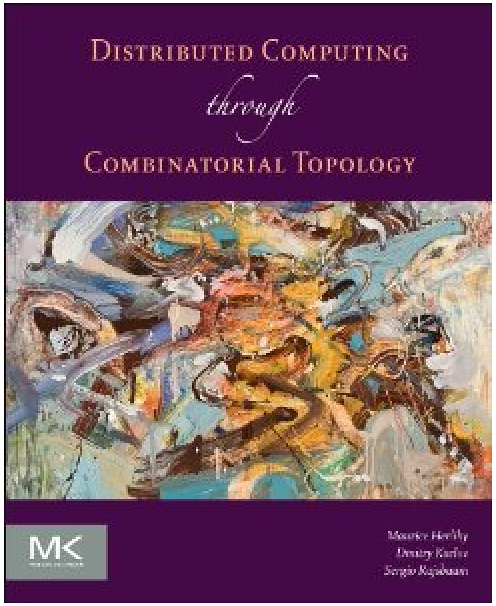
Manifold Protocols



Companion slides for

Distributed Computing

Through Combinatorial Topology

Maurice Herlihy & Dmitry Kozlov & Sergio Rajsbaum

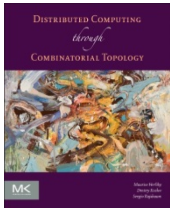
Kinds of Results

Impossibility

Task T cannot be solved in model M

Separation

Task T can be solved by a protocol for T', but not vice-versa



Road Map

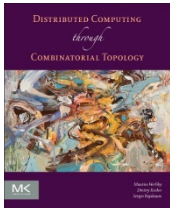
Manifolds

Immediate Snapshot Model

Sperner's Lemma and *k*-Set Agreement

Weak Symmetry-Breaking

Separation results



Road Map

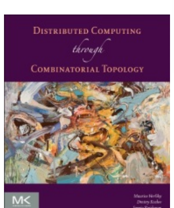
Manifolds

Immediate Snapshot Model

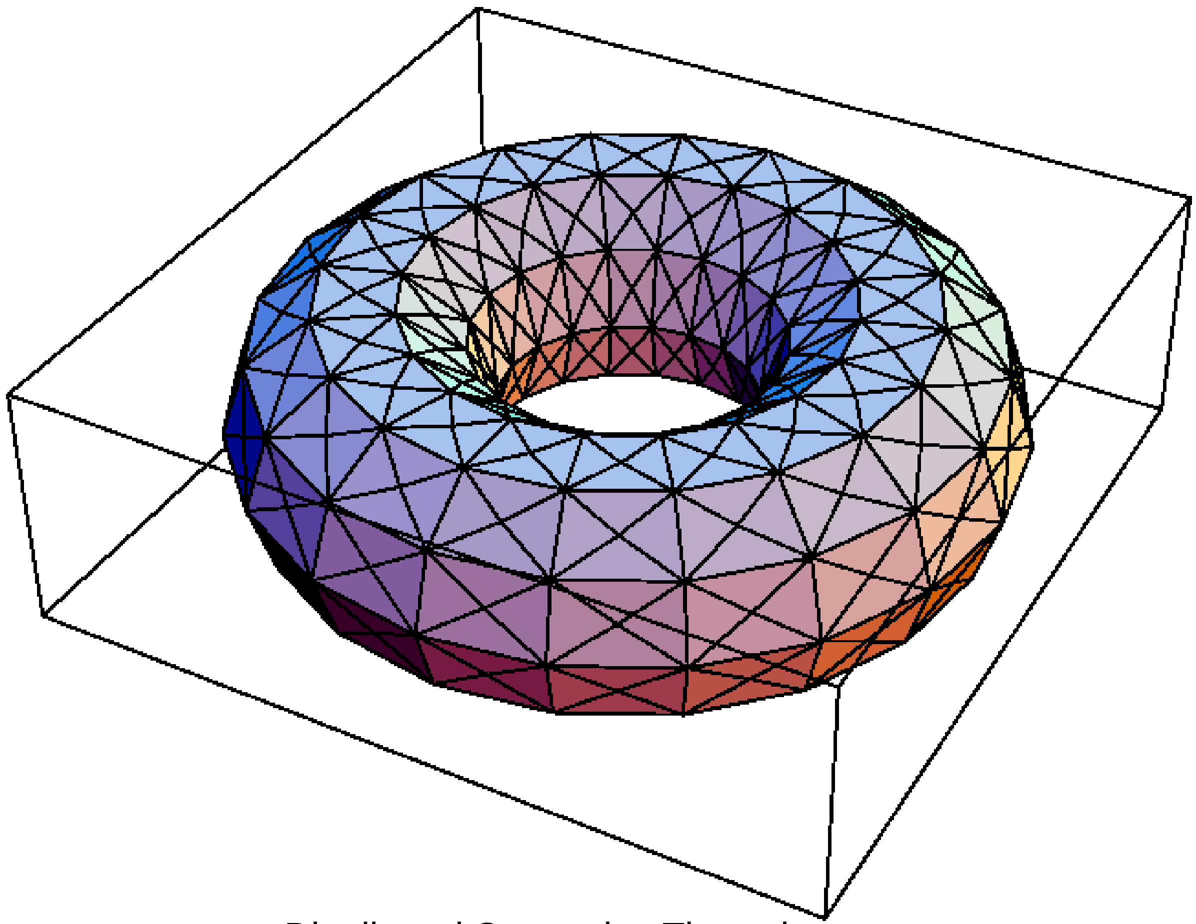
Sperner's Lemma and *k*-Set Agreement

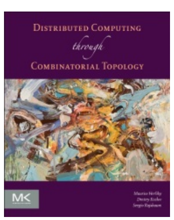
Weak Symmetry-Breaking

Separation results



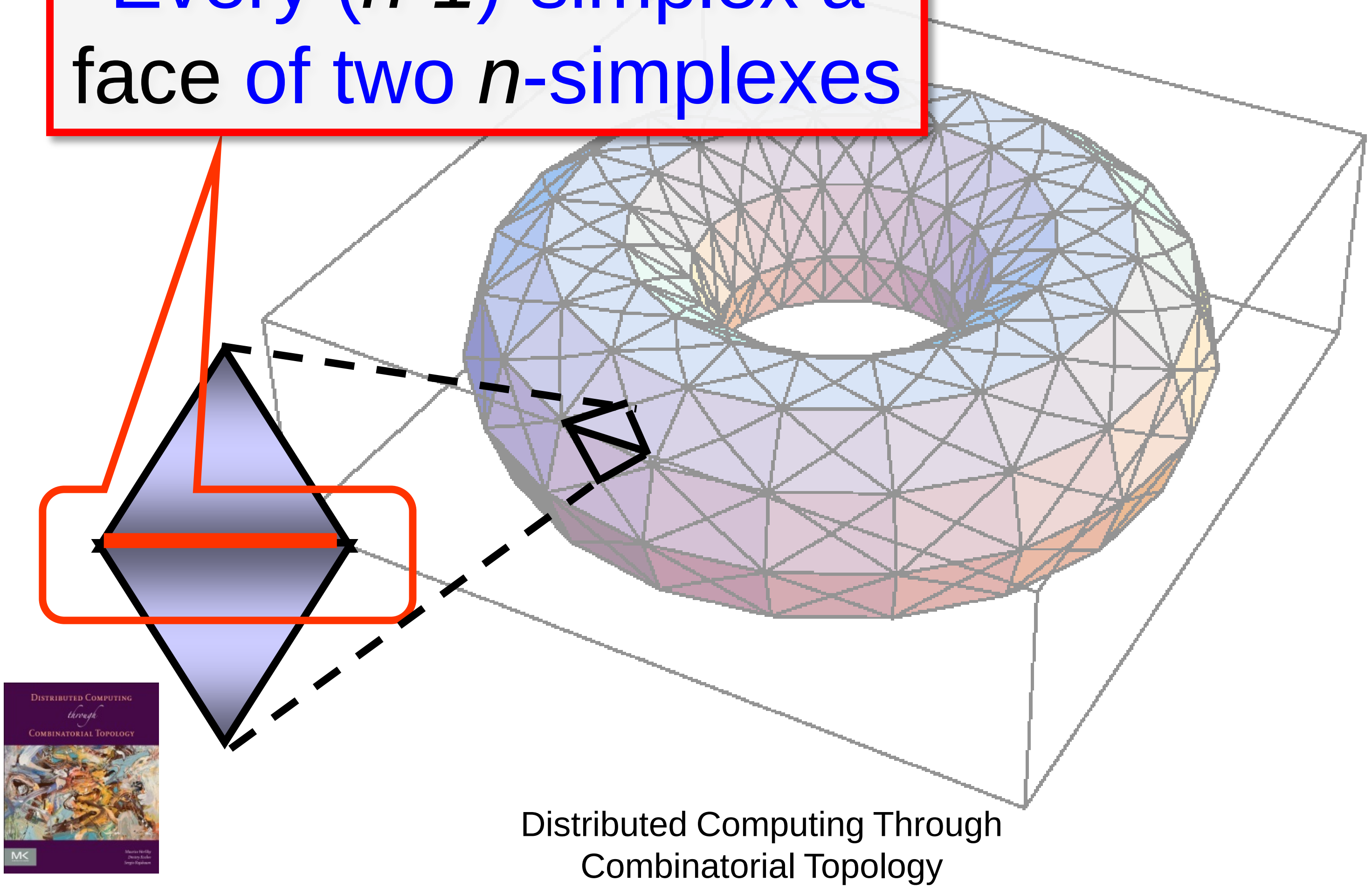
Simplicial Complex



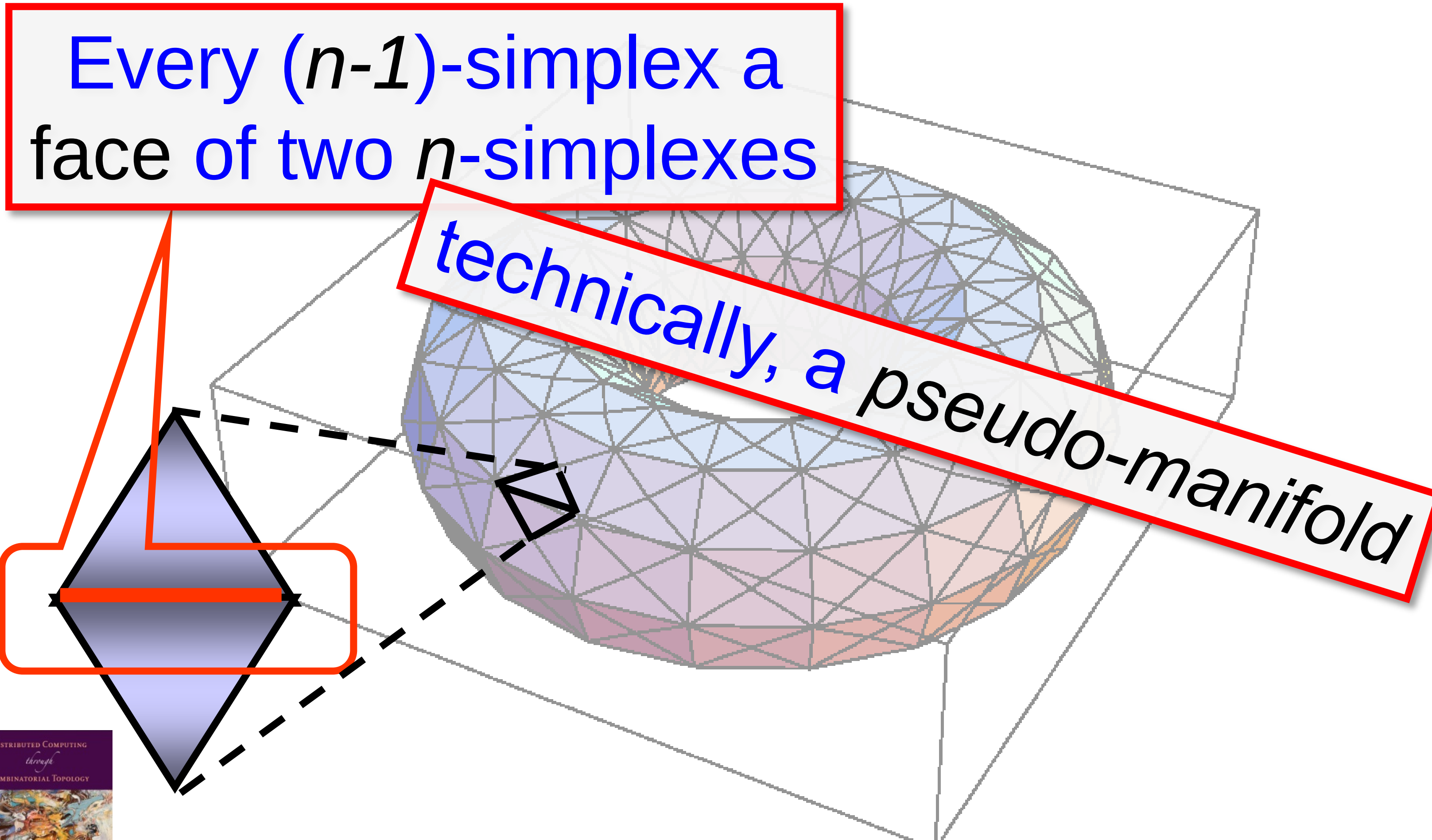


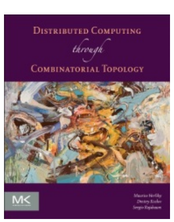
Manifolds

Every (*n-1*)-simplex a face of two *n*-simplexes



Manifolds

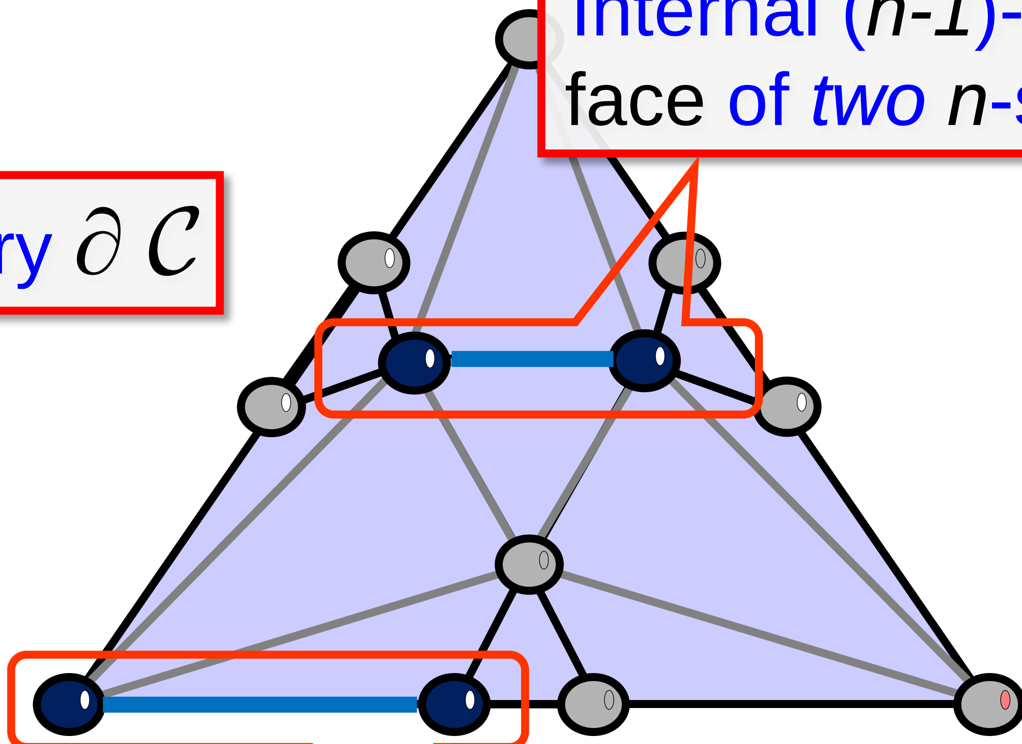

Every (*n-1*)-simplex a face of two *n*-simplexes
technically, a pseudo-manifold


Distributed Computing
through
Combinatorial Topology
MK

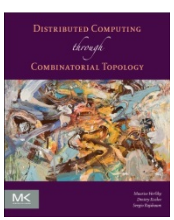
Manifold with Boundary

Internal (n-1)-simplex a face of *two* n-simplexes

Boundary $\partial\,\mathcal{C}$



Boundary (n-1)-simplex a face of *one* n-simplex



Why Manifolds?

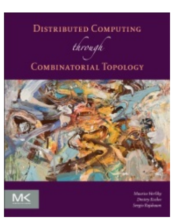
Nice combinatorial properties

Many useful theorems

Easy to prove certain claims

True, most complexes are not manifolds ….

Still a good place to start.



