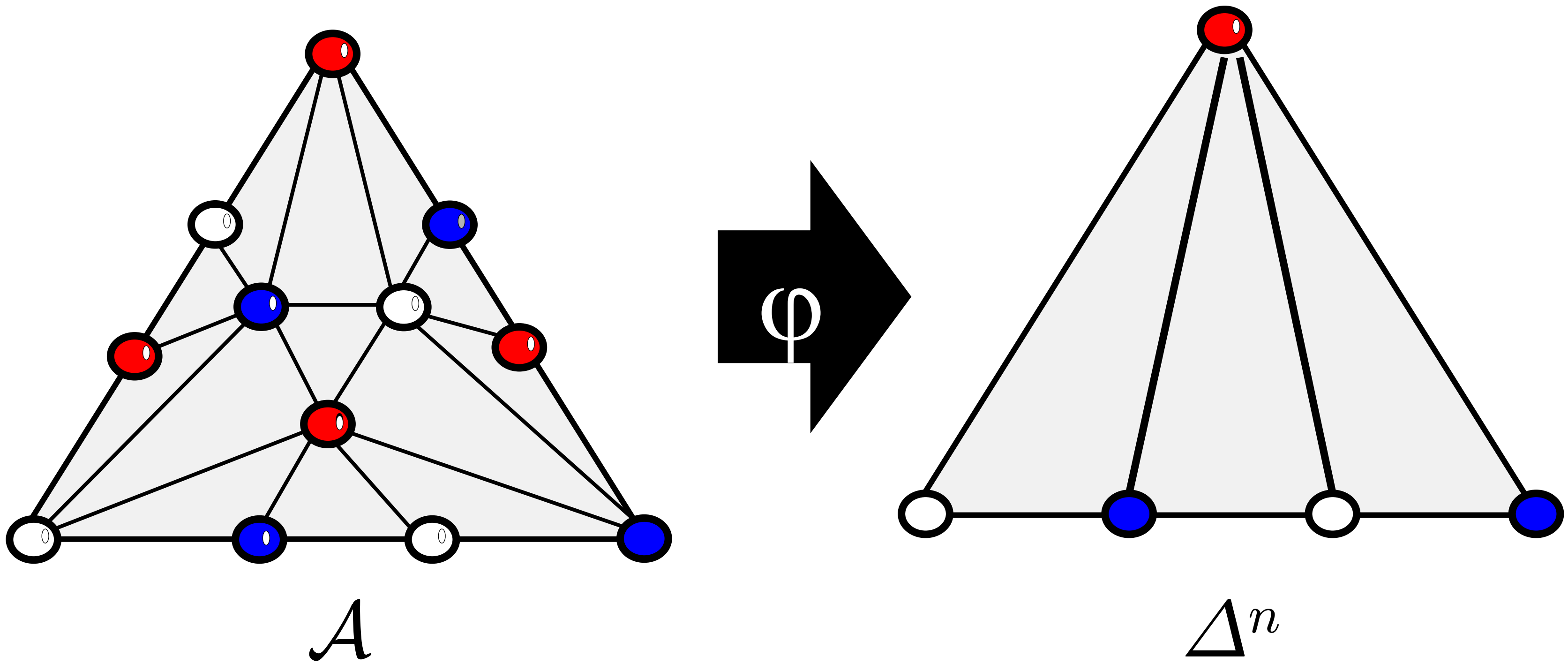
Chromatic Complex



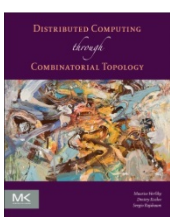
rigid simplicial map



Color-Preserving Simplicial Map



color of v = color of $\varphi(v)$



Road Map

Simplicial Complexes

Standard Constructions

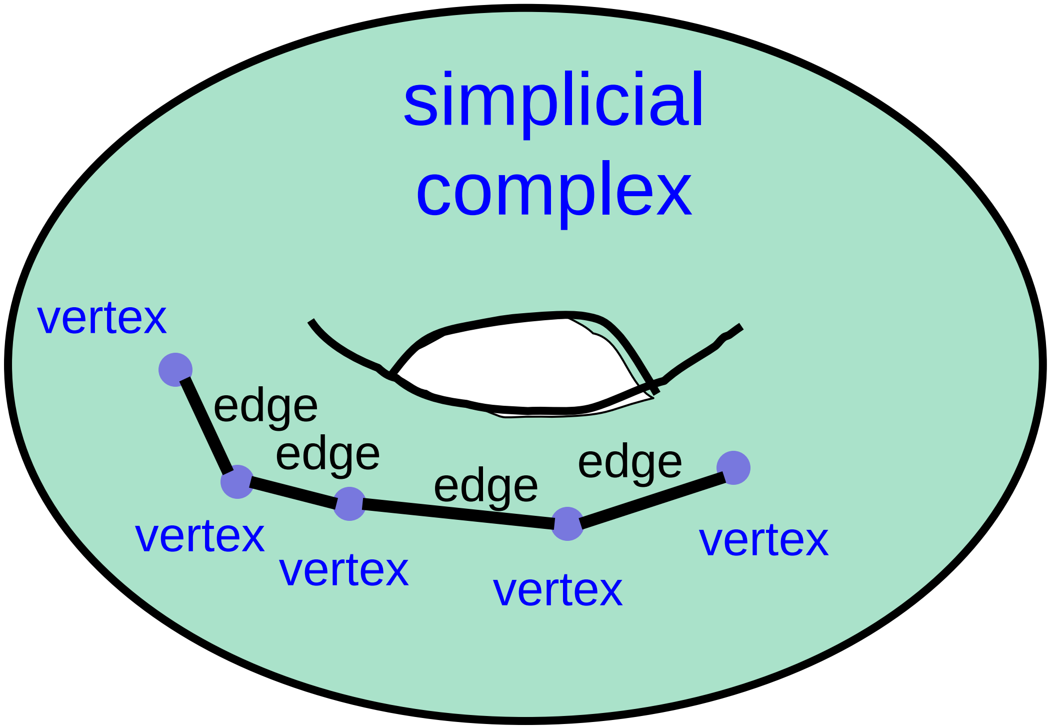
Carrier Maps

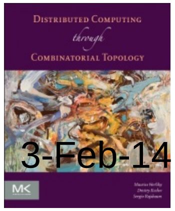
Connectivity

Subdivisions

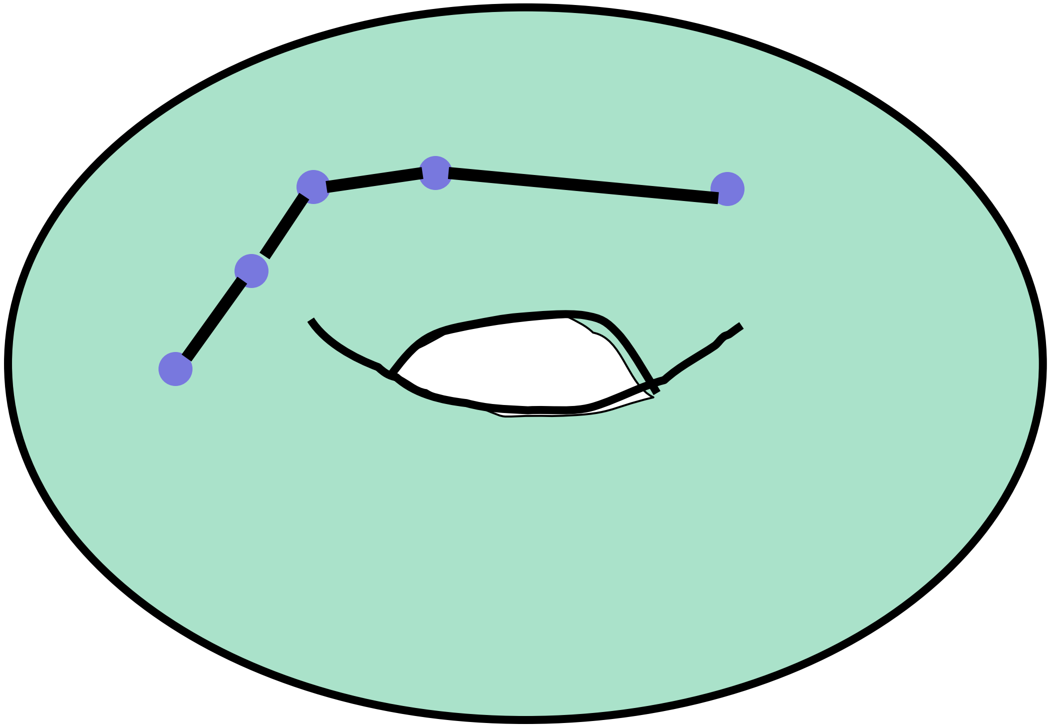
Simplicial & Continuous Approximations

A Path

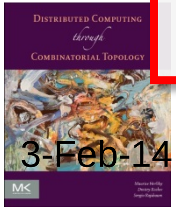

simplicial complex
vertex
edge
edge
edge
edge
vertex
vertex
vertex
vertex


Distributed Computing through Combinatorial Topology

Path Connected

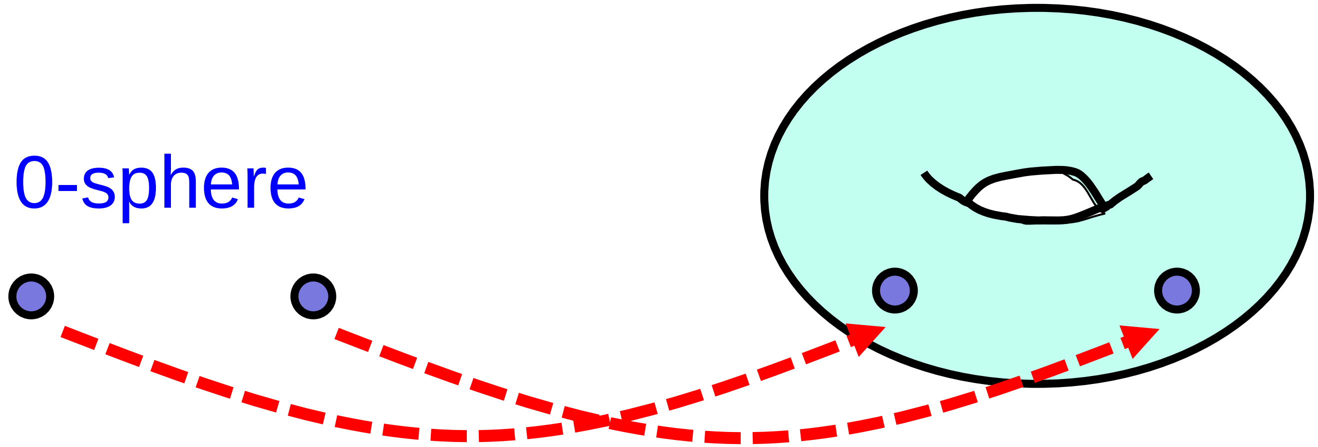


Any two vertexes can be linked by a path



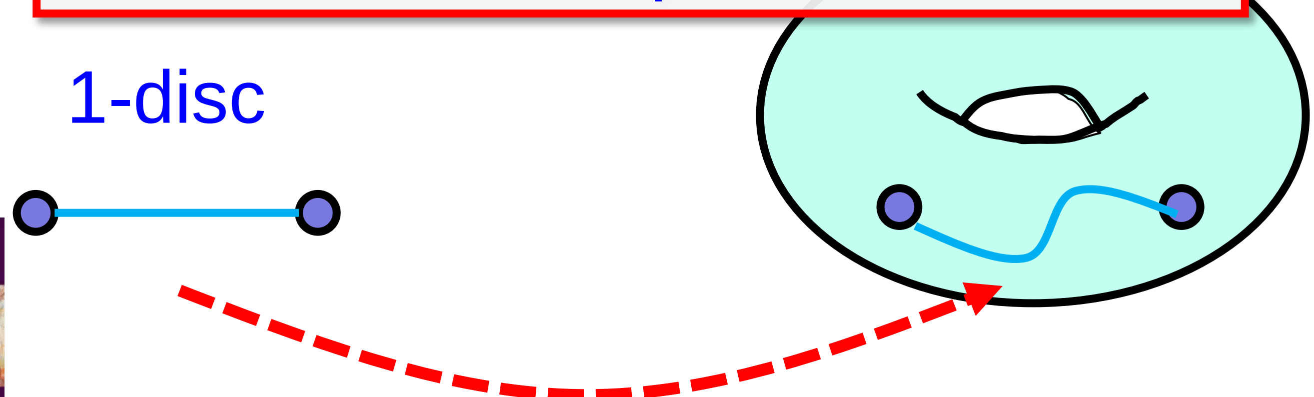
Rethinking Path Connectivity

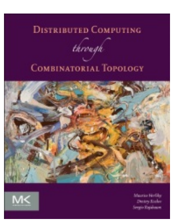
0-sphere



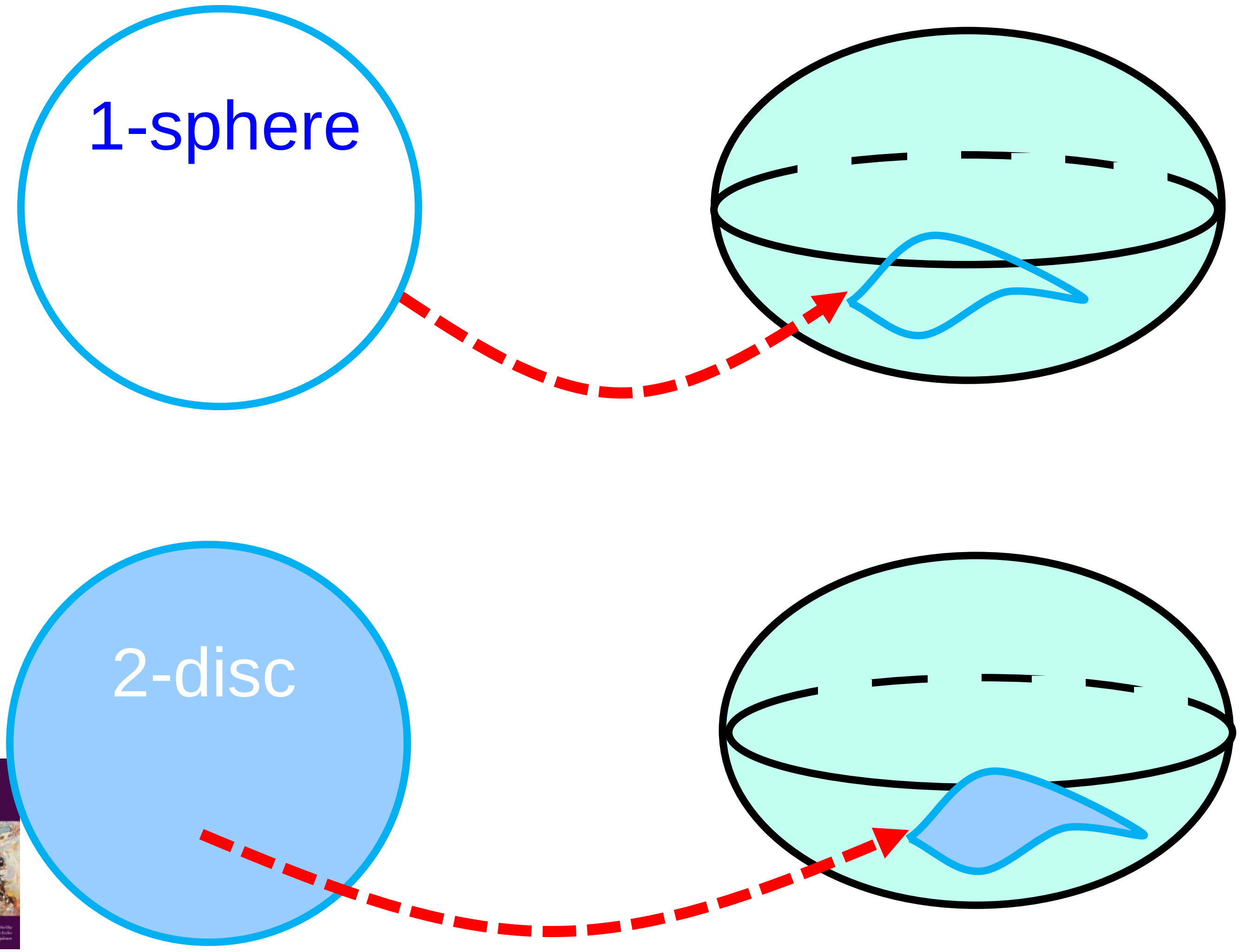
Let's call this complex *0-connected*

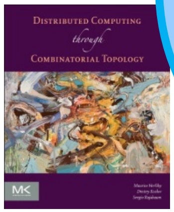
1-disc



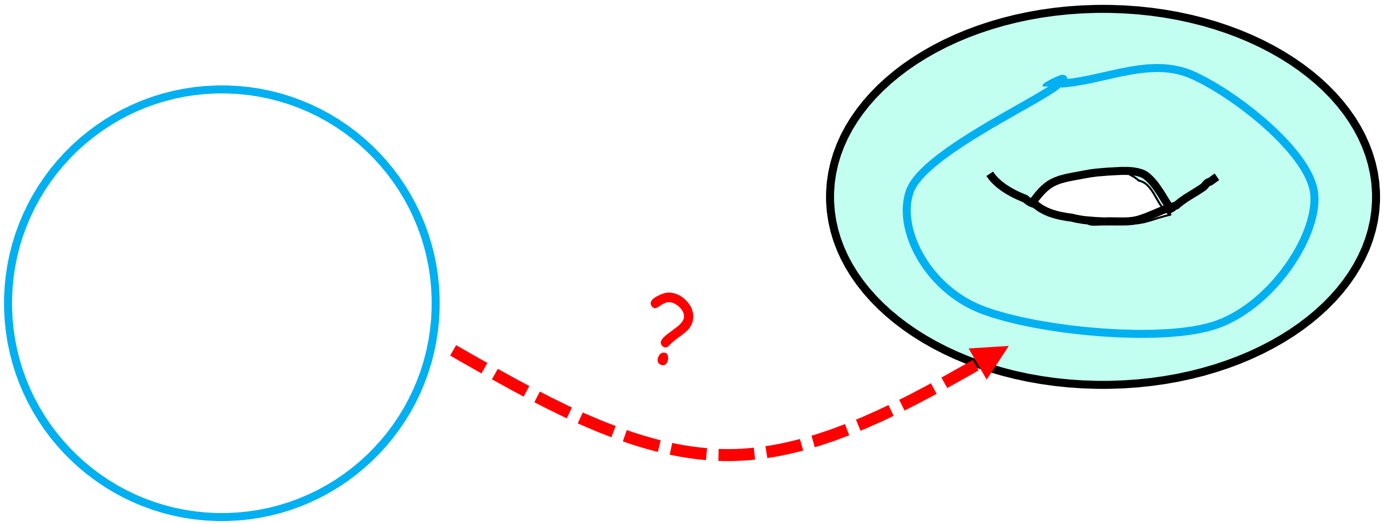


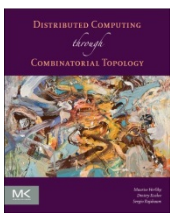
1-Connectivity





This Complex is not 1-Connected

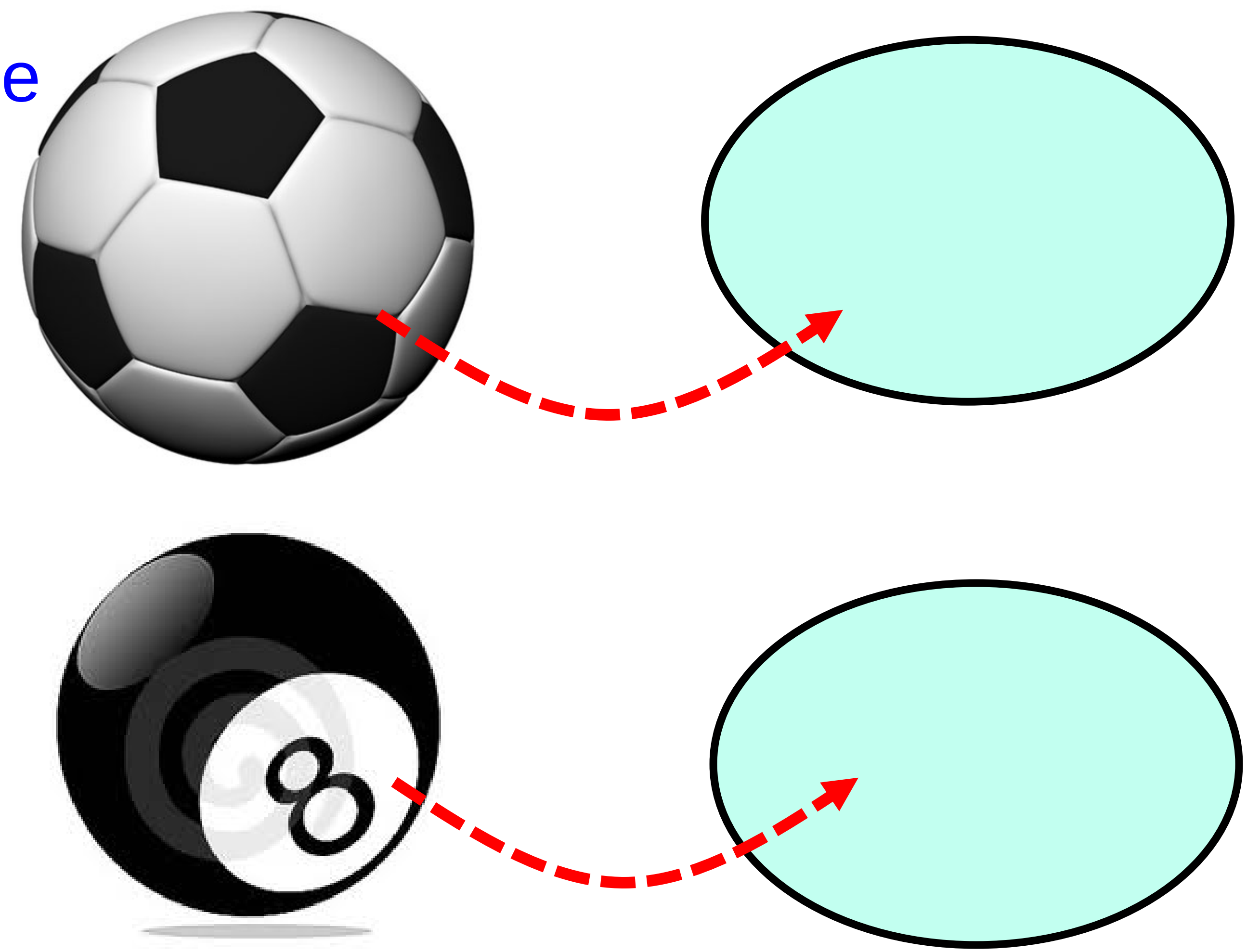