Road Map

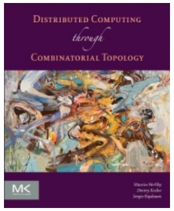
Manifolds

Immediate Snapshot Model

Sperner's Lemma and *k*-Set Agreement

Weak Symmetry-Breaking

Separation results



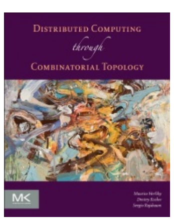
Immediate Snapshot Executions

Restricted form of Read-Write memory

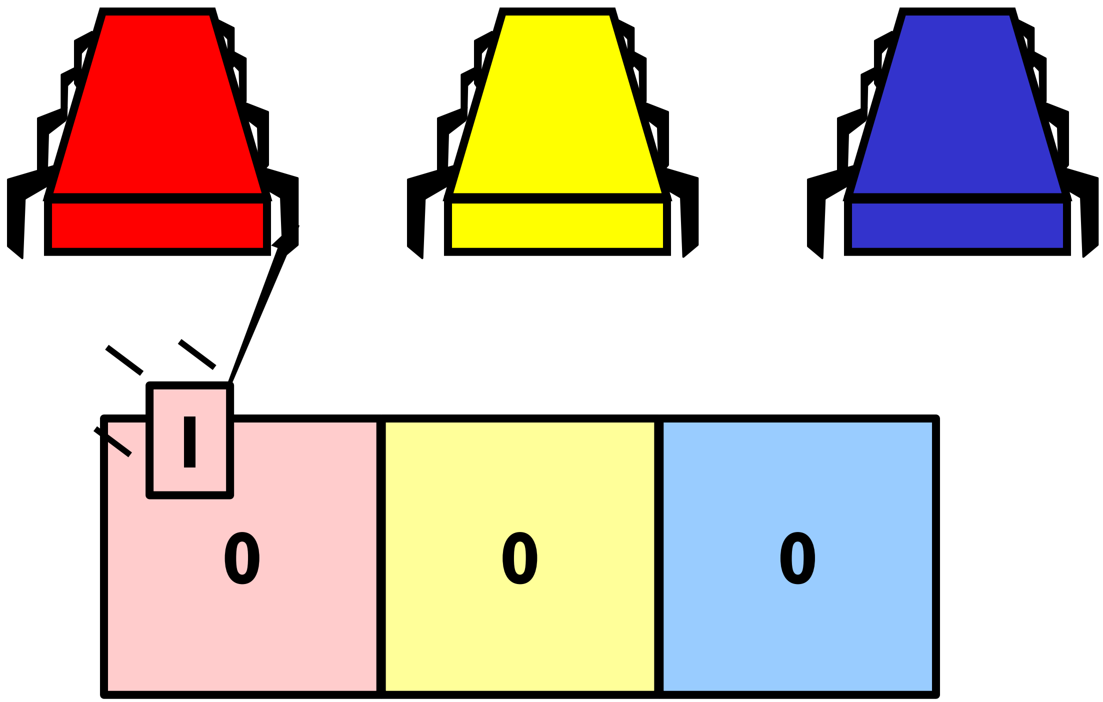
Protocol complexes are manifolds

(Equivalent to regular R-W memory)

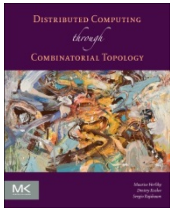
But we will not prove it yet.



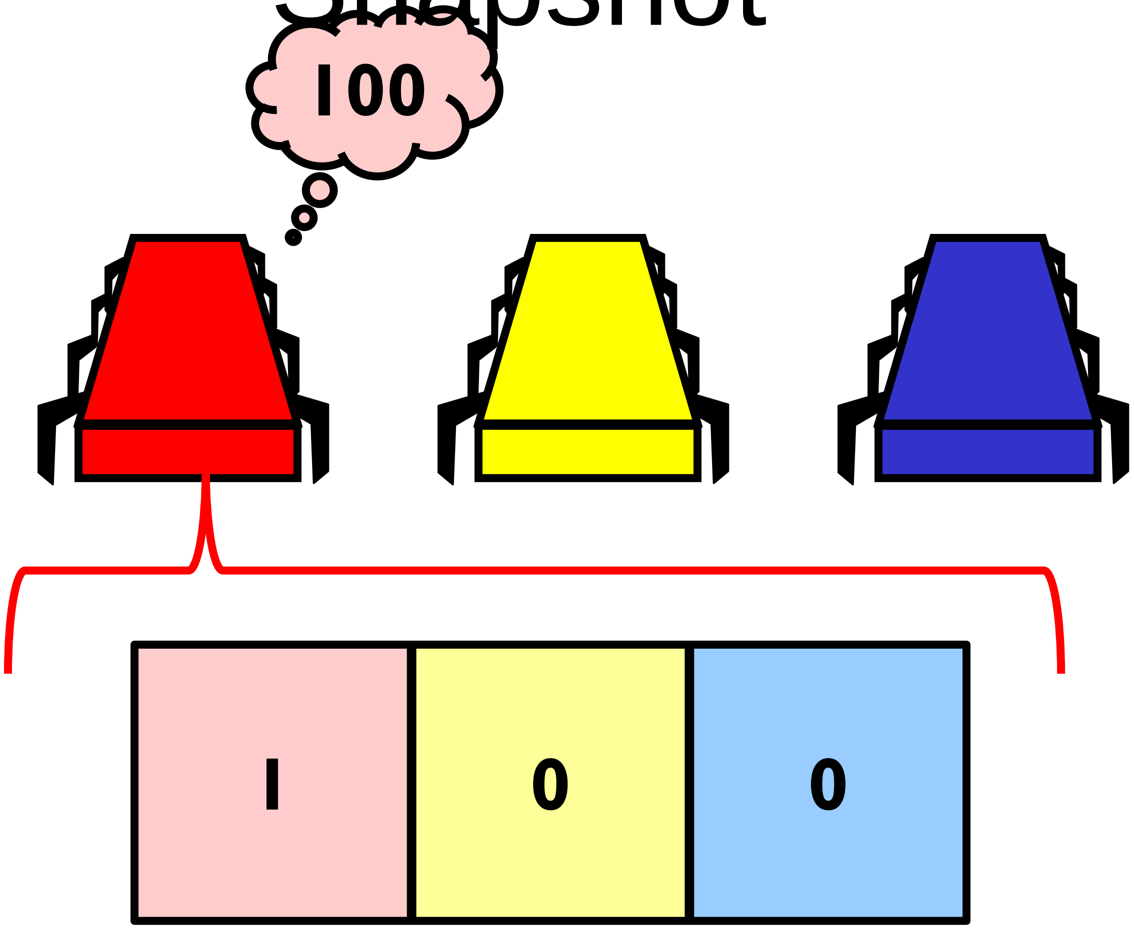
Write



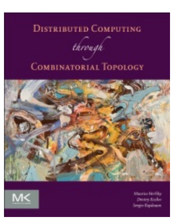
Single-writer, multi-reader variables



Snapshot



Single-writer, multi-reader variables



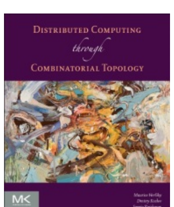
Immediate Snapshot Executions

Pick a set of processes

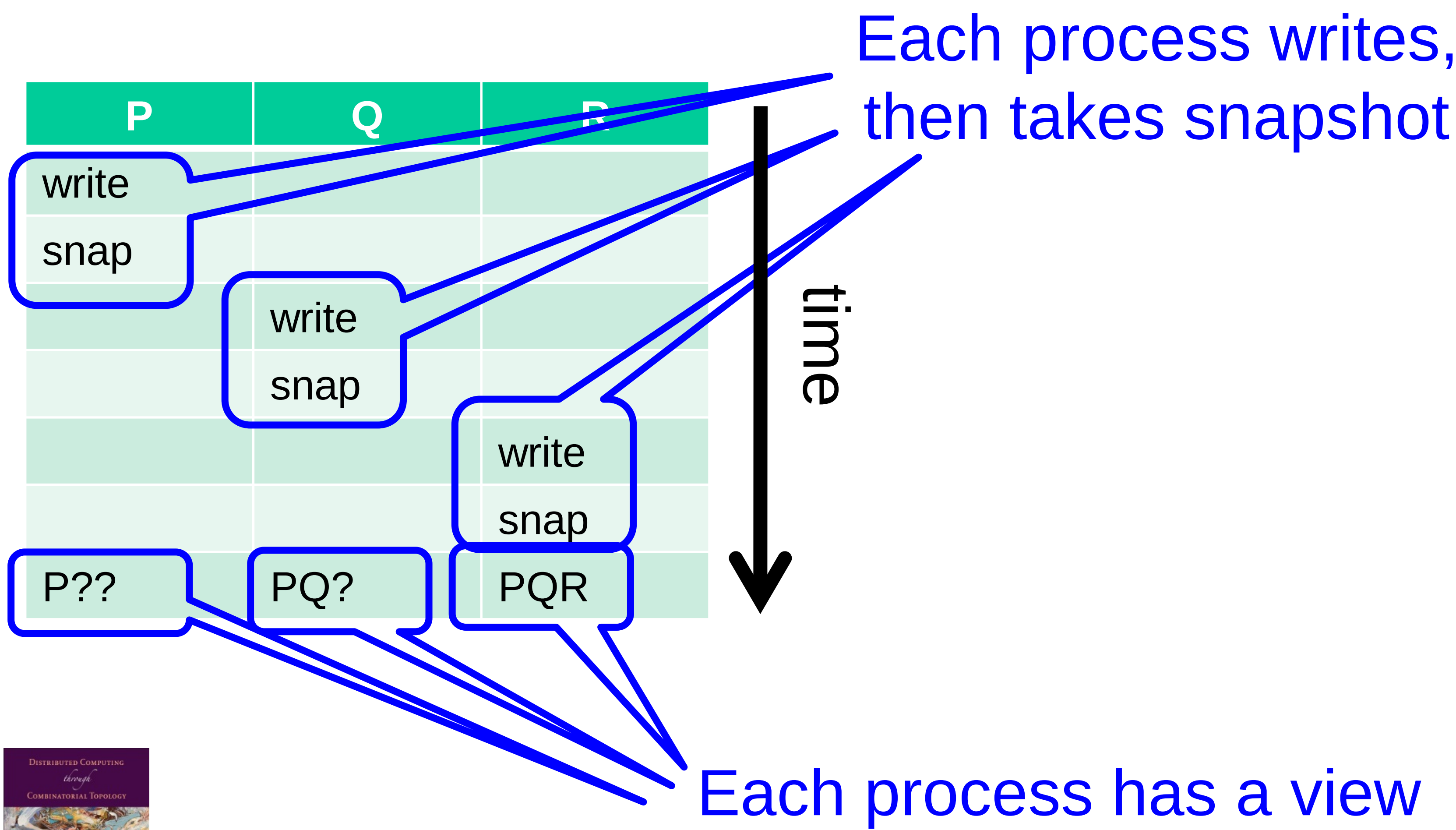
write together

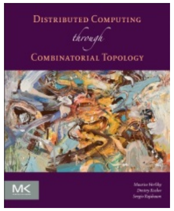
snapshot together

Repeat with another set



Example Executions



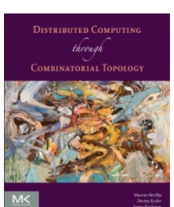


Example Executions

P	Q	R
write		
snap		
	write	
	snap	
		write
		snap
P??	PQ?	PQR

Moving last process one round earlier,

Changes this view from PQ? to PQR

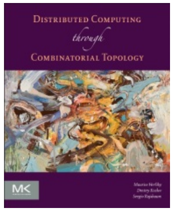


Example Executions

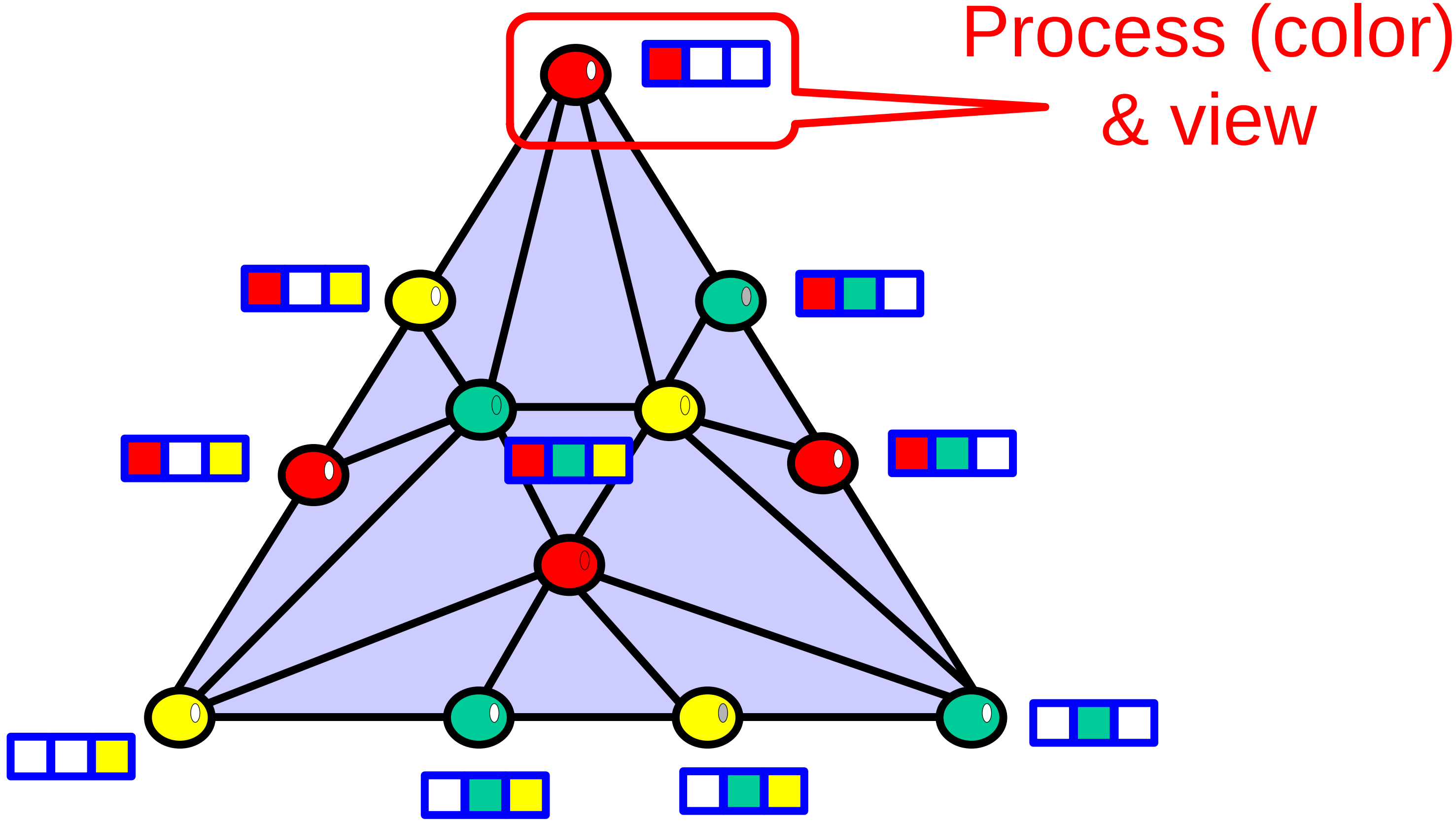
P	Q	R
write		
snap		
	write	write
	snap	snap
		write
		snap
P??	PQR	PQR

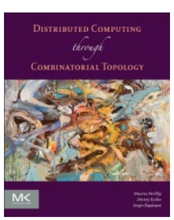
Moving last processes one round earlier,