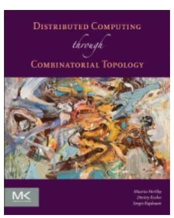


2-Connectivity

2-sphere

3-disk





n-connectivity

$\mathcal{C}$ is *n*-connected, if, for *m* $\leq$ *n*, every continuous map of the *m*-sphere

$$f : S^m \to \mathcal{C}$$

can be extended to a continuous map of the (*m*+*1*)-disk

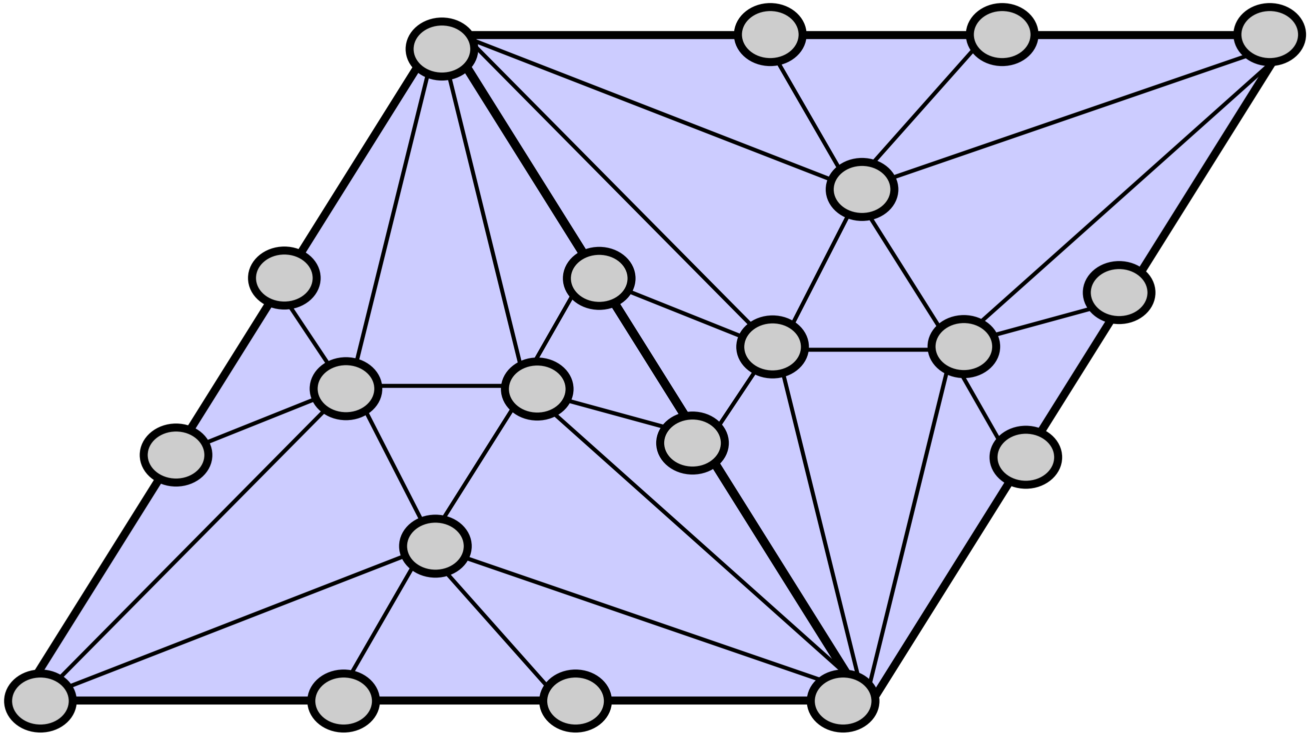
$$f : D^{m+1} \to \mathcal{C}$$

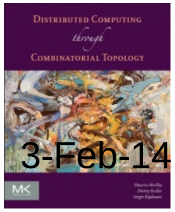
(-*1*)-connected is non-empty

Road Map

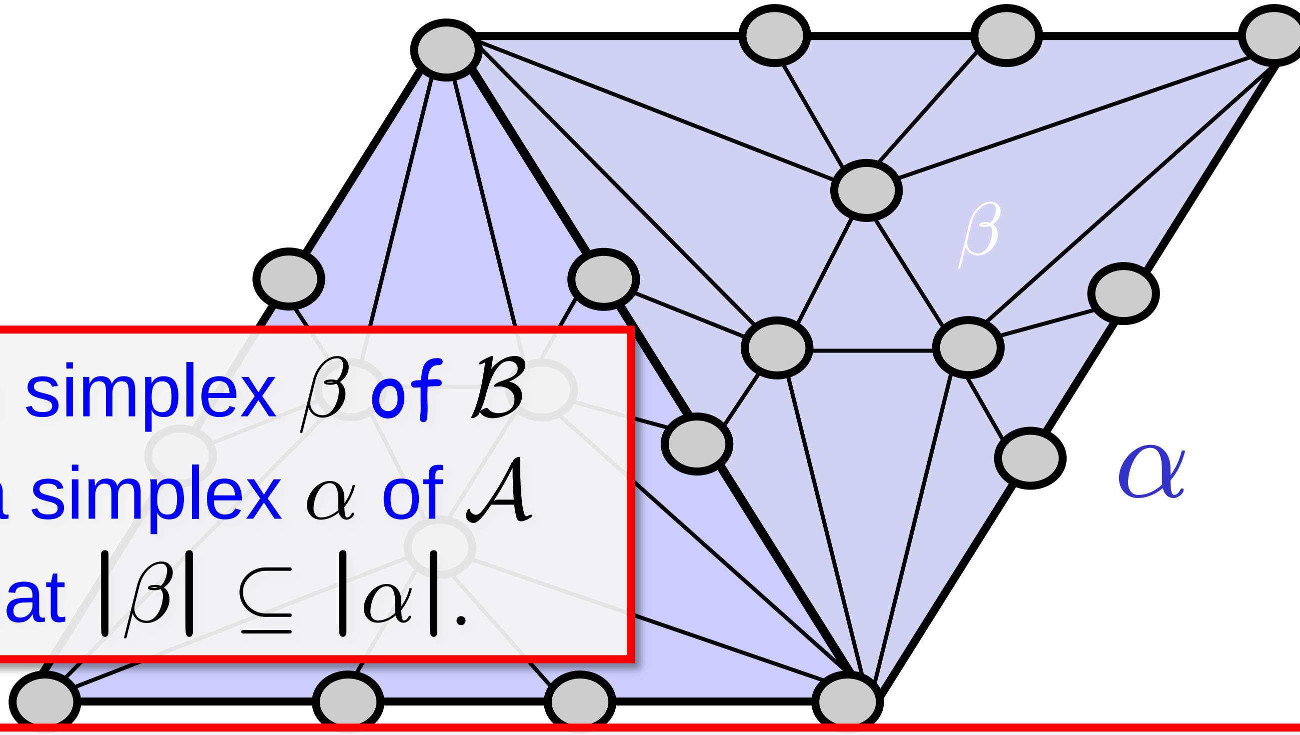
- Simplicial Complexes
- Standard Constructions
- Carrier Maps
- Connectivity
- **Subdivisions**
- Simplicial & Continuous Approximations

Subdivisions





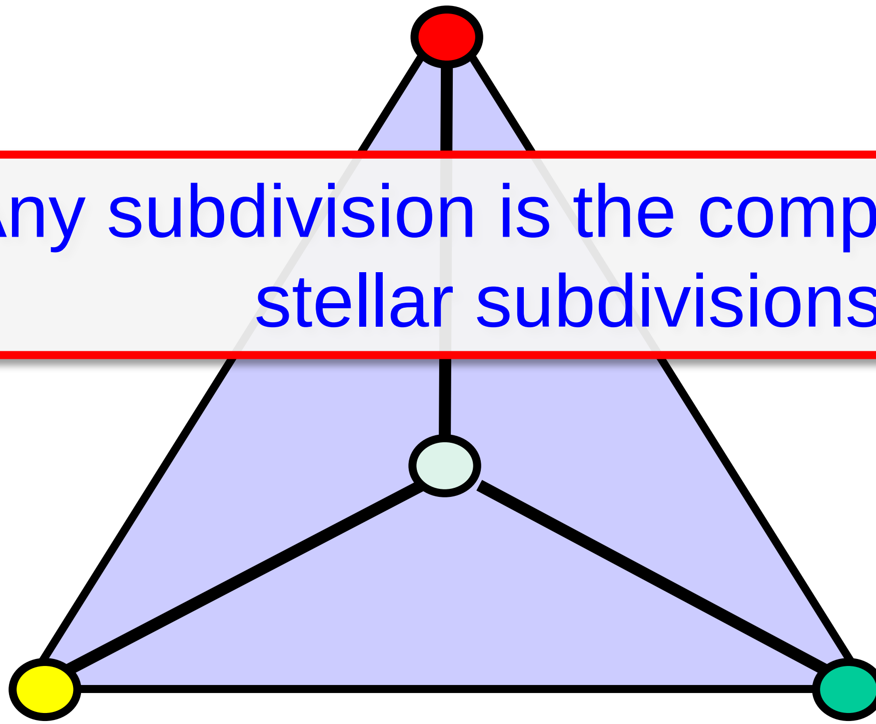
$\mathcal{B}$ is a subdivision of $\mathcal{A}$ if …



For each simplex β **of** $\mathcal{B}$ there is a simplex α of $\mathcal{A}$ such that $|\beta| \subseteq |\alpha|$.

For each simplex α of $\mathcal{A}$, $|\alpha|$ is the union of a finite set of geometric simplexes of $\mathcal{B}$.

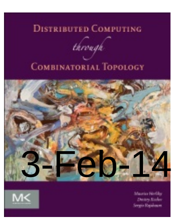
Stellar Subdivision



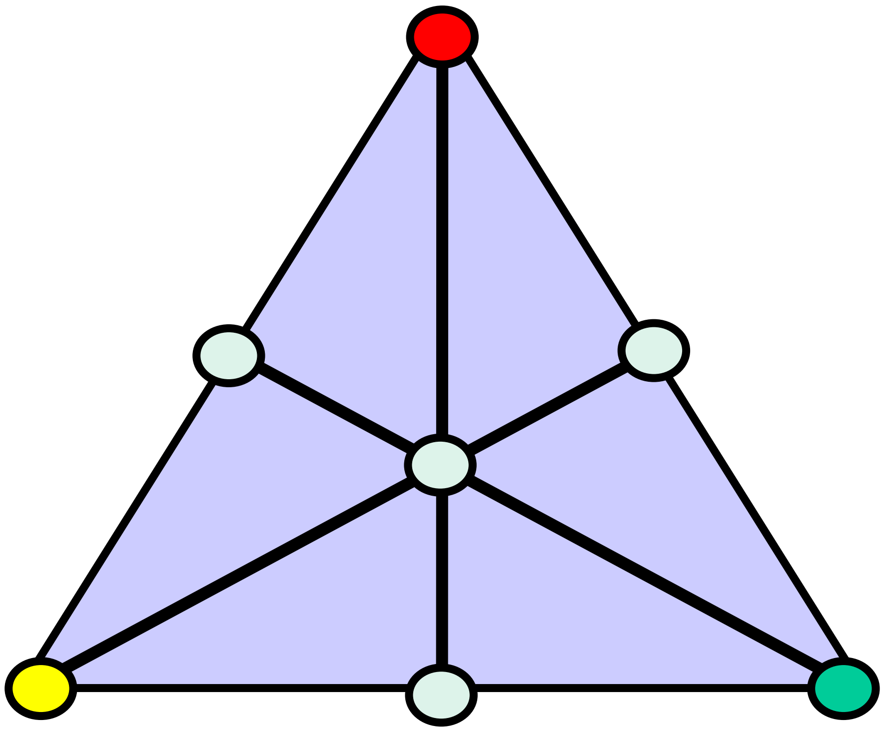
Any subdivision is the composition of stellar subdivisions