Changes this view from P?? to PQR



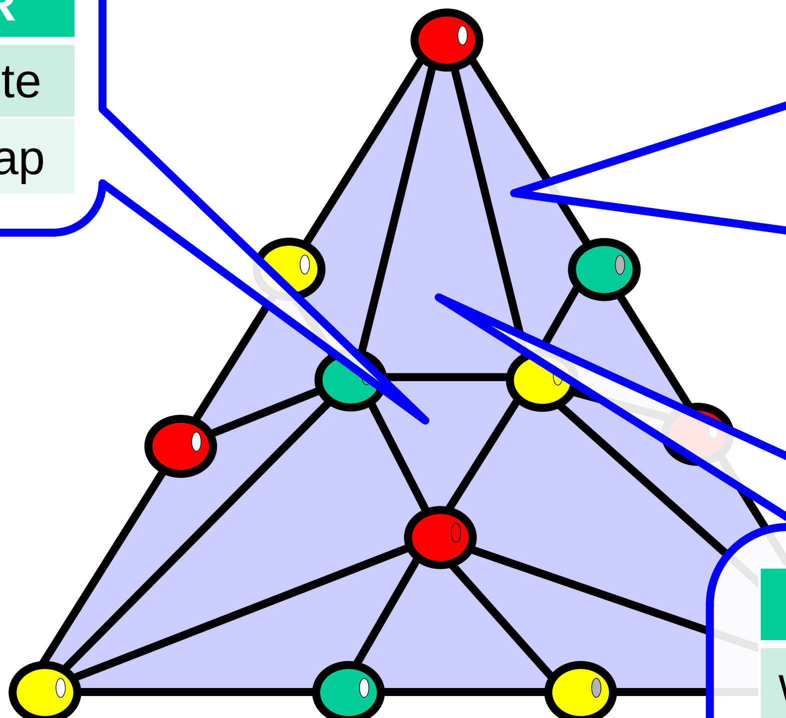
Protocol Complex





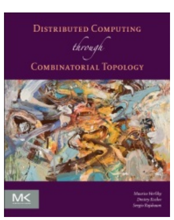
Protocol Comple

P	Q	R
write	write	write
snap	snap	snap

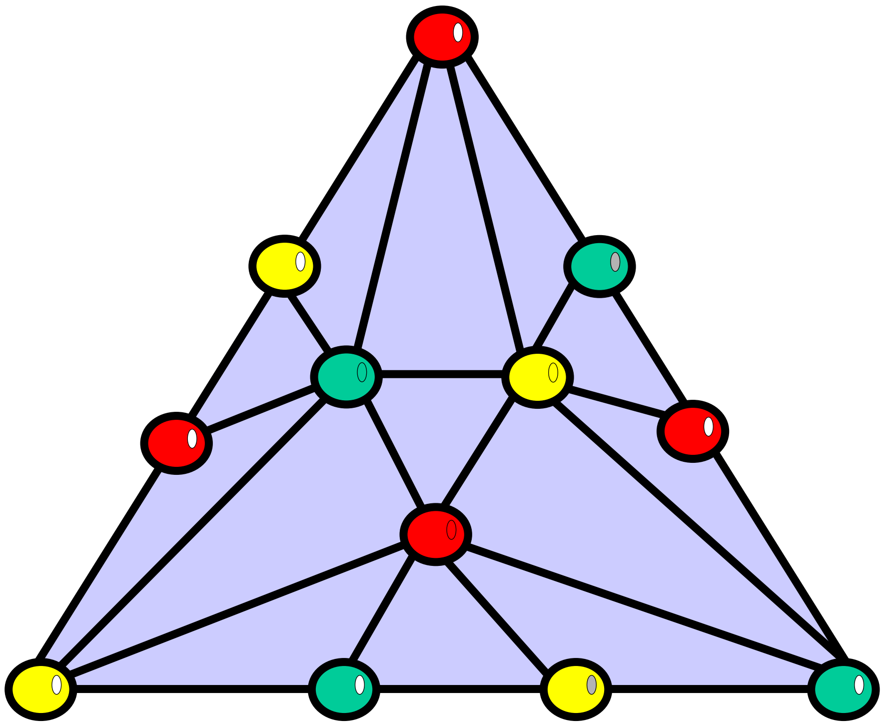


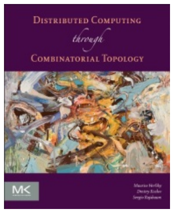
P	Q	R
write		
snap		
	write	
	snap	
		write
		snap

P	Q	R
write		
snap		
	write	write
	snap	snap



Standard Chromatic Subdivision





Combinatorial Definition (I)

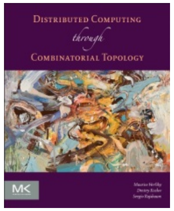
Input simplex σ

Protocol complex $\mathcal{IS}(\sigma)$

Vertex (P_i, σ_i)

Process name

view $\sigma_i \subseteq \sigma$



Combinatorial Definition (II)

Each process's write appears in its own view

$P_i \in \text{names}(\sigma_i)$.

Snapshots are ordered

$\sigma_i \subseteq \sigma_j$ or vice-versa

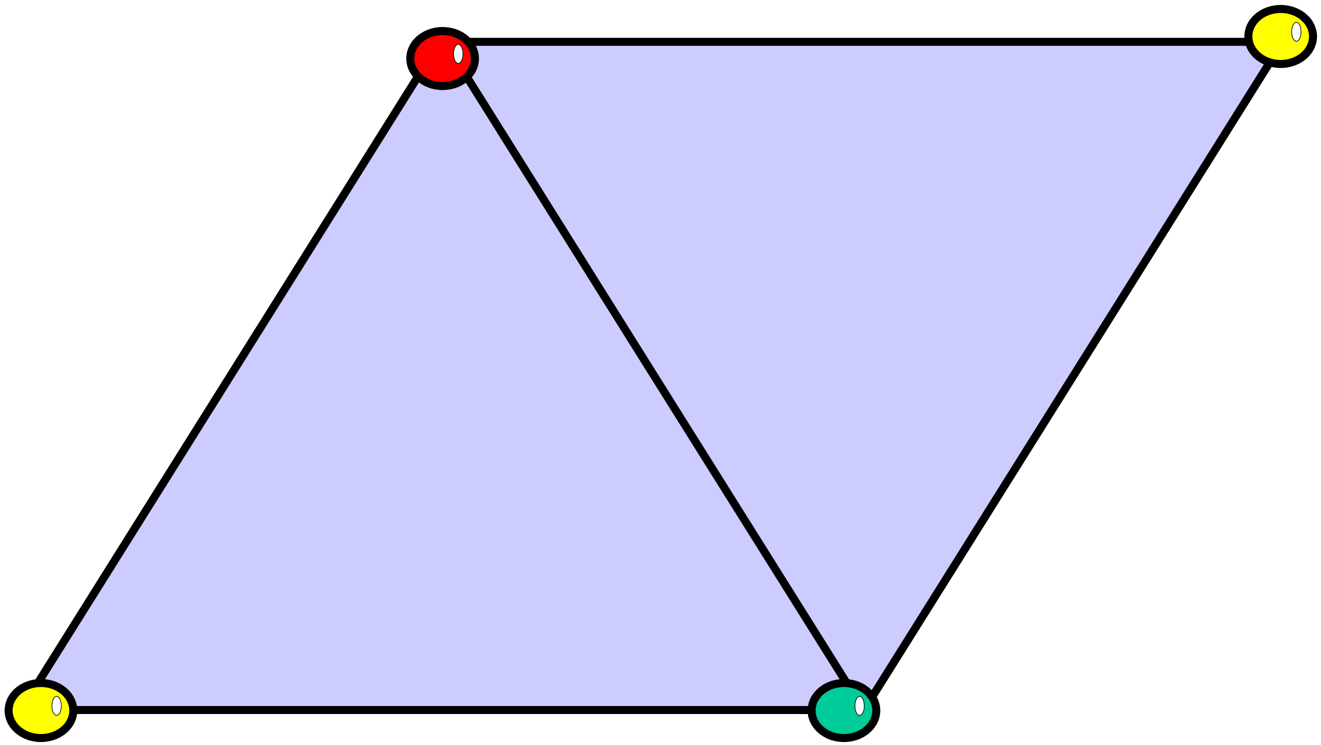
Each snapshot ordered immediately after write

if $P_i \in \text{ids}(\sigma_j)$, then $\sigma_i \subseteq \sigma_j$.

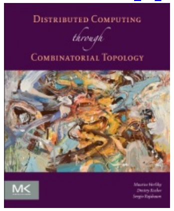
If P_j saw P_i write …

then P_j's snapshot not later than P_j's.

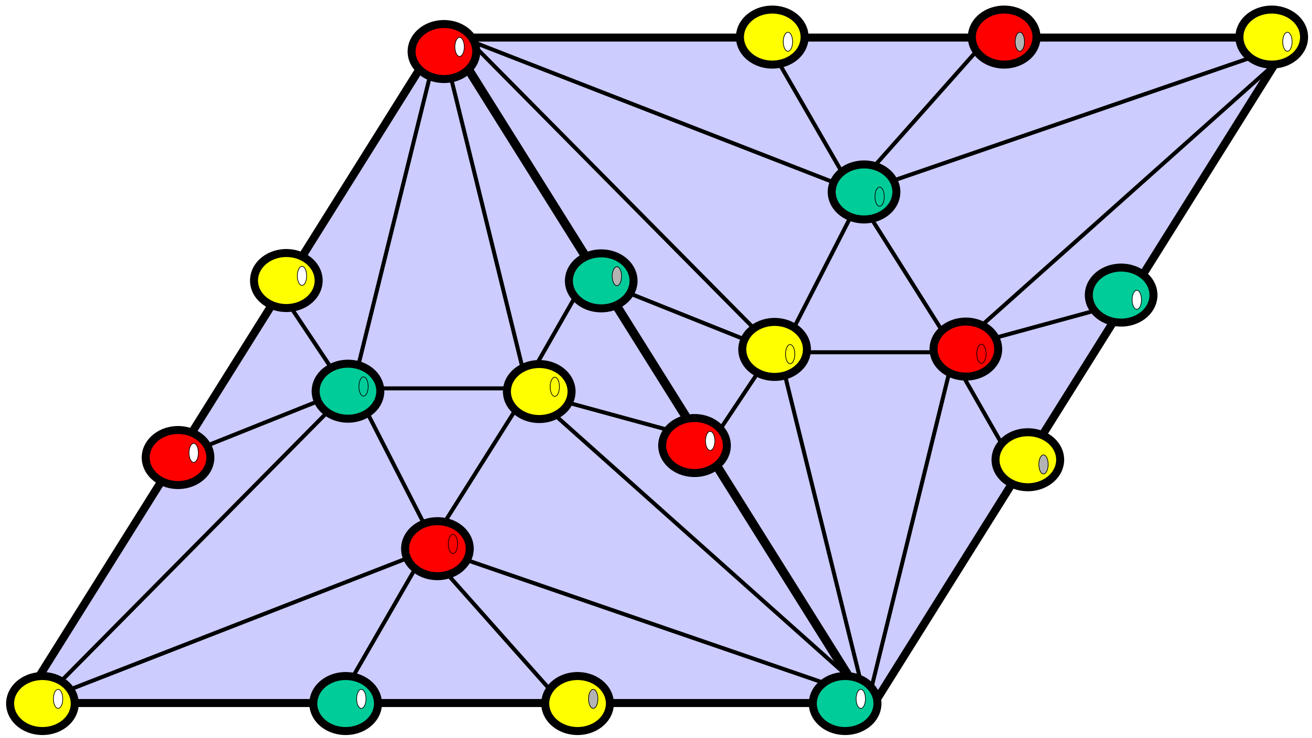
Manifold Theorem



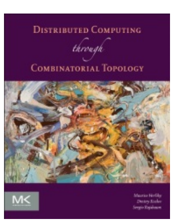
If the input complex $\mathcal{I}$ is a manifold …



Manifold Theorem

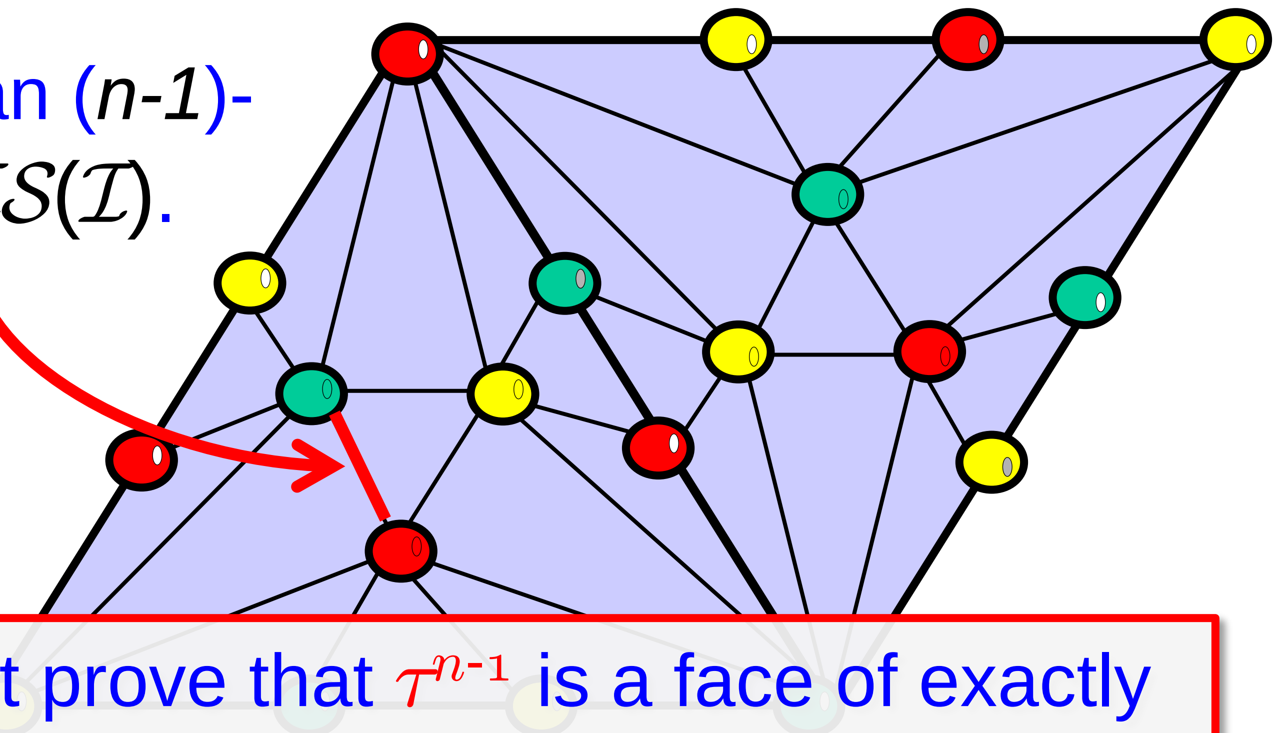


... so is the immediate snapshot protocol complex $\mathcal{IS}(\mathcal{I})$.

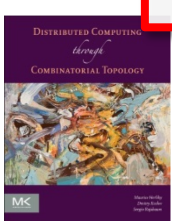


Proof of Manifold Theorem

Let τ^{n-1} be an (*n*-*1*)-simplex of $\mathcal{IS}(\mathcal{I})$.



Must prove that τ^{n-1} is a face of exactly one or two *n*-simplexes of $\mathcal{IS}(\mathcal{I})$.



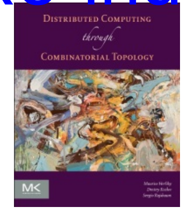
Without Loss of Generality …

Simplex of $\mathcal{I}$ seen by last process

$$\tau^{n-1} = \{\langle P_0, \sigma_0\rangle, \ldots, \langle P_{n-1}, \boxed{\sigma_{n-1}}\rangle\}$$

Where $\sigma_i \subseteq \sigma_{i+1}$ for $0 \leq i < n$

Re-index processes in execution order
(ignoring ties)



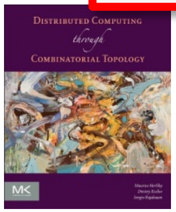
Proof Strategy

Count the ways we can extend…

$$\tau^{n\text{-}1} = \{(P_0, \sigma_0), \ldots, (P_{n\text{-}1}, \sigma_{n\text{-}1})\}$$

$$\tau^{n} = \{(P_0, \sigma_0), \ldots, (P_{n\text{-}1}, \sigma_{n\text{-}1}\rangle, (P_n, \sigma_n)\}$$

Where σ_n is an n-simplex of $\mathcal{IS}(\sigma)$



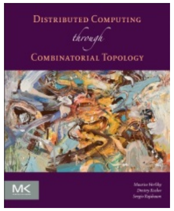
Cases

$$\tau^{n-1} = \{\langle P_0, \sigma_0\rangle, \ldots, \langle P_{n-1}, \boxed{\sigma_{n-1}}\rangle\}$$

dim *n-1* or dim *n*

3 cases

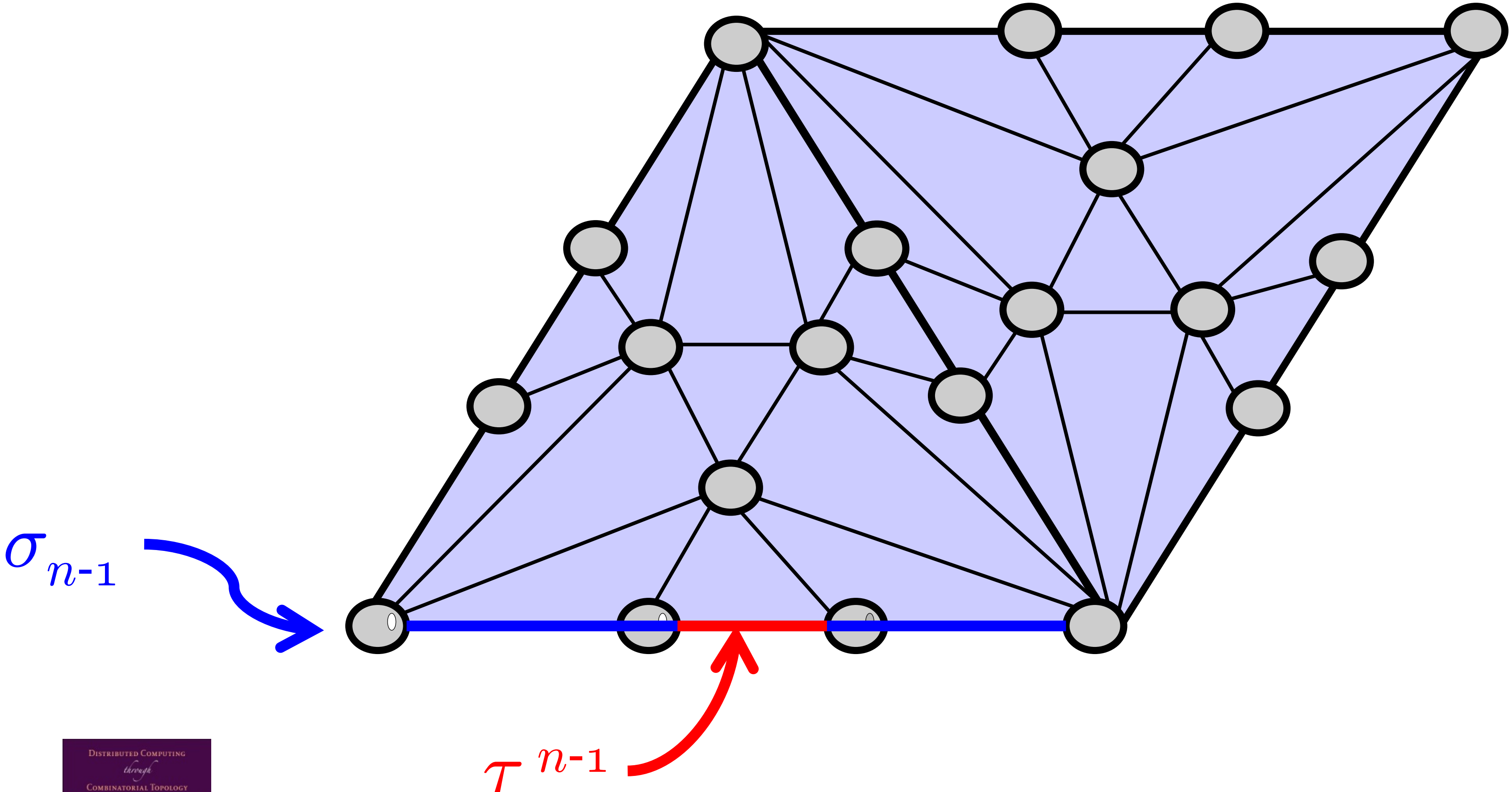
Boundary or internal

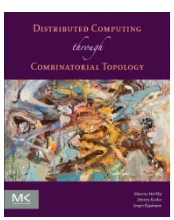


Case One

$$\tau^{n-1} = \{\langle P_0, \sigma_0\rangle, \ldots, \langle P_{n-1}, \boxed{\sigma_{n-1}}\rangle\}$$