σ

Stel σ



Barycentric Subdivision



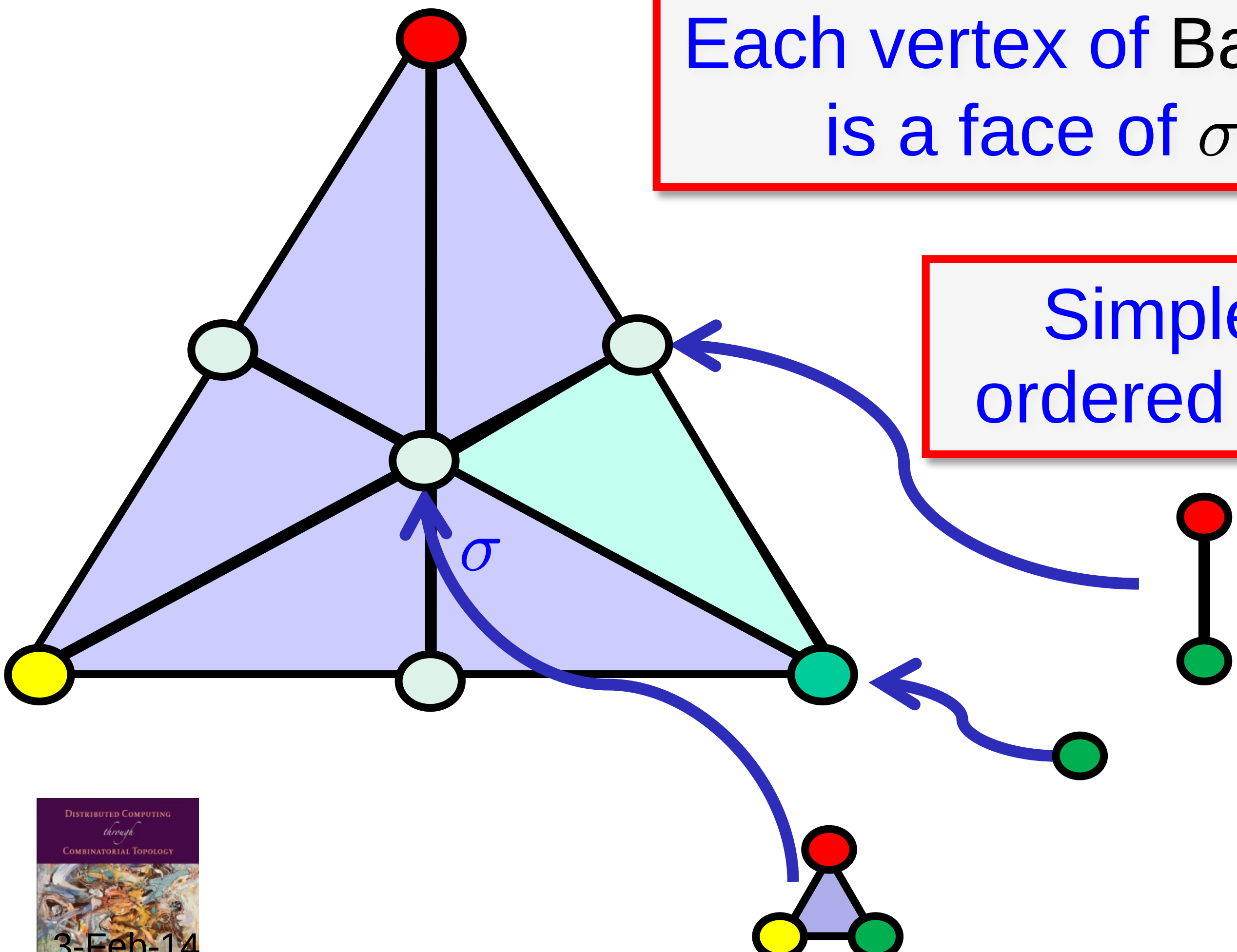
σ

Bary σ

Barycentric Subdivision

Each vertex of Bary σ is a face of σ

Simplex = faces ordered by inclusion

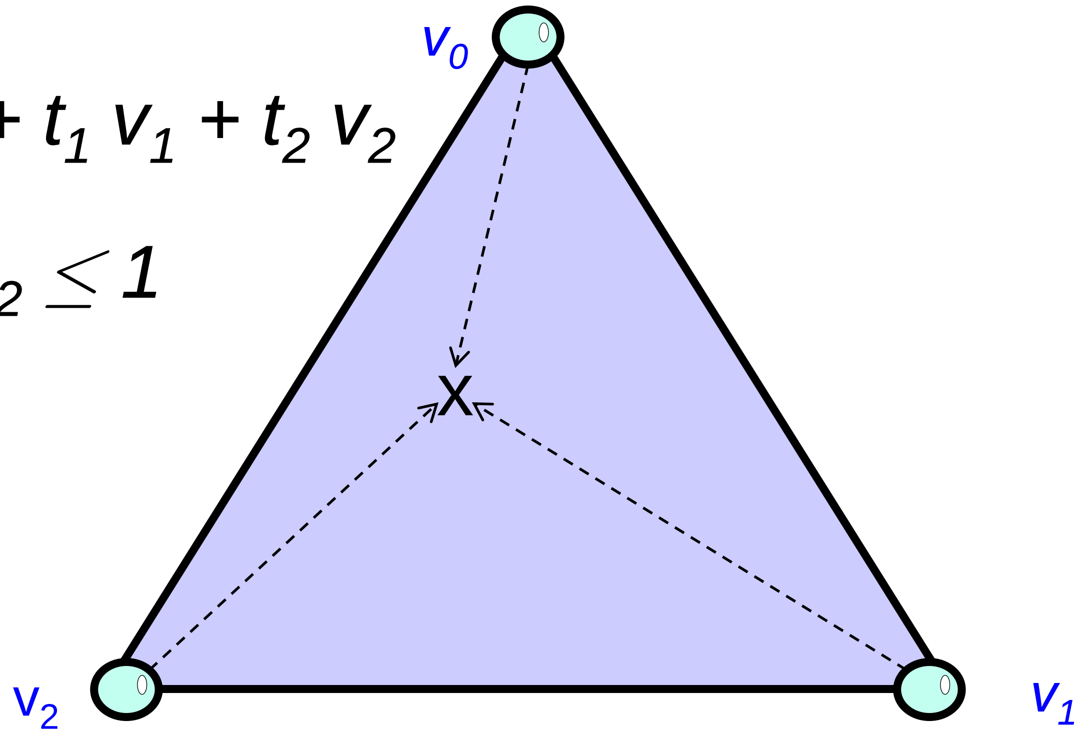


Barycentric Coordinates

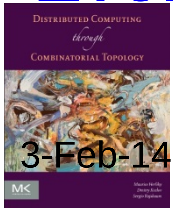
$x = t_0 v_0 + t_1 v_1 + t_2 v_2$

$0 \leq t_0, t_1, t_2 \leq 1$

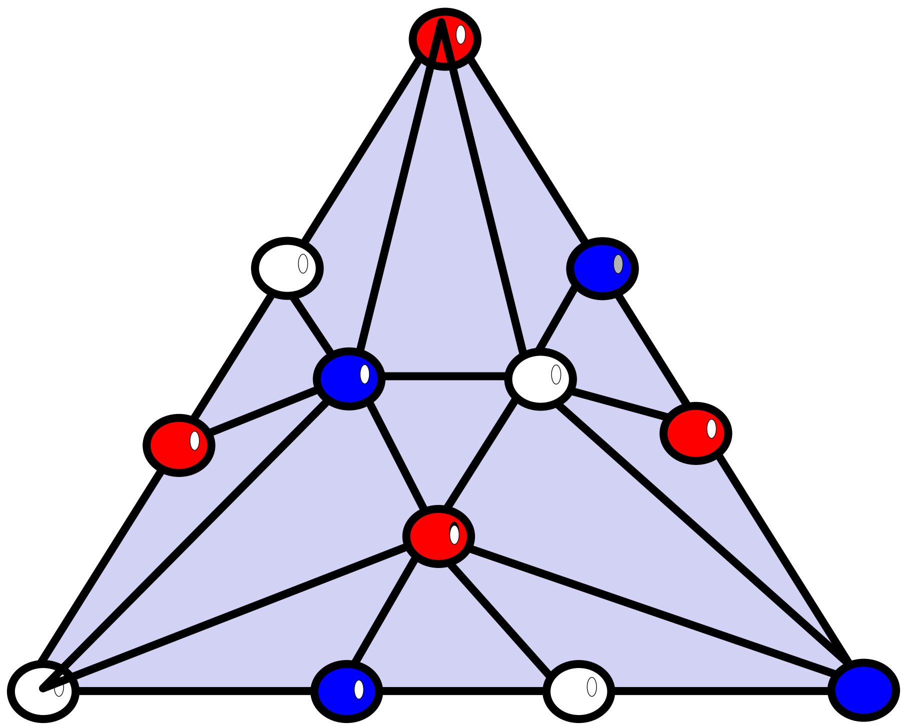
$\sum t_i = 1$



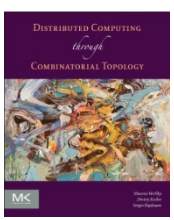
Every point of $|\mathcal{C}|$ has a unique representation using barycentric coordinates



Standard Chromatic Subdivision



$\mathrm{Ch}\ \sigma$



Chromatic form of Barycentric

Road Map

- Simplicial Complexes
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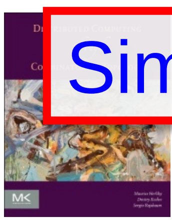
- Connectivity
- Subdivisions
- **Simplicial & Continuous Approximations**



From Simplicial to Continuous

simplicial

$$\phi : \mathcal{A} \to \mathcal{B}$$

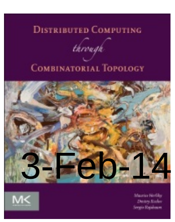
$$f(x) = \sum_i t_i \cdot |\phi(s_i)|$$

continuous

$$f : |\mathcal{A}| \to |\mathcal{B}|$$

extend over barycentric coordinates

(piece-wise linear map)