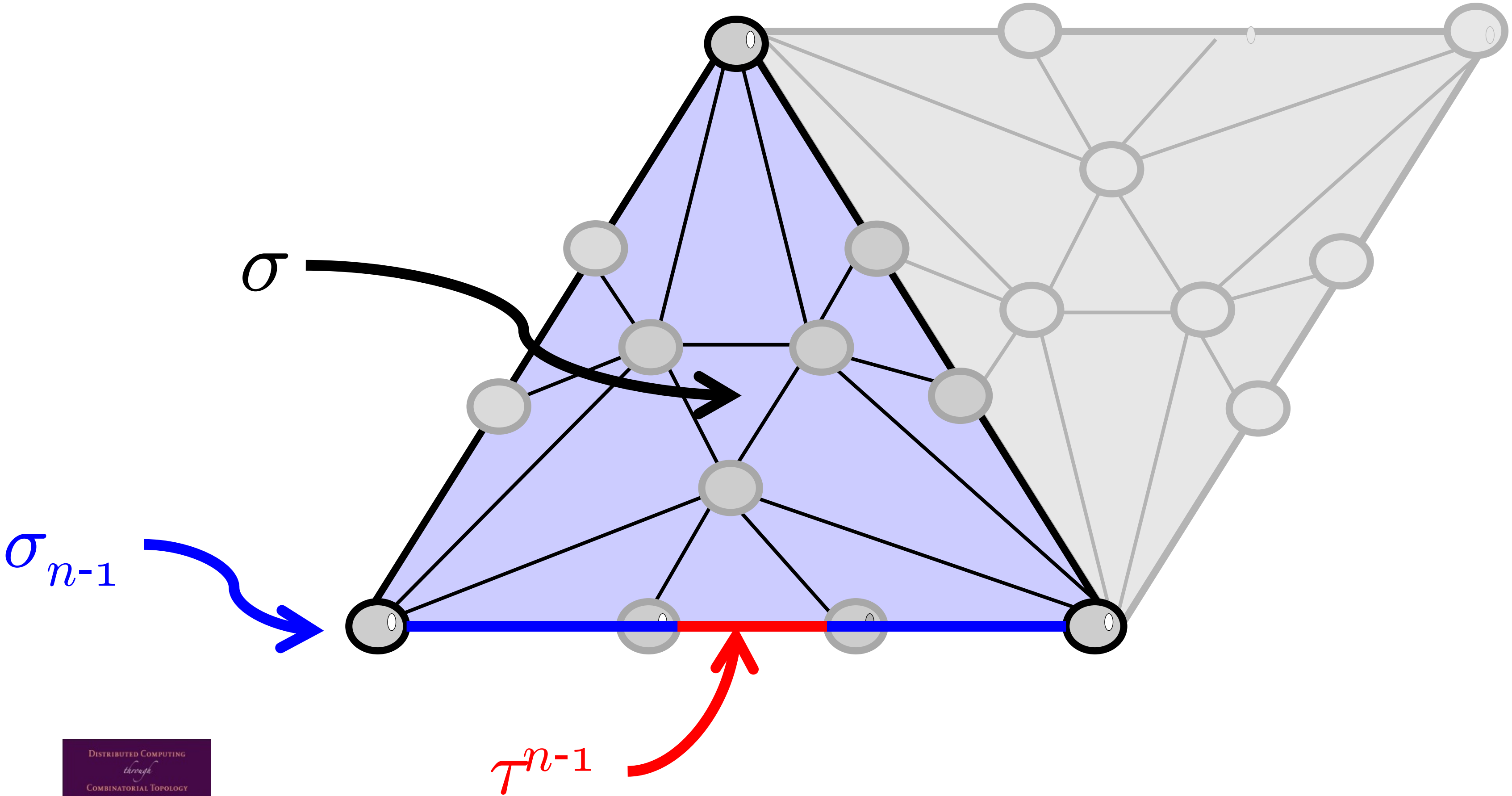
Boundary
(*n-1*)-simplex

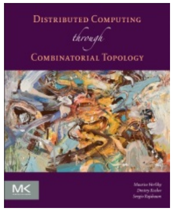




Case One

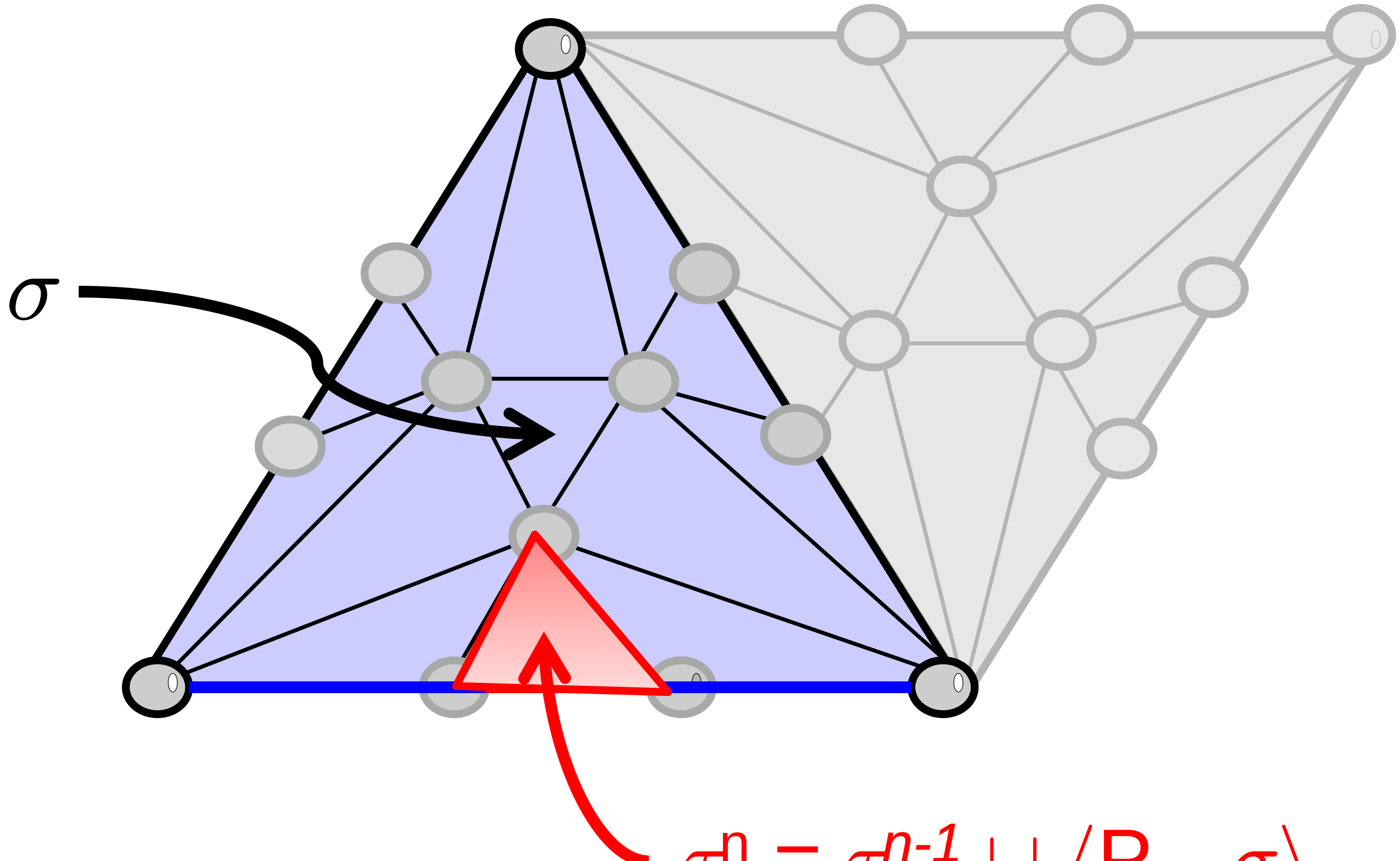
Exactly one *n*-simplex σ of $\mathcal{I}$ contains $\sigma_{n\text{-}1}$



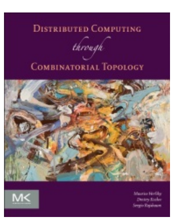


Case One

Exactly one *n*-simplex τ^n of $\mathcal{IS}(\mathcal{I})$ contains τ^{n-1}



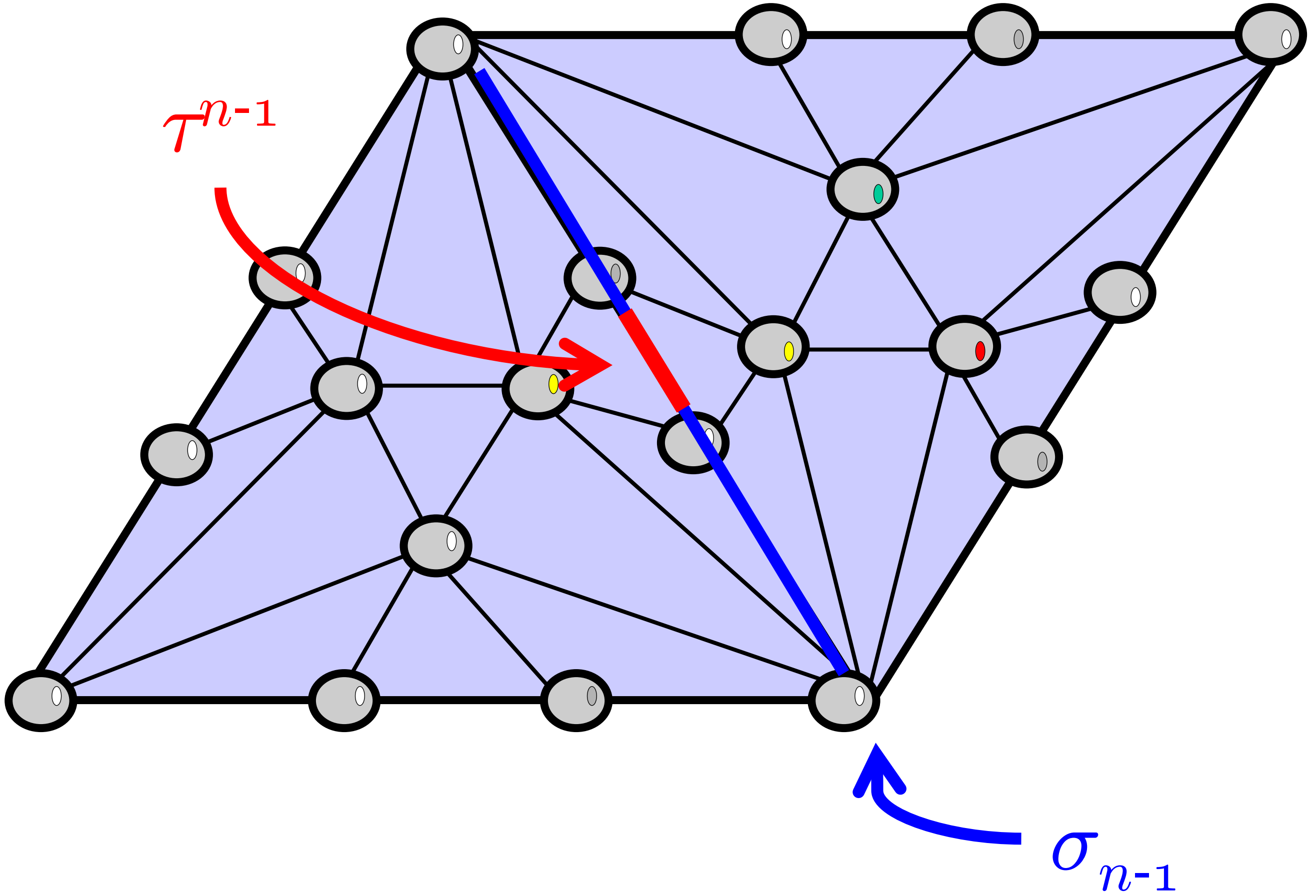
$\tau^n = \tau^{n-1} \cup \langle P_n, \sigma \rangle$

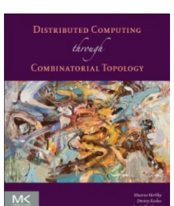


Case Two

$$\tau^{n-1} = \{\langle P_0, \sigma_0\rangle, \ldots, \langle P_{n-1}, \boxed{\sigma_{n-1}}\rangle\}$$

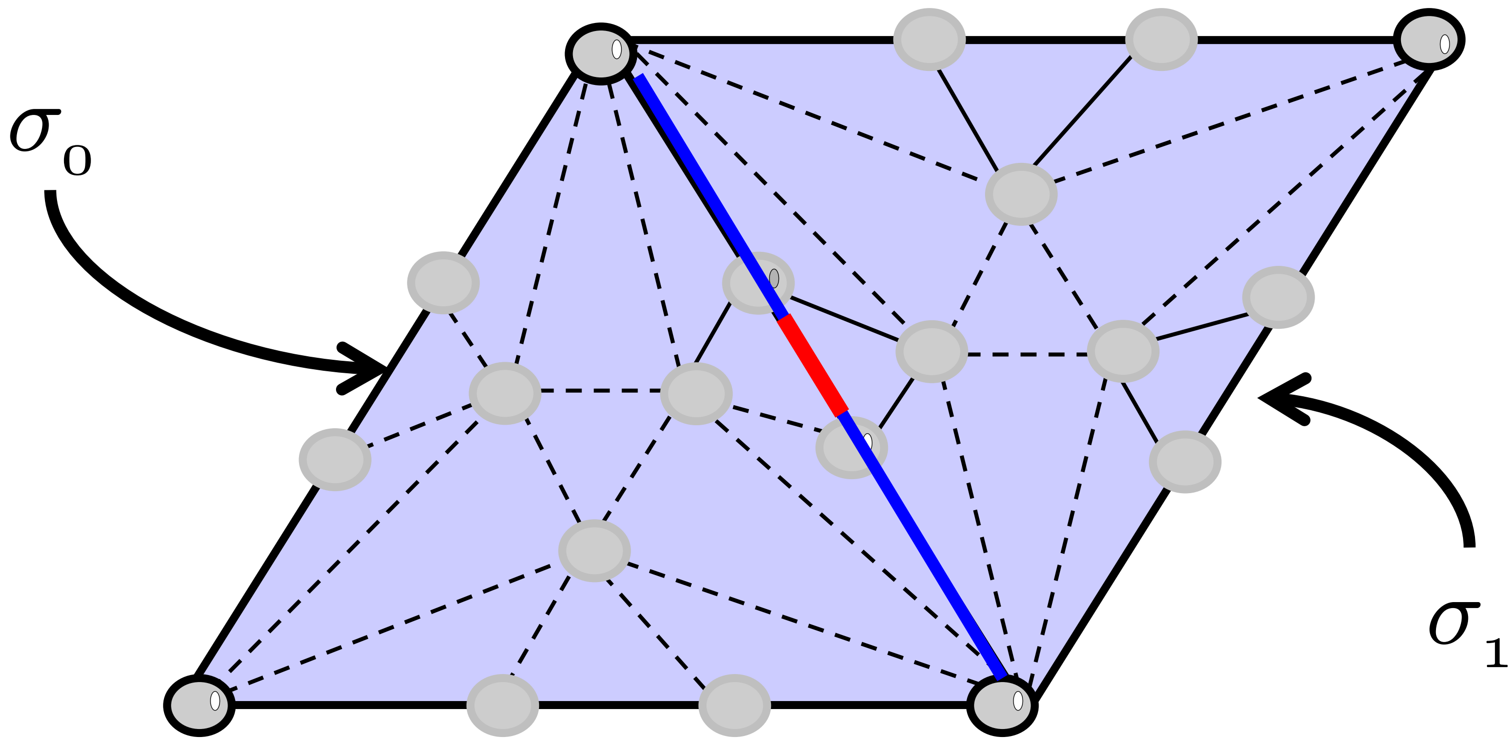
Internal
(*n-1*)-simplex

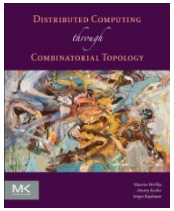




Case Two

Exactly two *n*-simplexes σ_0 & σ_1 of $\mathcal{I}$ contain $\sigma_{n\text{-}1}$