Maps

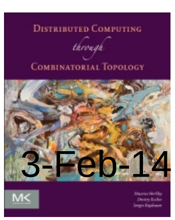
simplicial

$$\phi : \mathcal{A} \to \mathcal{B}$$

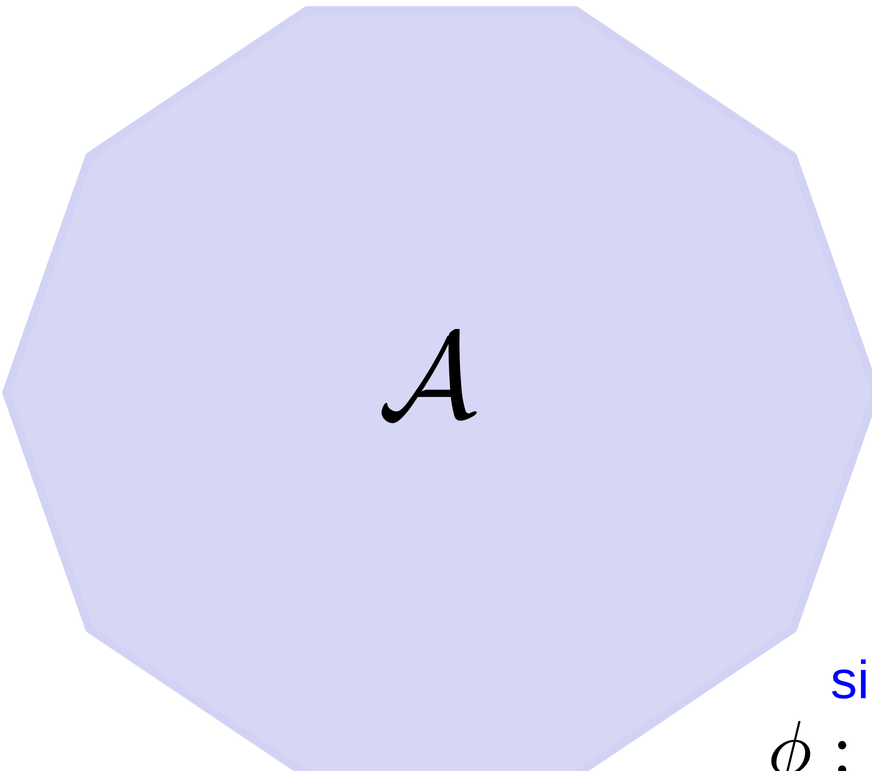
Simplicial Approximation Theorem

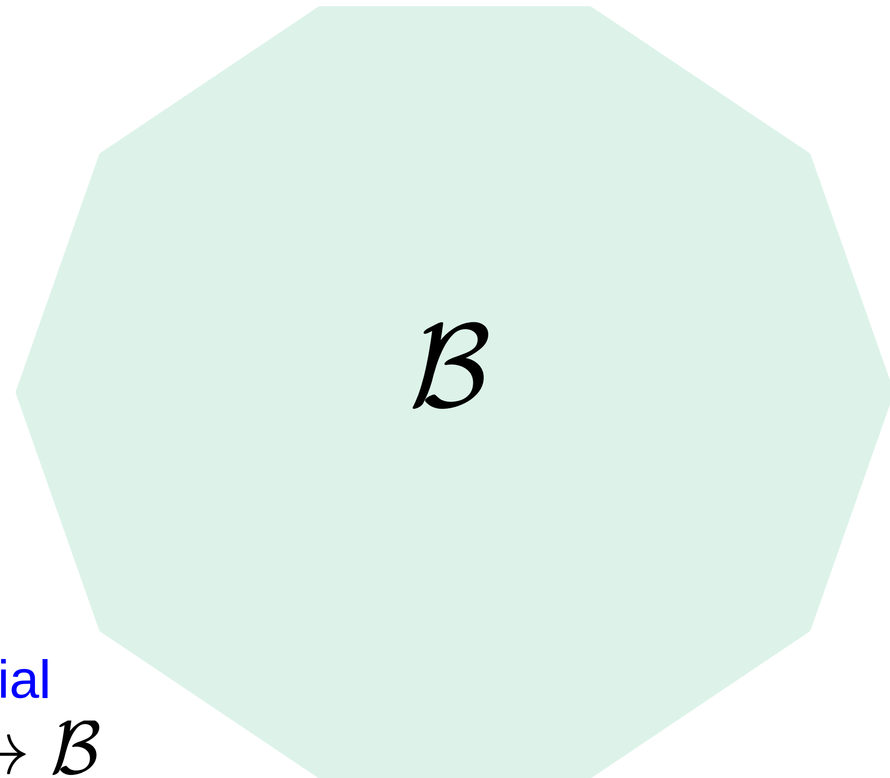
continuous

$$f : |\mathcal{A}| \to |\mathcal{B}|$$



Simplicial Approximation



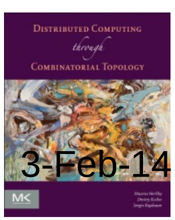


simplicial

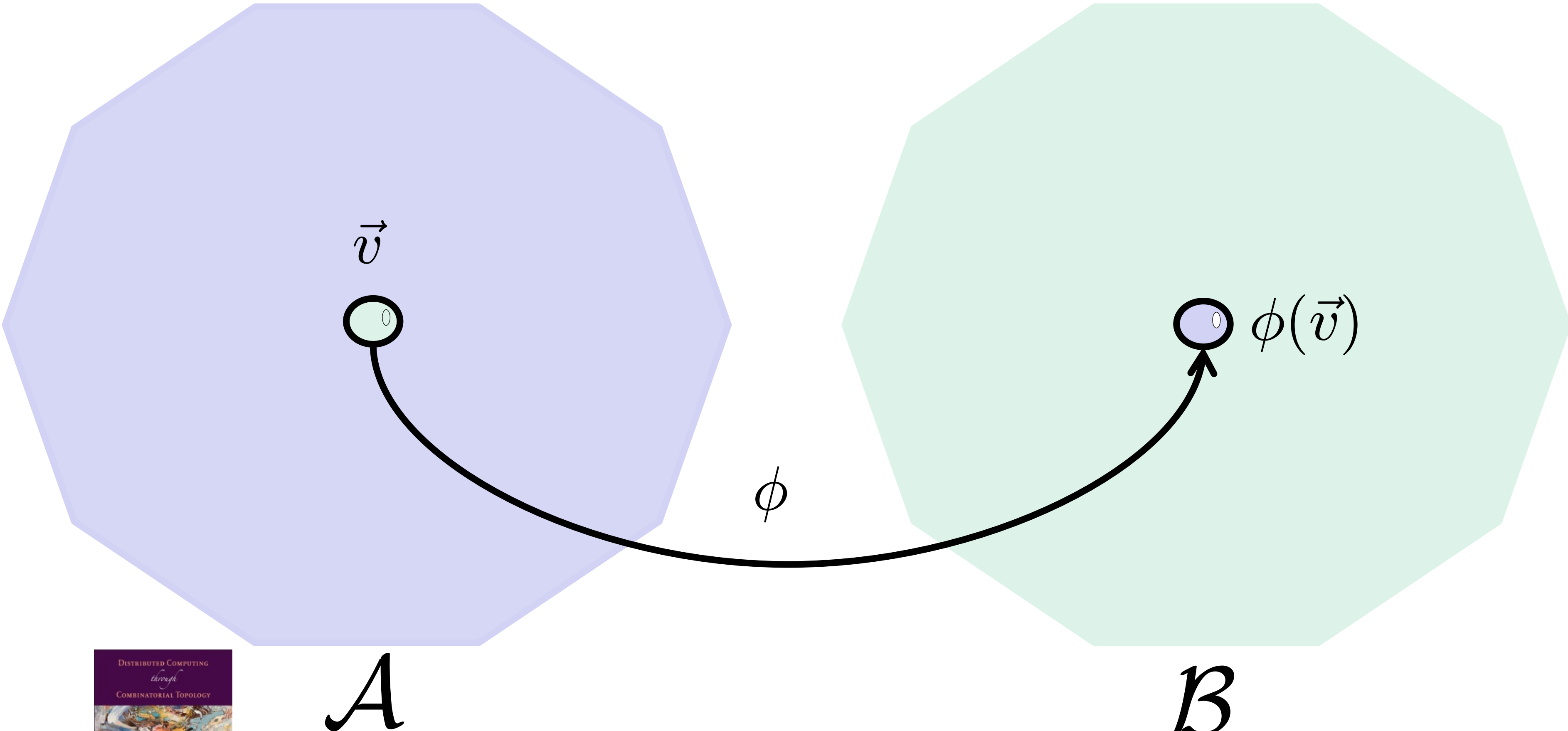
$$\phi : \mathcal{A} \to \mathcal{B}$$

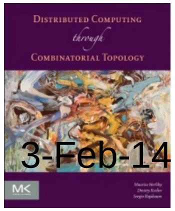
continuous

$$f : |\mathcal{A}| \to |\mathcal{B}|$$

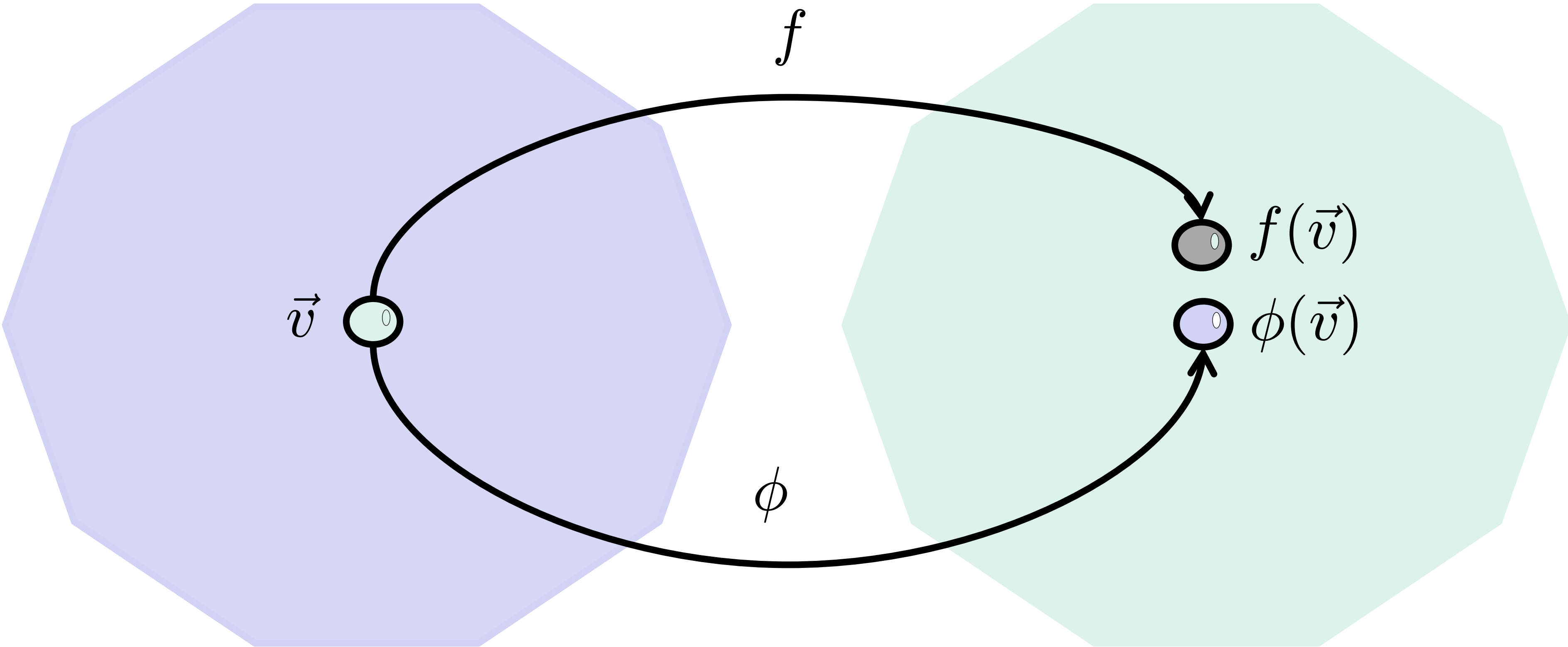


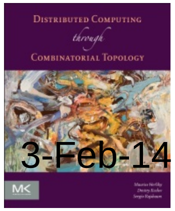