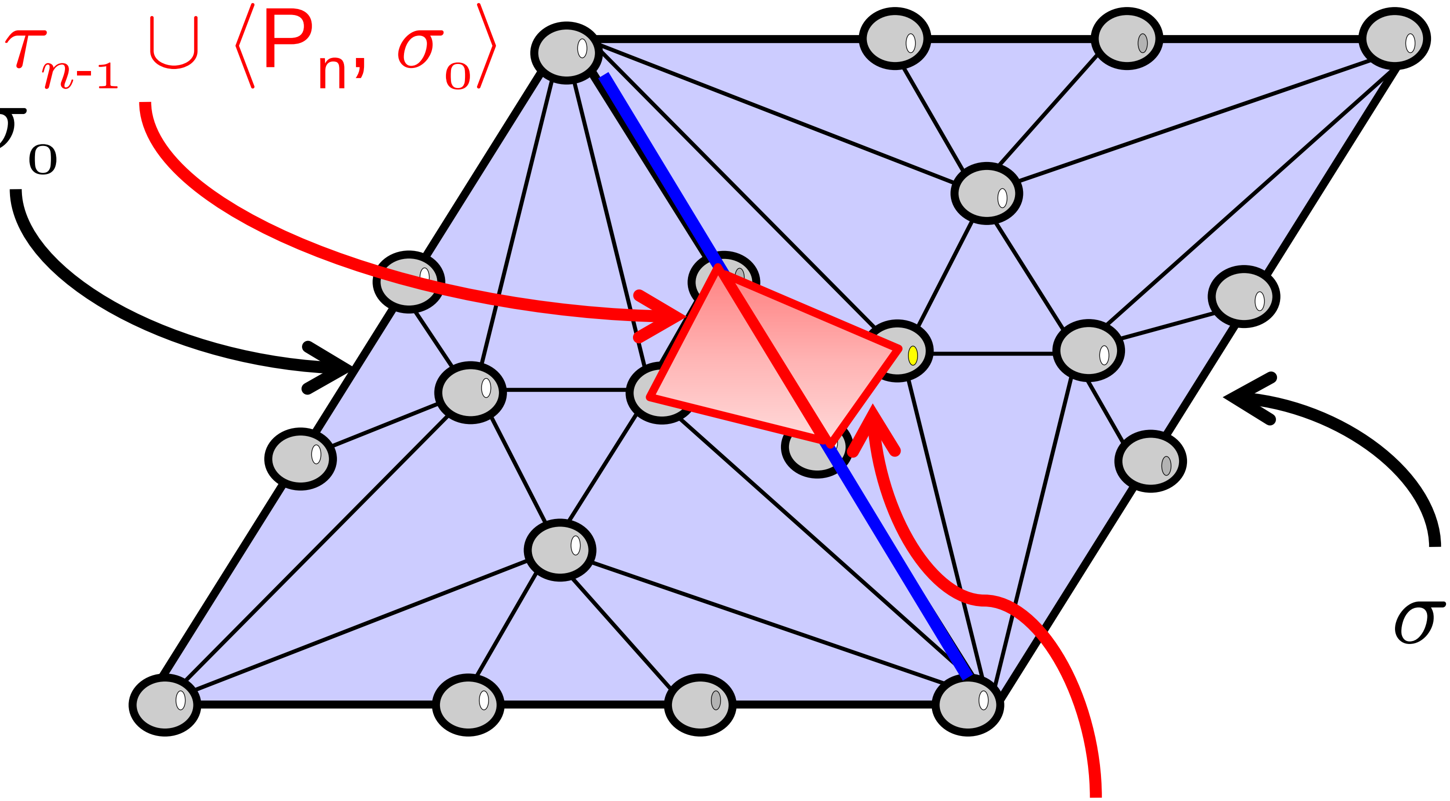


Case Three

Exactly 2 *n*-simplexes τ_0^n & τ_1^n of $\mathcal{IS}(\mathcal{I})$ contain τ^{n-1}

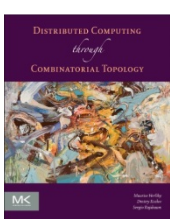
$\tau^{n}_{0} = \tau_{n-1} \cup \langle P_n, \sigma_0 \rangle$

σ_0



σ_1

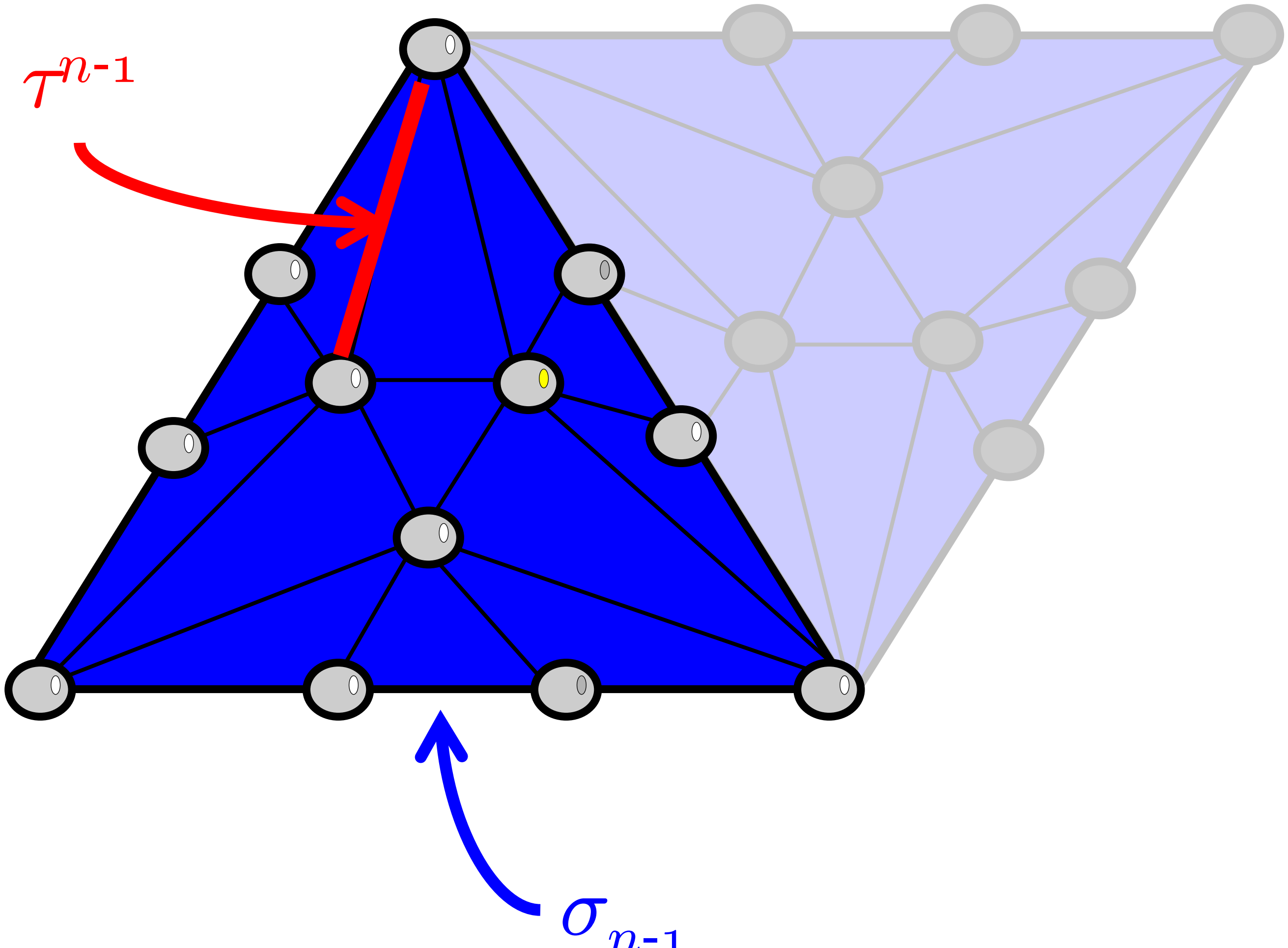
$\tau^{n}_{1} = \tau_{n-1} \cup \langle P_n, \sigma_1 \rangle$

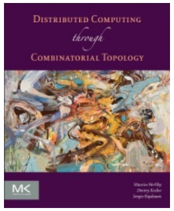


Case Three

$$\tau^{n-1} = \{\langle P_0, \sigma_0 \rangle, \ldots, \langle P_{n-1}, \boxed{\sigma_{n-1}} \rangle\}$$

n-simplex





Case Three

Suppose

$P_n \in \text{ids}(\sigma_k)$

(P_k saw P_n's write)

$\Rightarrow$ $\sigma_n \subseteq \sigma_k$

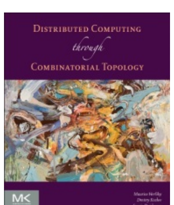
$P_n \notin \text{ids}(\sigma_{k\text{-}1})$

($P_{k\text{-}1}$ did not see P_n's write)

Special case: k=0, $\sigma_{-1} = \emptyset$

$\Rightarrow$ $\sigma_{k\text{-}1} \subset \sigma_n$

Either $\sigma_{k\text{-}1} \subset \sigma_n \subset \sigma_k$ or $\sigma_{k\text{-}1} \subset \sigma_n = \sigma_k$

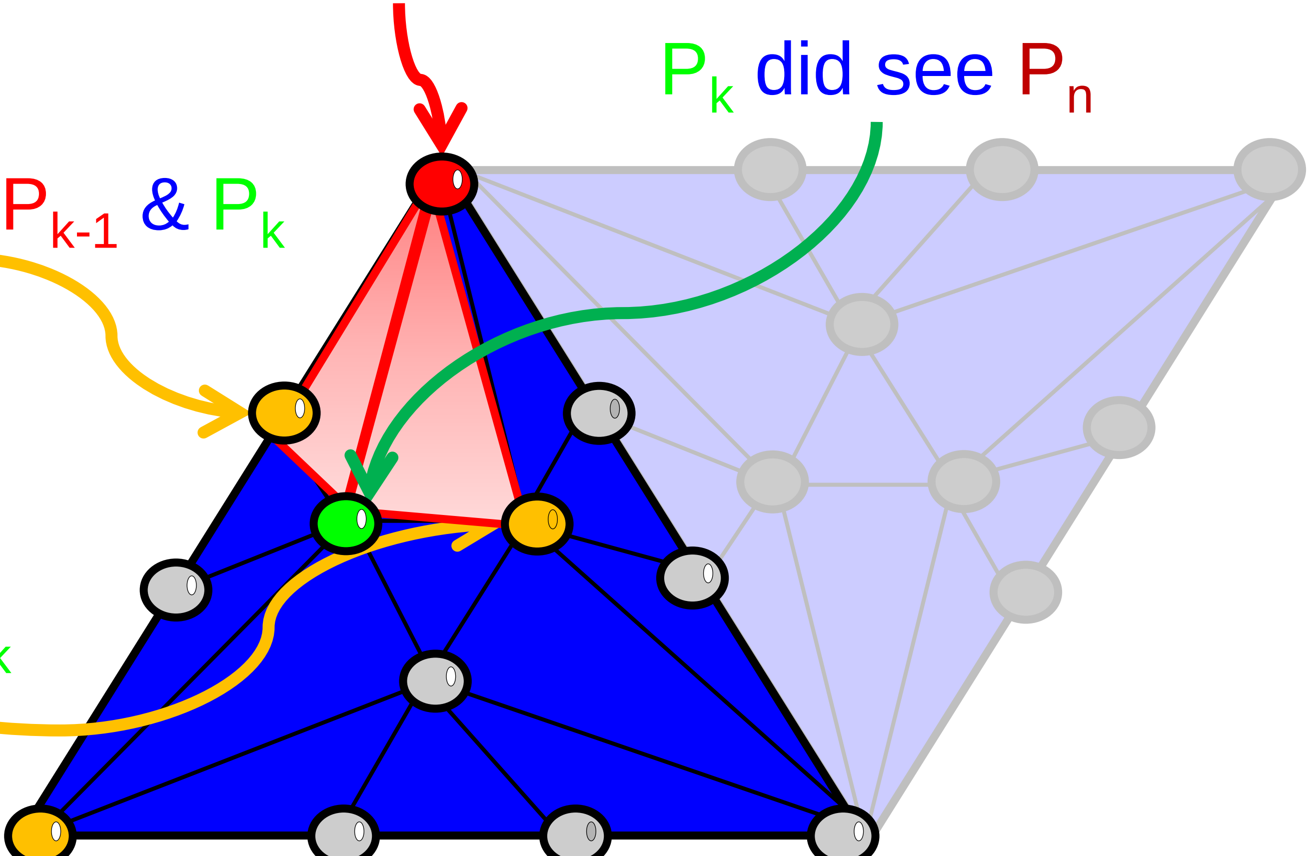


Case Three

P_{k-1} did not see P_n

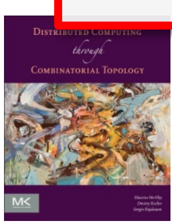
P_k did see P_n

P_n between P_{k-1} & P_k



P_n with P_k

This completes the proof.



Manifold Protocols

A protocol $\mathcal{M}(\mathcal{I})$ is a *manifold protocol* if

If $\mathcal{I}$ is a manifold, so is $\mathcal{M}(\mathcal{I})$

$\mathcal{M}(\partial\mathcal{I}) = \partial\mathcal{M}(\mathcal{I})$.