Simplicial Approximation



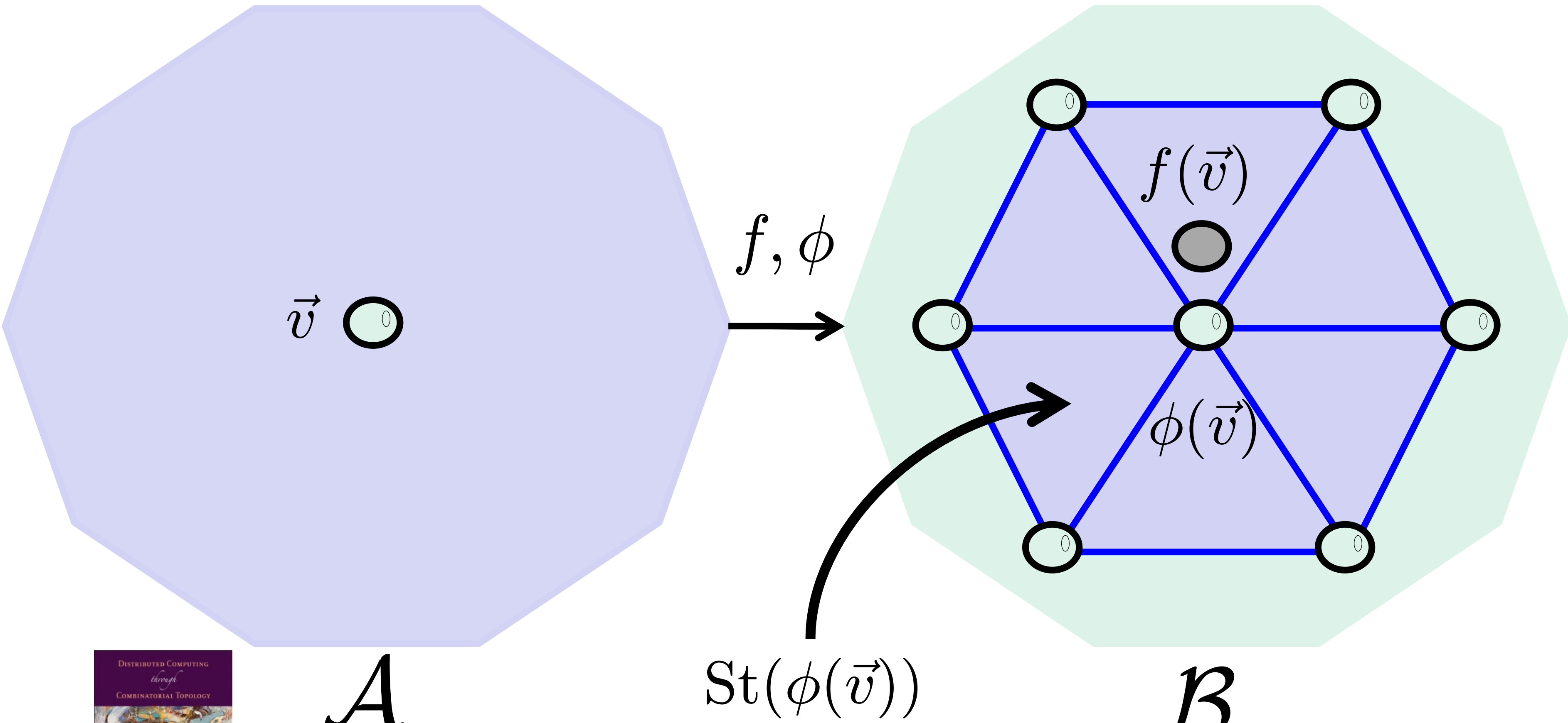


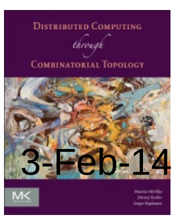
Simplicial Approximation



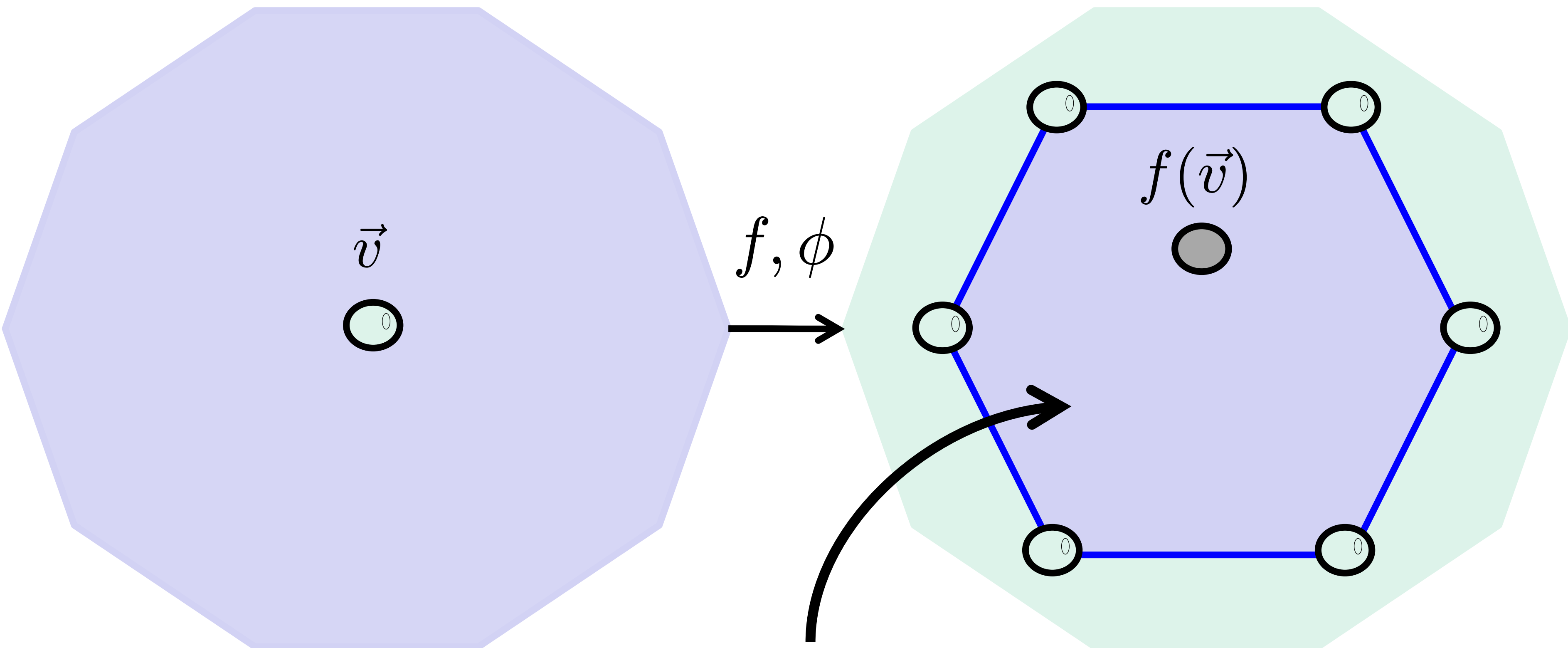


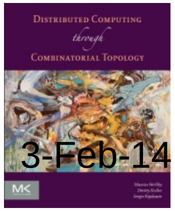
Simplicial Approximation



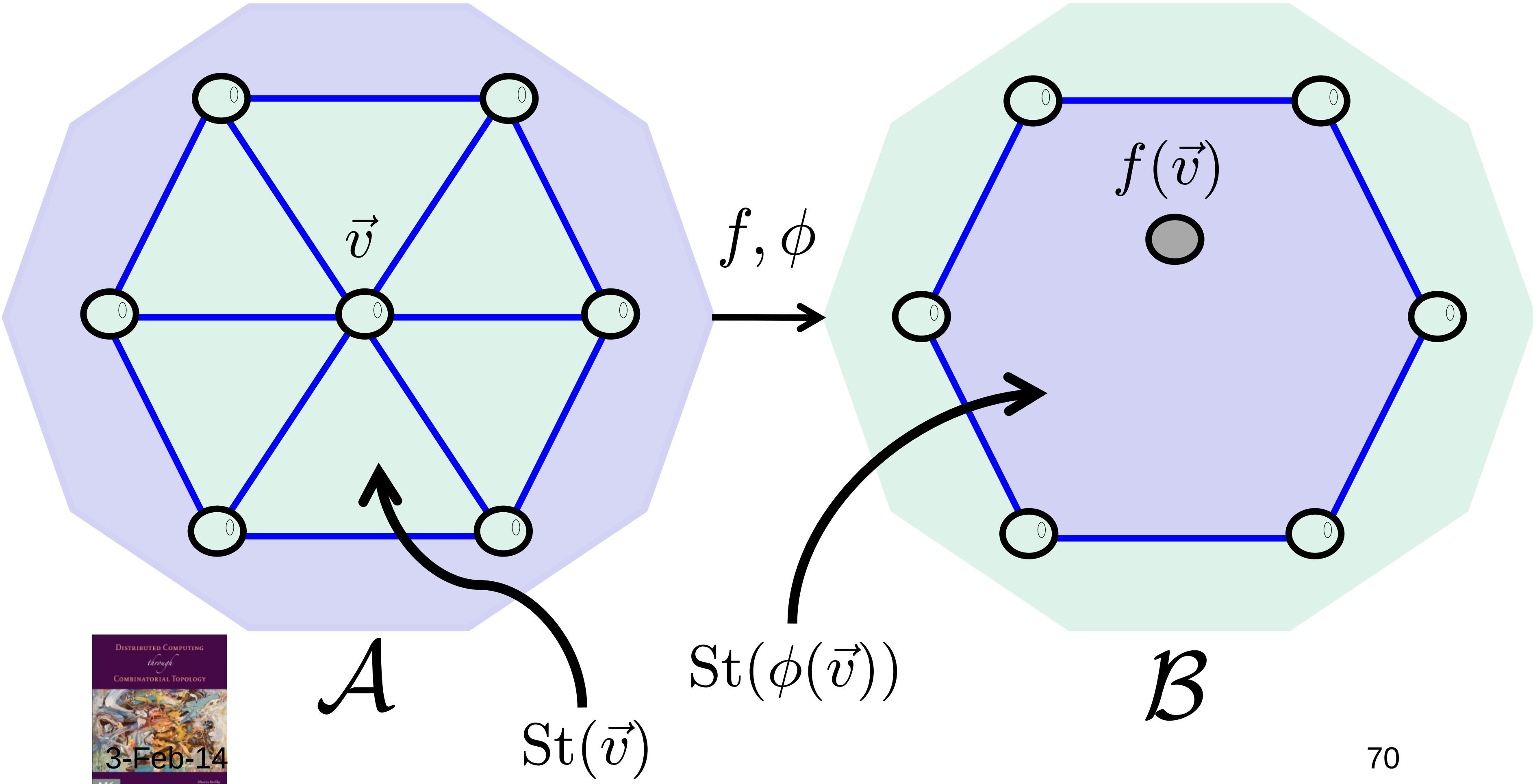


Simplicial Approximation

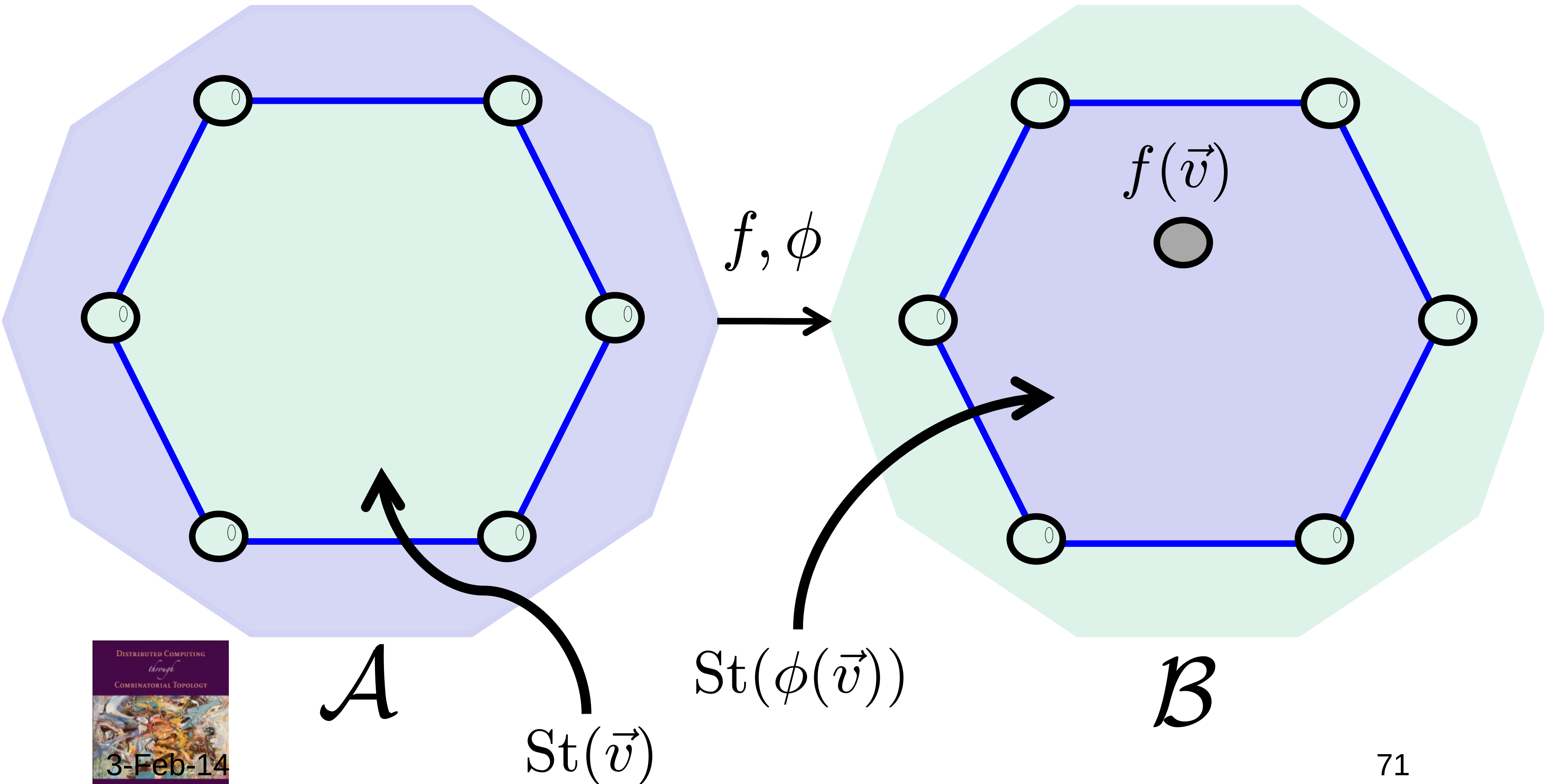




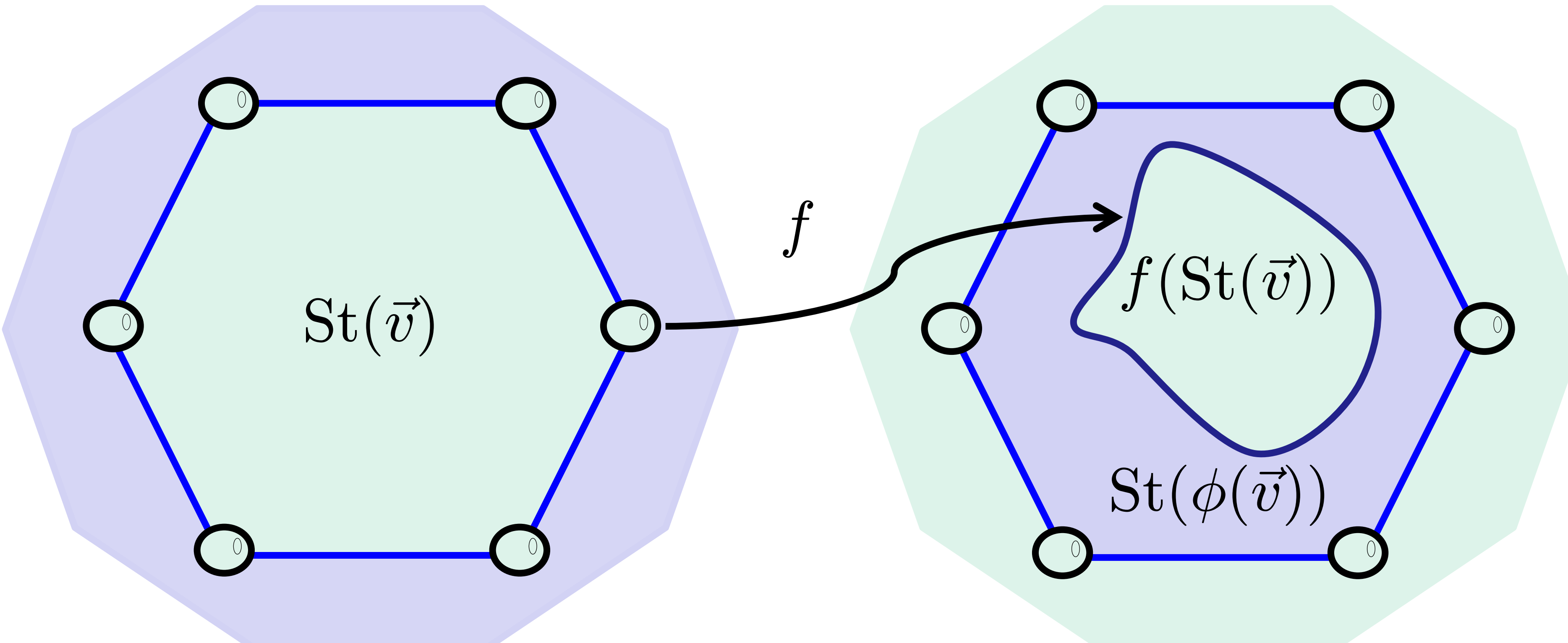
Simplicial Approximation

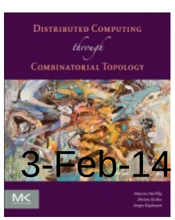


Simplicial Approximation



Simplicial Approximation



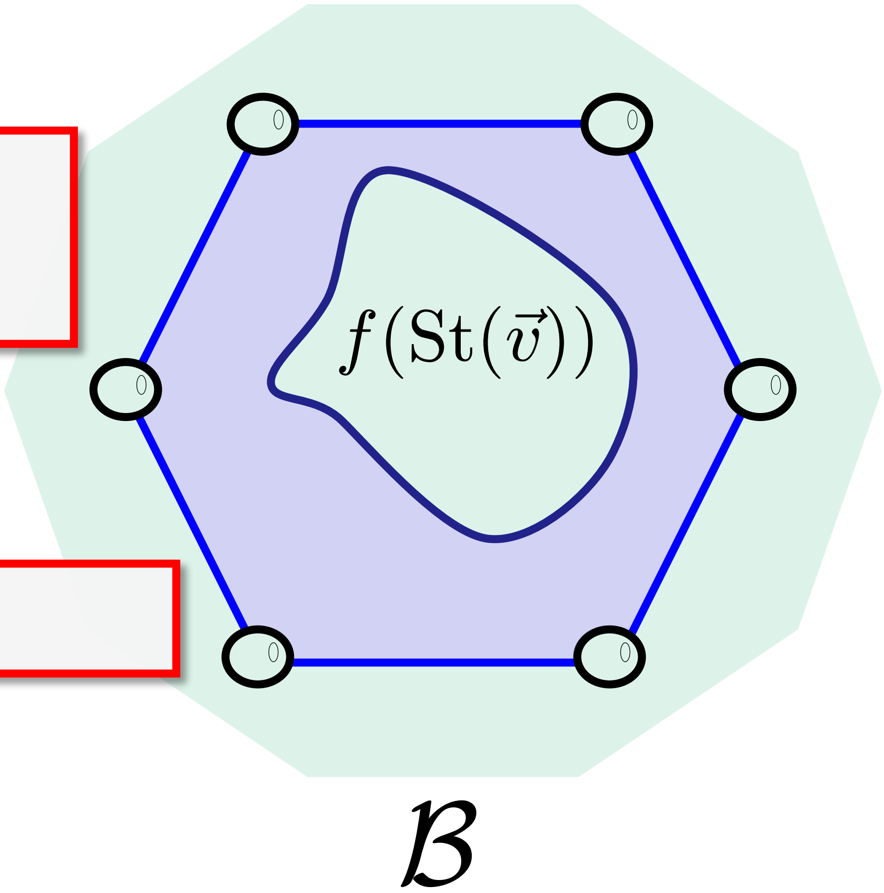


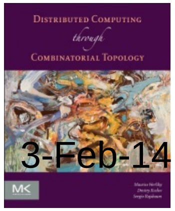
Simplicial Approximation

ϕ is a simplicial approximation of f if …

for every v in $\mathcal{A}$ …

$$f(\mathrm{St}(\vec{v})) \subseteq \mathrm{St}(\phi(\vec{v}))$$





Simplicial Approximation Theorem

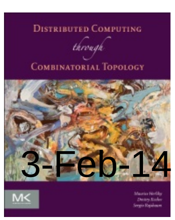
- Given a continuous map

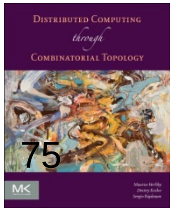
$$f : |\mathcal{A}| \to |\mathcal{B}|$$

- there is an N such that f has a simplicial approximation

$$\phi : \boxed{\mathrm{Bary}^N} \mathcal{A} \to \mathcal{B}$$

Actually Holds for most other subdivisions….

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