Example: immediate snapshot

Important: closed under composition

Road Map

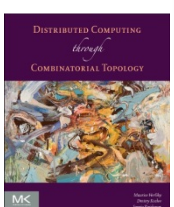
Manifolds

Immediate Snapshot Model

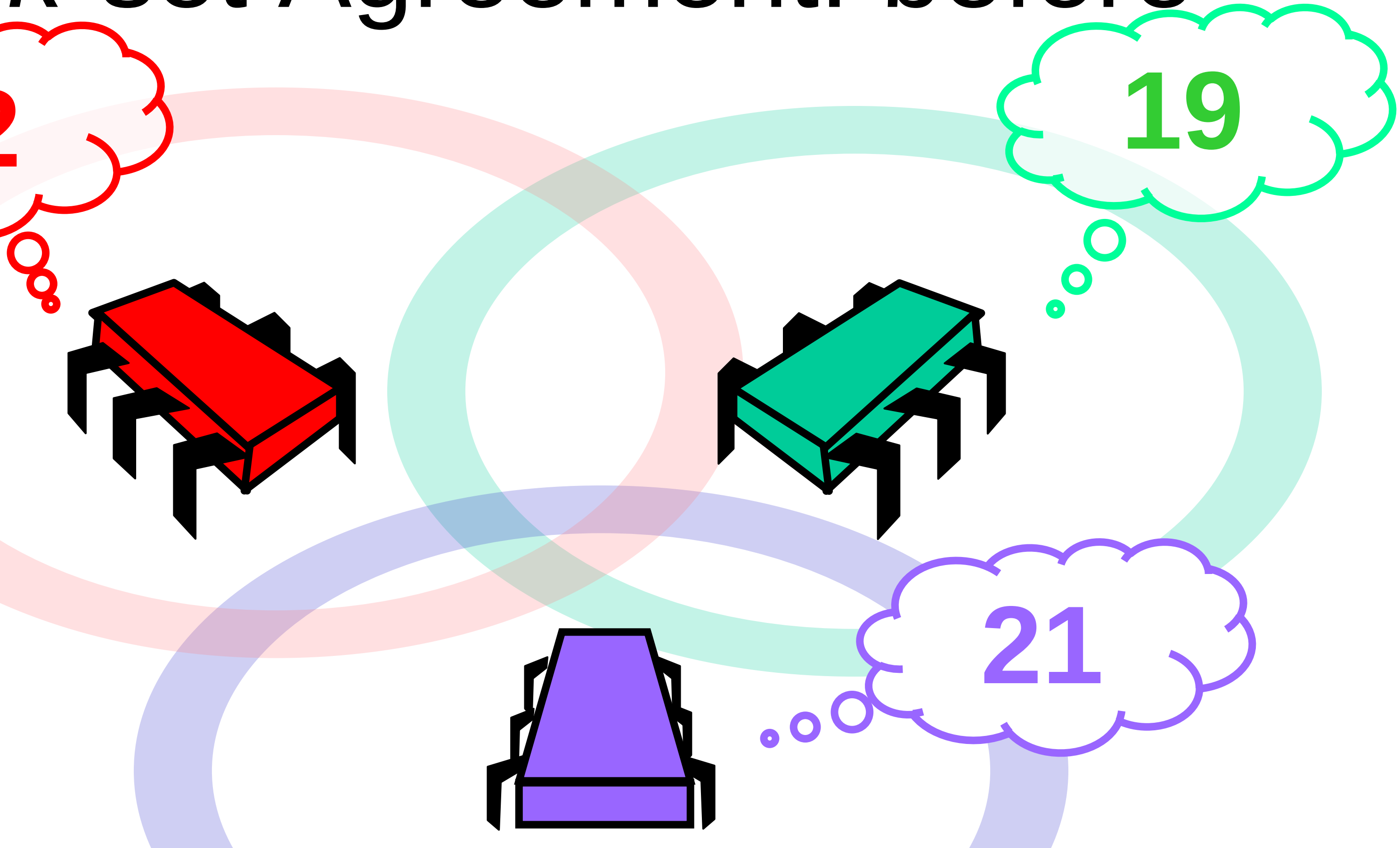
Sperner’s Lemma and *k*-Set Agreement

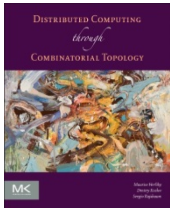
Weak Symmetry-Breaking

Separation results

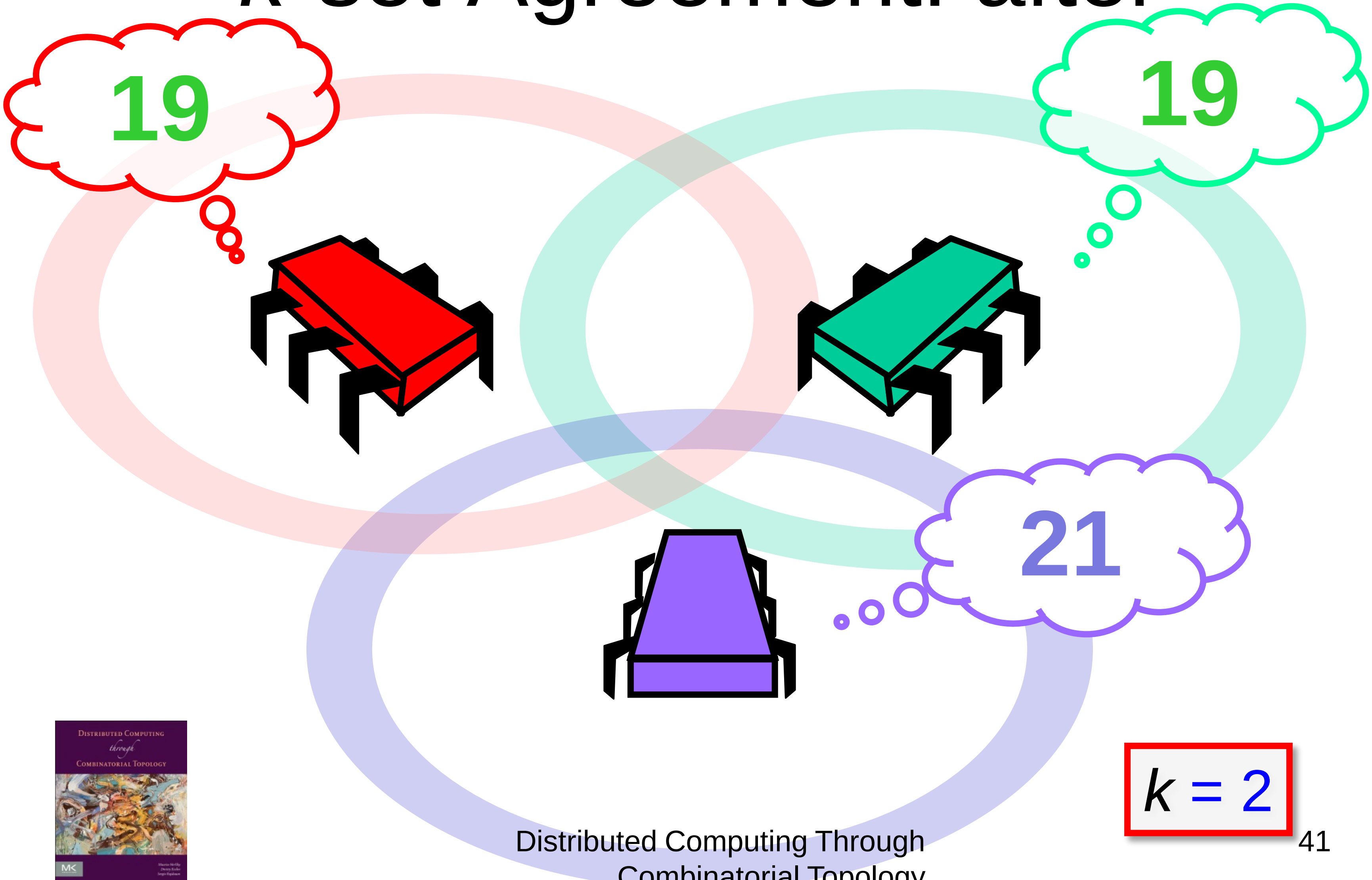


k-set Agreement: before





k-set Agreement: after



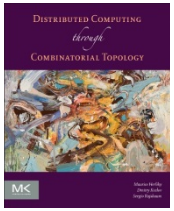
$k = 2$

Theorem

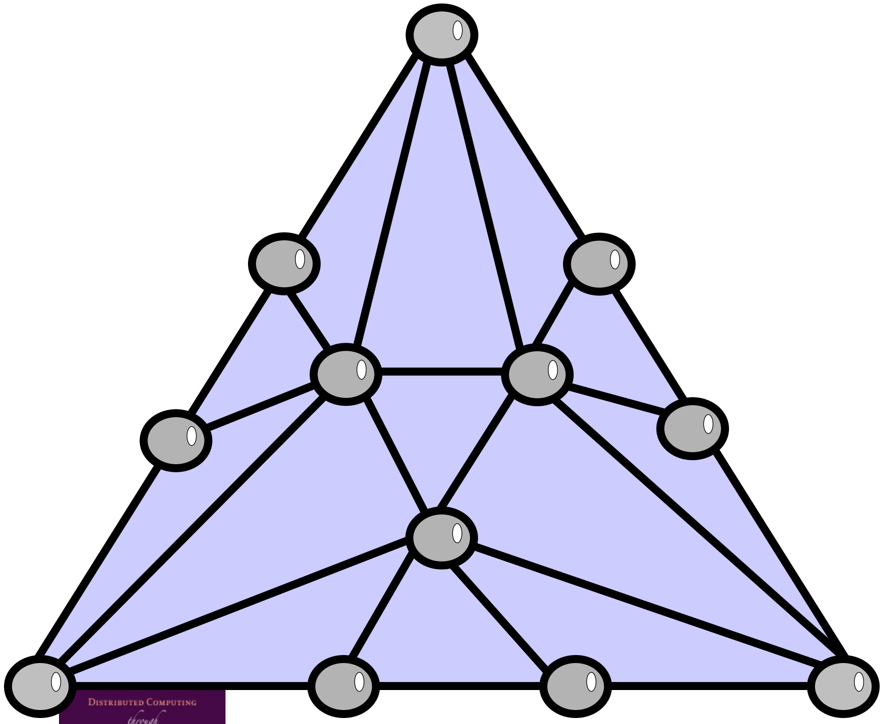
No Manifold Protocol can solve *n*-set agreement

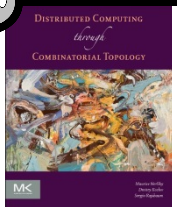
Including Immediate Snapshot

Including read-write memory



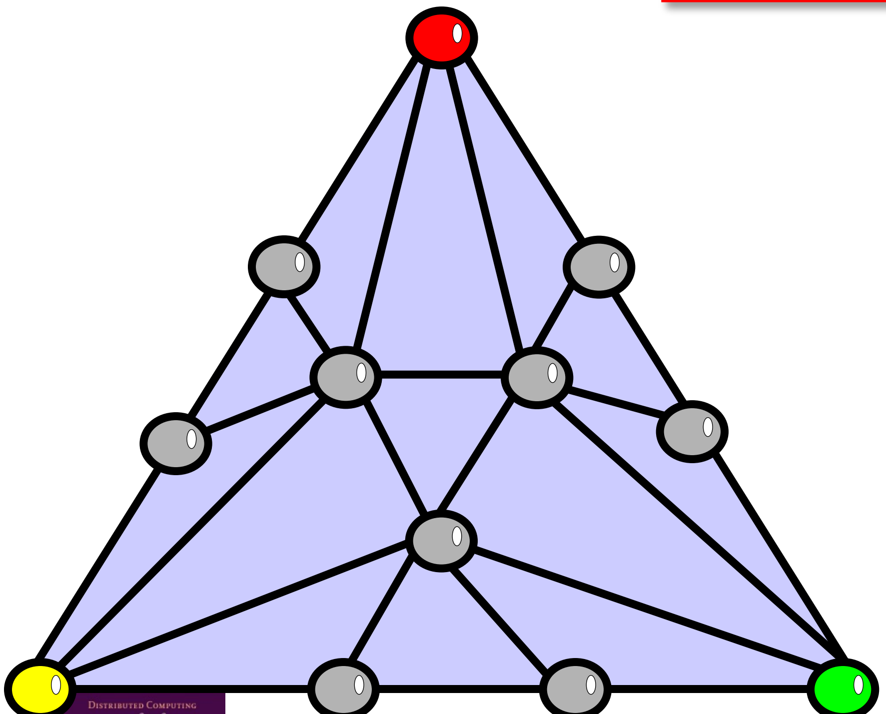
Sperner Coloring

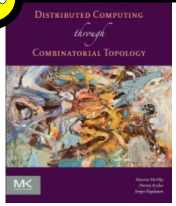




Sperner Coloring

“Corners” have distinct colors

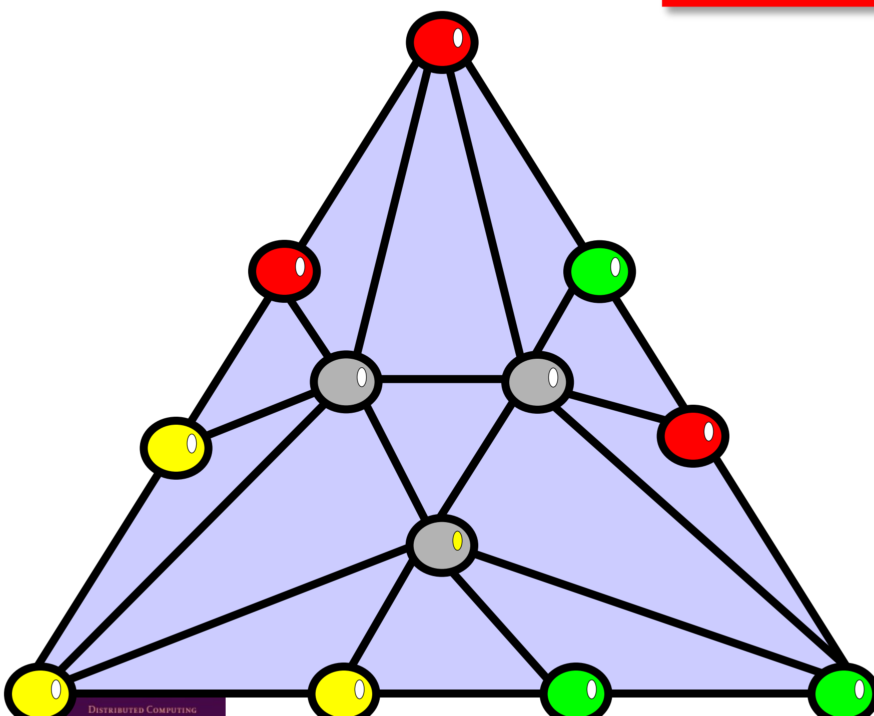


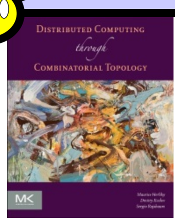


Sperner Coloring

“Corners” have distinct colors

Edge vertexes have corner colors



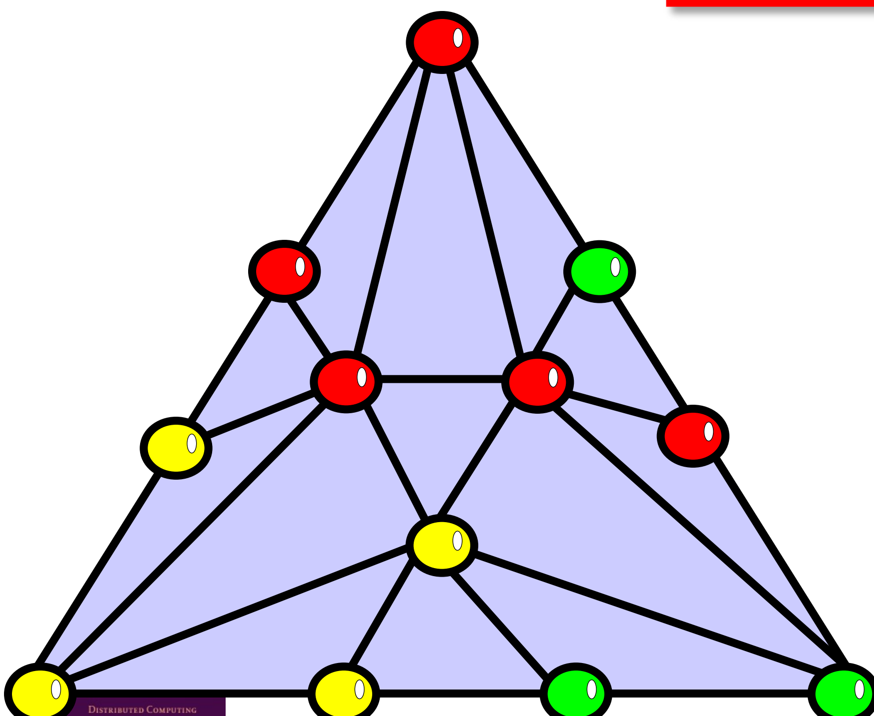


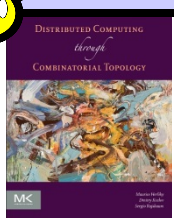
Sperner Coloring

“Corners” have distinct colors

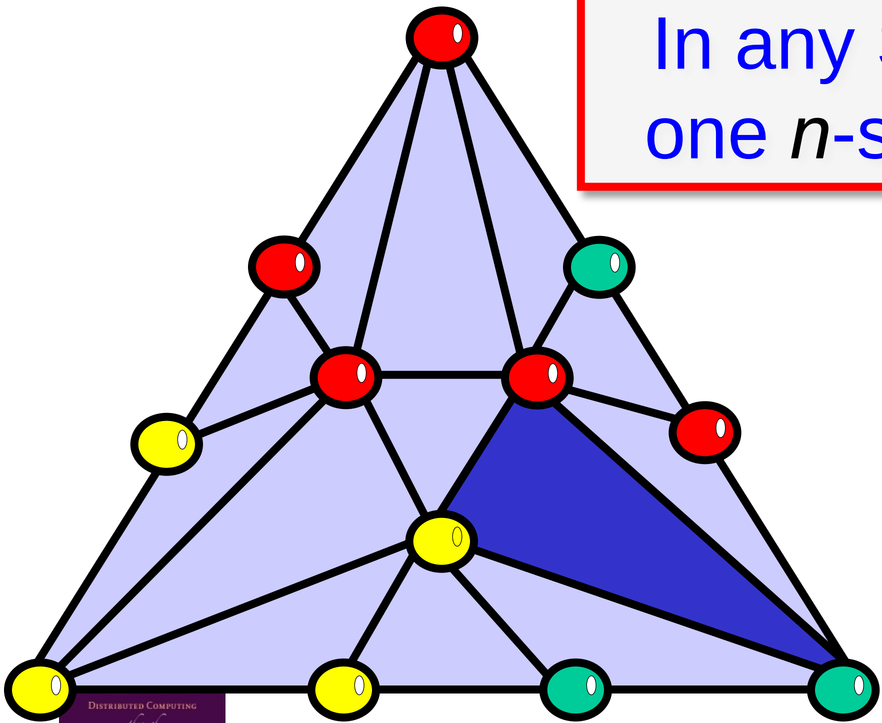
Edge vertexes have corner colors

Every vertex has face boundary colors

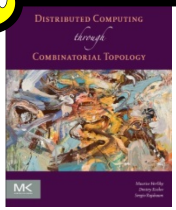




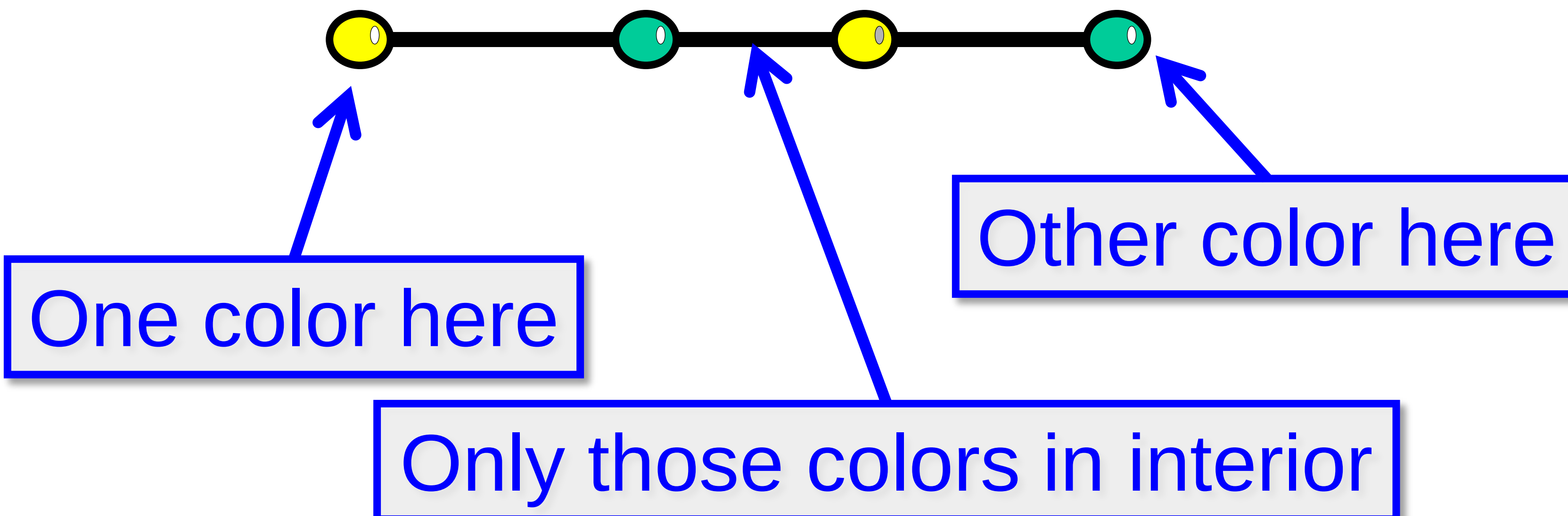
Sperner's Lemma

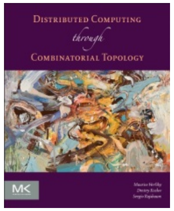


In any Sperner coloring, at least one *n*-simplex has all *n+1* colors



Sperner Coloring for Manifolds with Boundary (base)



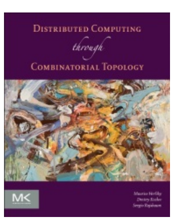


Sperner Coloring for Manifolds with Boundary (inductive)

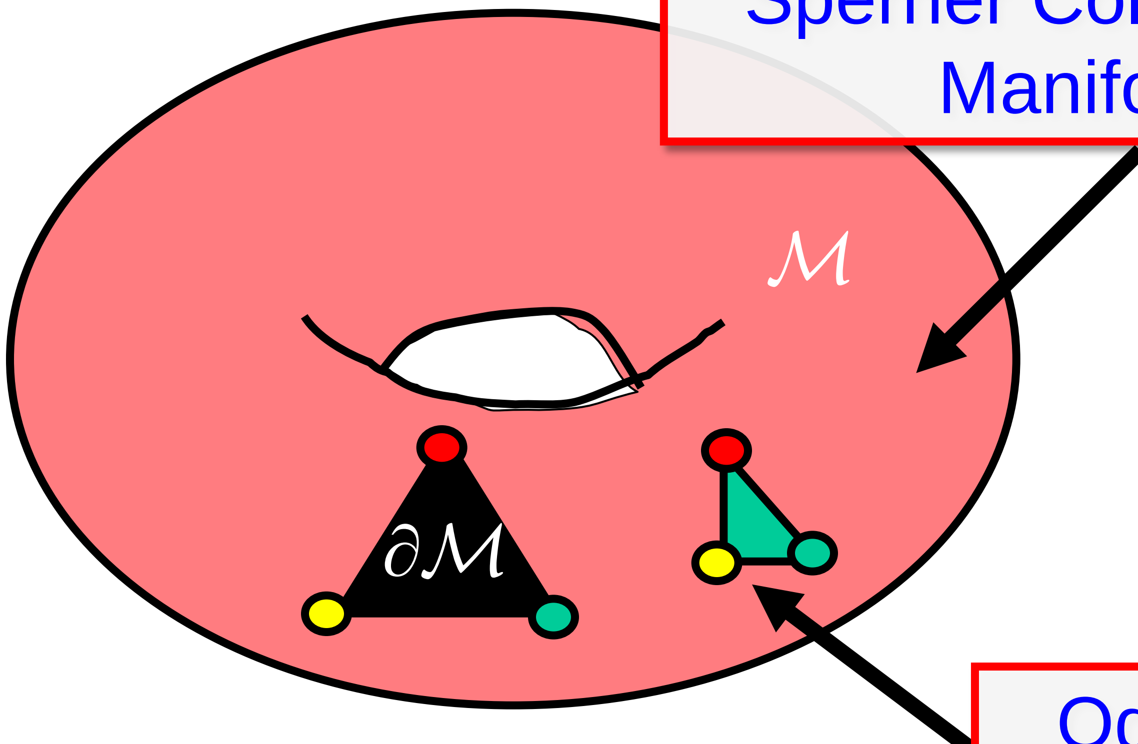
$\mathcal{M}$ is n-manifold colored with Π

$$\partial\, \mathcal{M} = \cup_i\, \mathcal{M}_i$$

Where $\mathcal{M}_i$ is non-empty $(n\text{-}1)$-manifold Sperner-colored with $\Pi \setminus \{i\}$

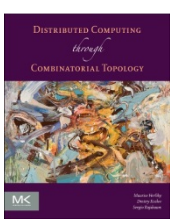


Sperner's Lemma for Manifolds



Sperner Coloring on Manifold

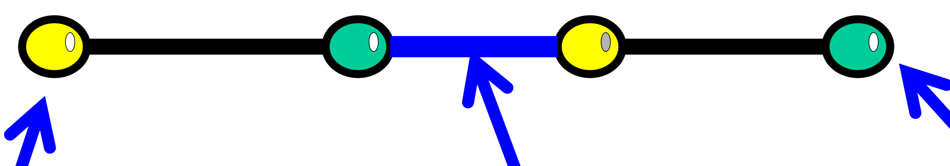
Odd number of *n*-simplexes have all *n+1* colors



Proof of Sperner's Lemma

Induction on *n*

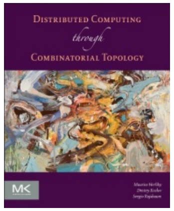
Base



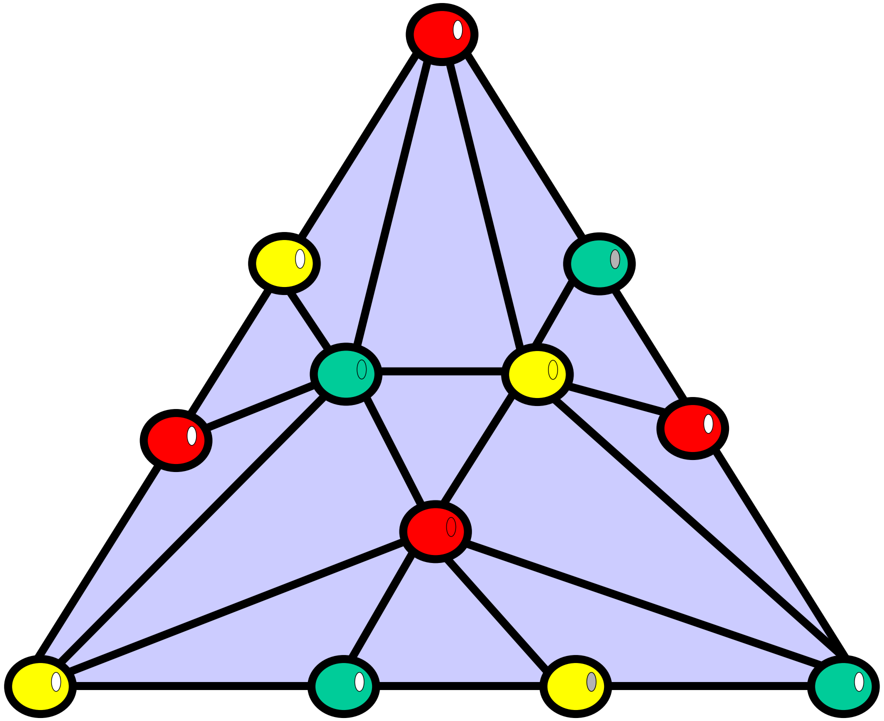
One color here

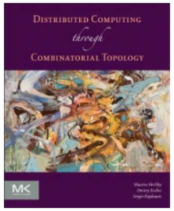
Odd number of changes

Other color here

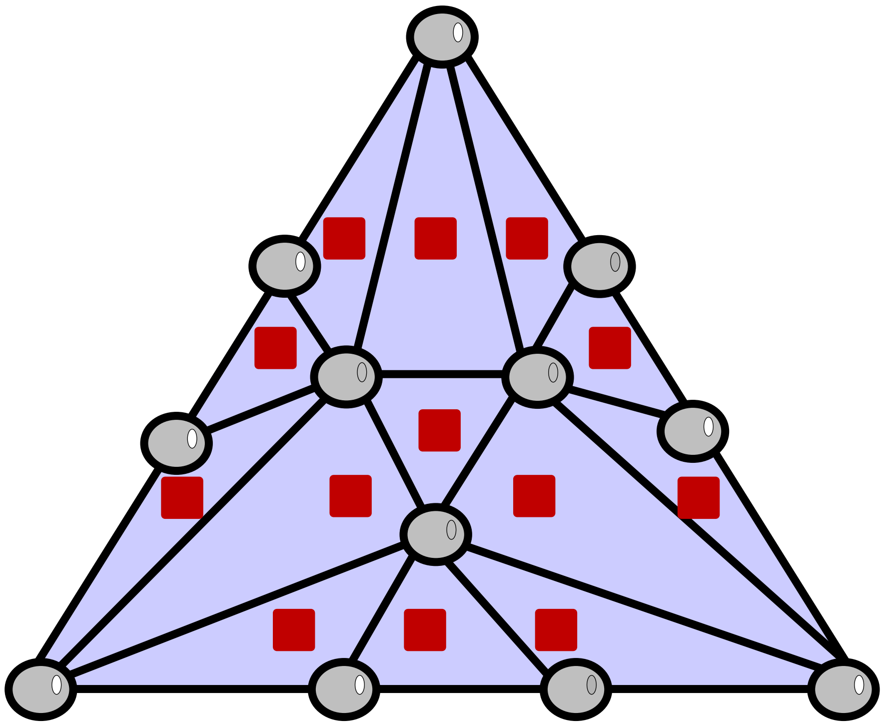


Induction Step

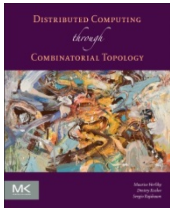




Dual Graph

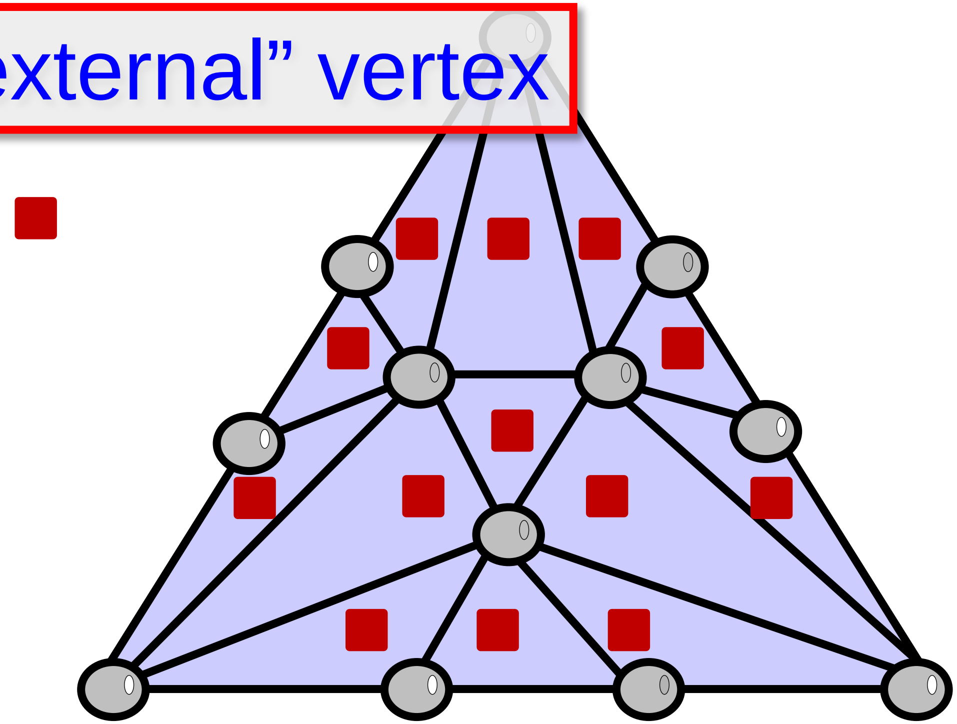


One vertex per *n*-simplex …

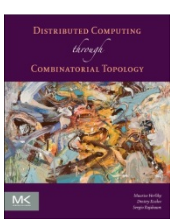


Dual Graph

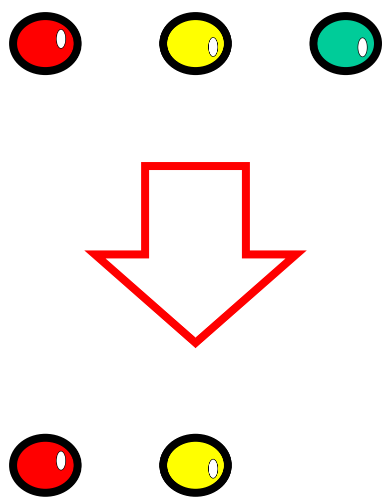
One “external” vertex



One vertex per *n*-simplex …

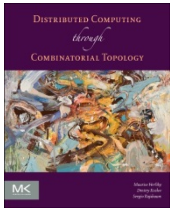


Discard One Color



n+1 colors

"restricted" *n* colors

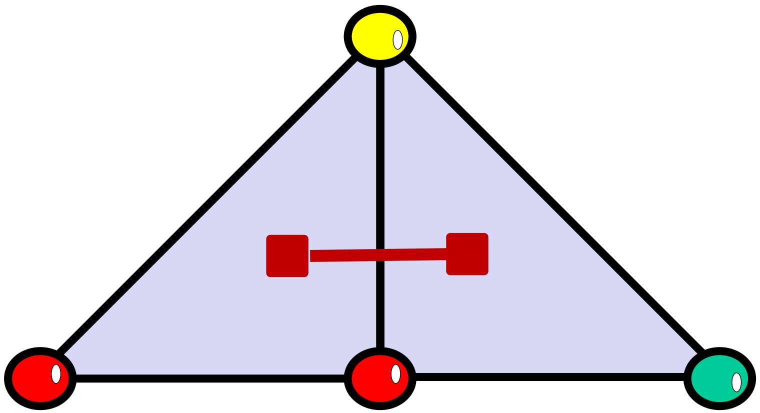


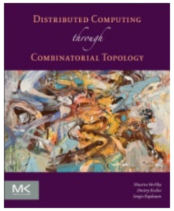
Edges

If two *n*-simplexes share an (*n-1*)-face …

with restricted *n* colors …

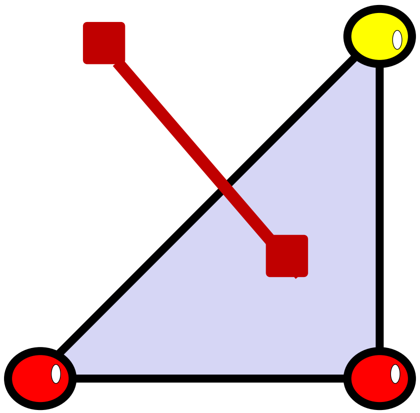
add an edge.

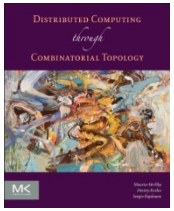




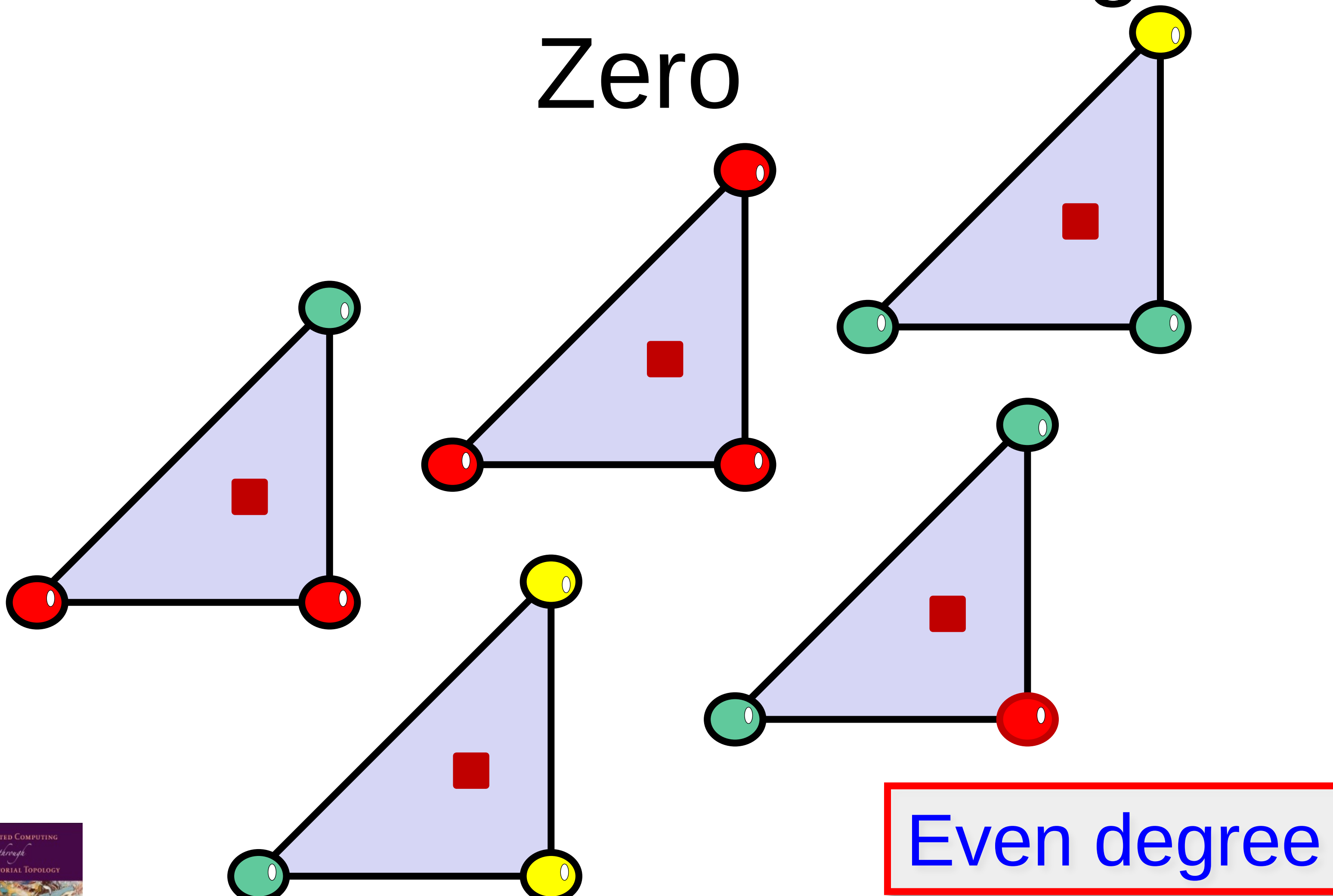
Edges

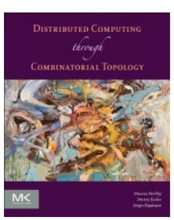
Same for external vertex



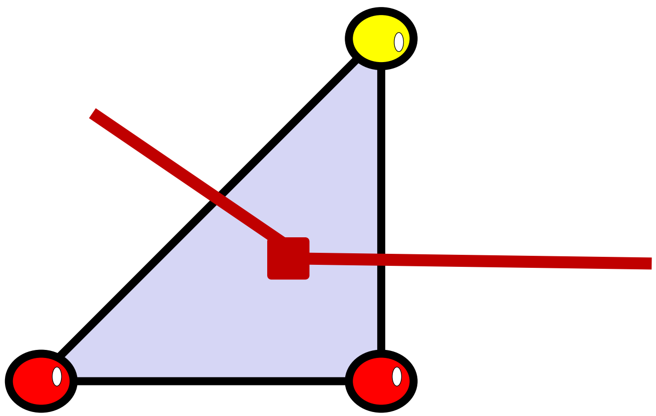


Some Vertexes have Degree Zero



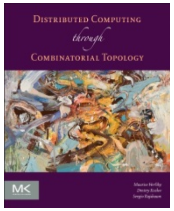


Some Vertexes have Degree Two

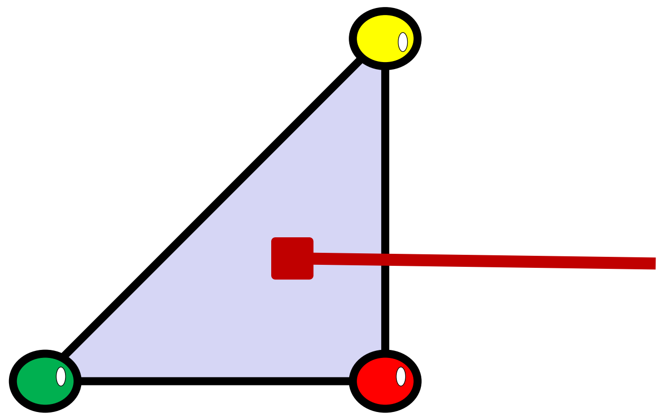


Only *n* restricted colors

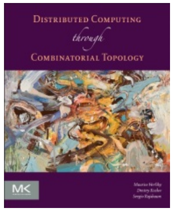
Even degree



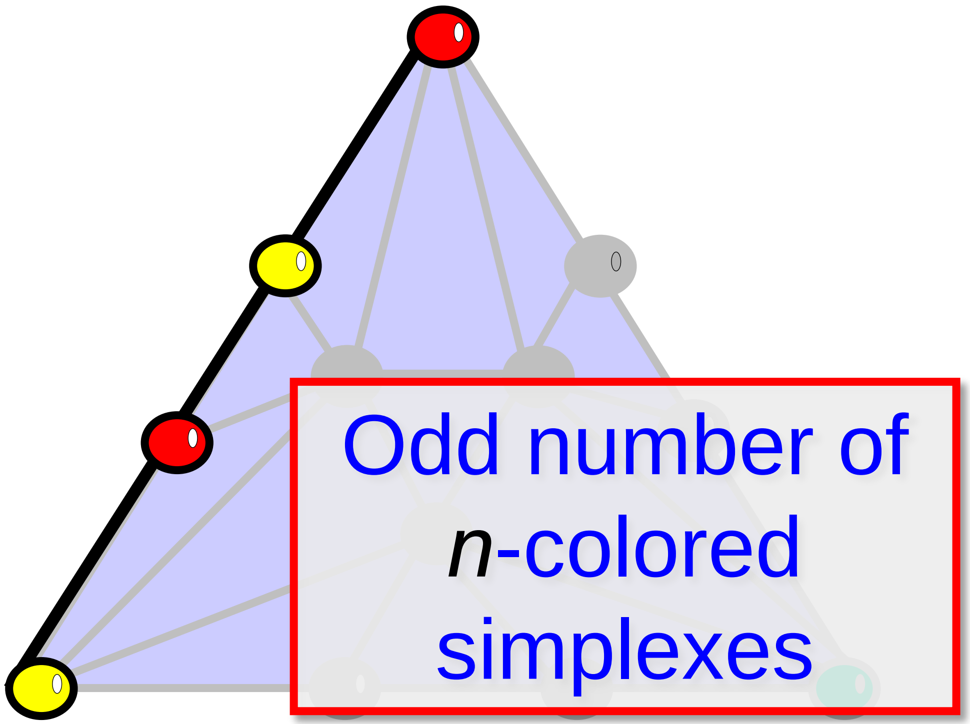
All $n+1$ Colors $\Rightarrow$ Degree 1

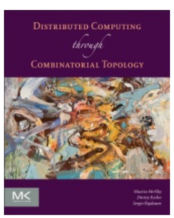


Odd degree

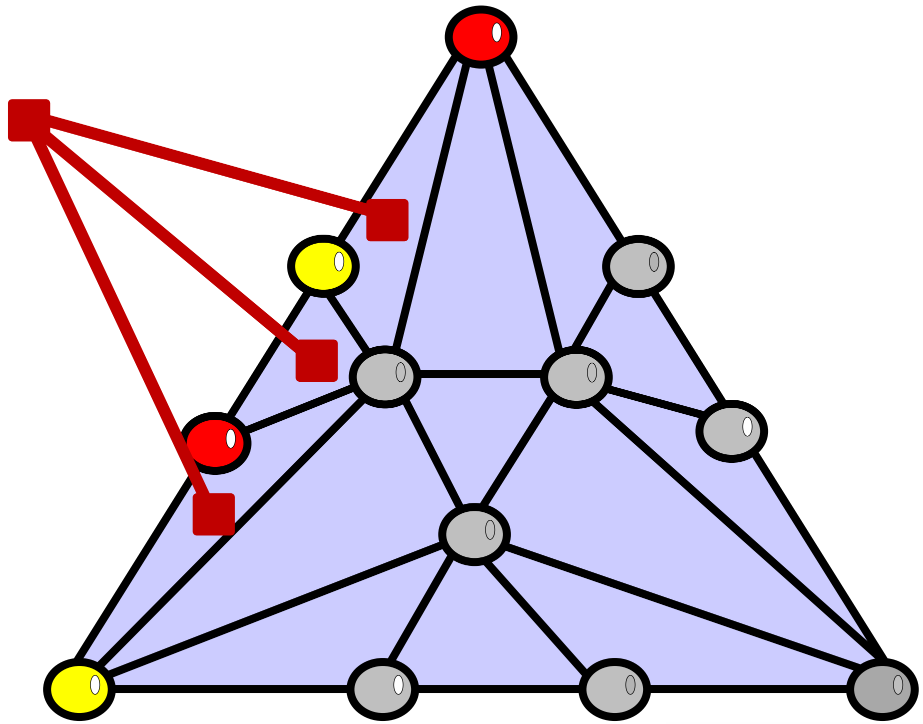


Induction Hypothesis

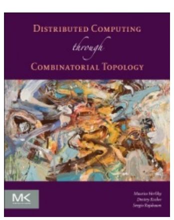




Induction Hypothesis



External vertex has odd degree



Degrees

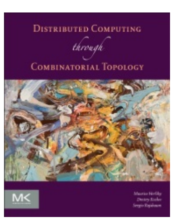
Odd

One external vertex

n-simplexes with *n+1* colors?

Even

All the rest



Degrees

Odd

One external vertex

Every graph

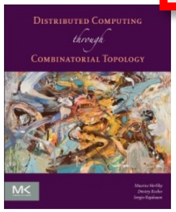
sum of degrees is *even*…

Even

each edge contributes 2

All the rest

Even number of *odd* degree vertexes



Degrees

Odd

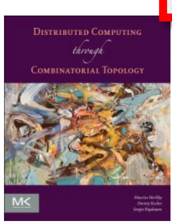
One external vertex

n-simplexes with n+1 colors?

Even

All the rest

Even number of *odd* degree vertexes



Parity

Odd

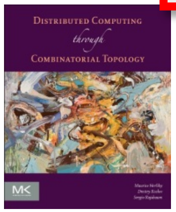
Even

One external vertex

n-simplexes with *n+1* colors?

Odd

Even number of *odd* degree vertexes



Parity

Odd

Even

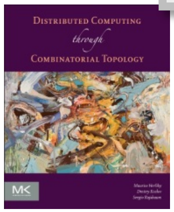
One external vertex

n-simplexes with *n+1* colors?

Odd

Even number of *odd* degree vertexes

QED



No Manifold Task can solve n-Set Agreement

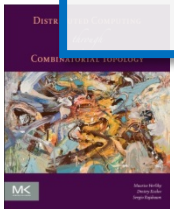
Assume protocol exists:

Run manifold task protocol

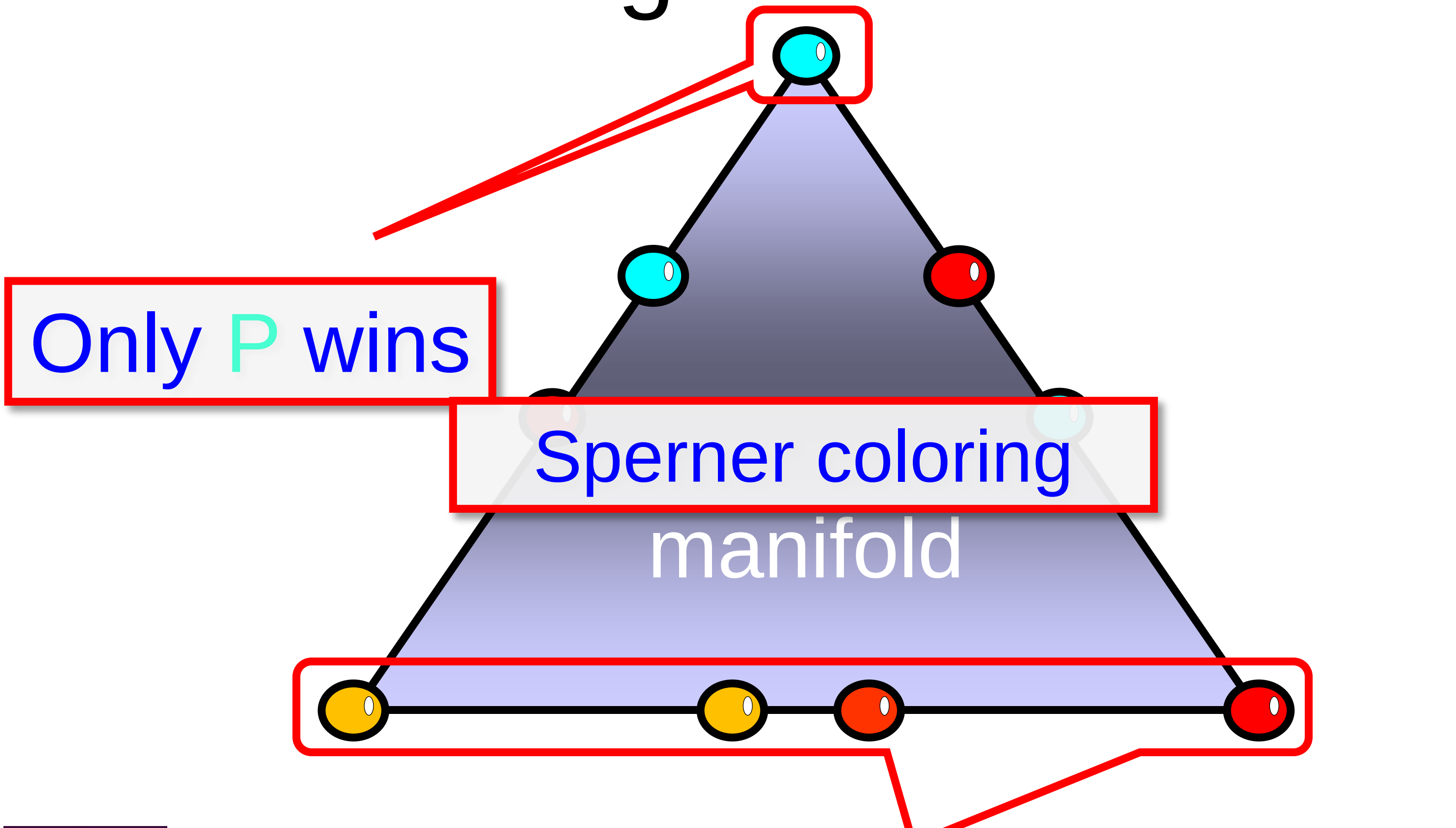
Choose value based on vertex

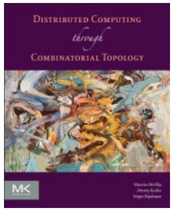
Idea:

Color vertex with “winning” process name …

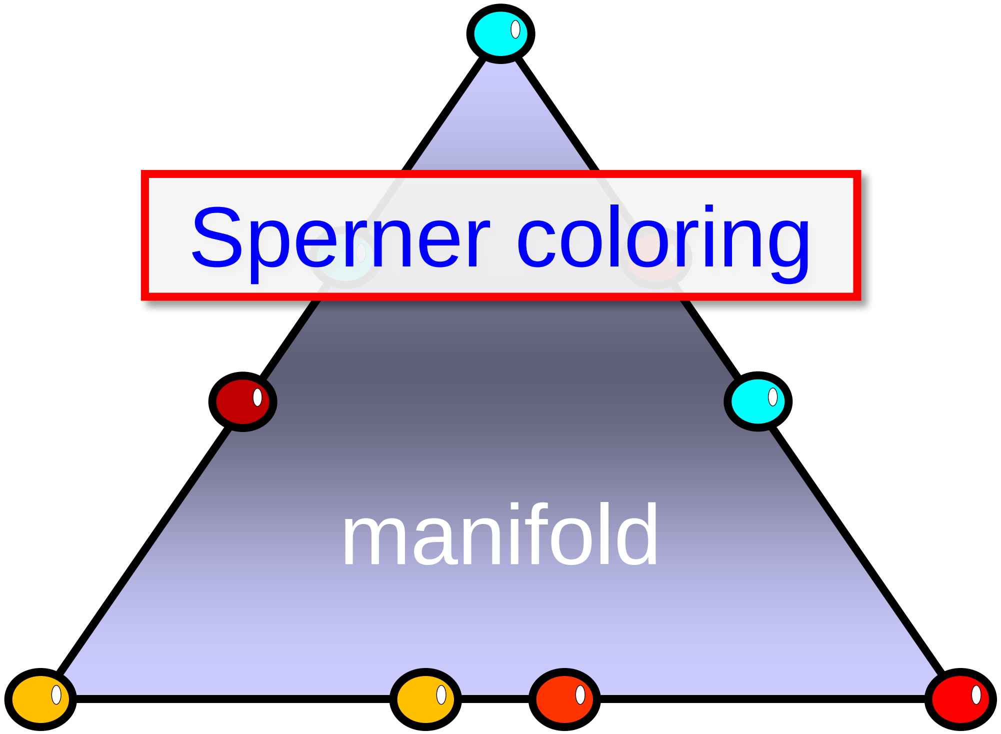


Manifold Task for *n*-Set Agreement

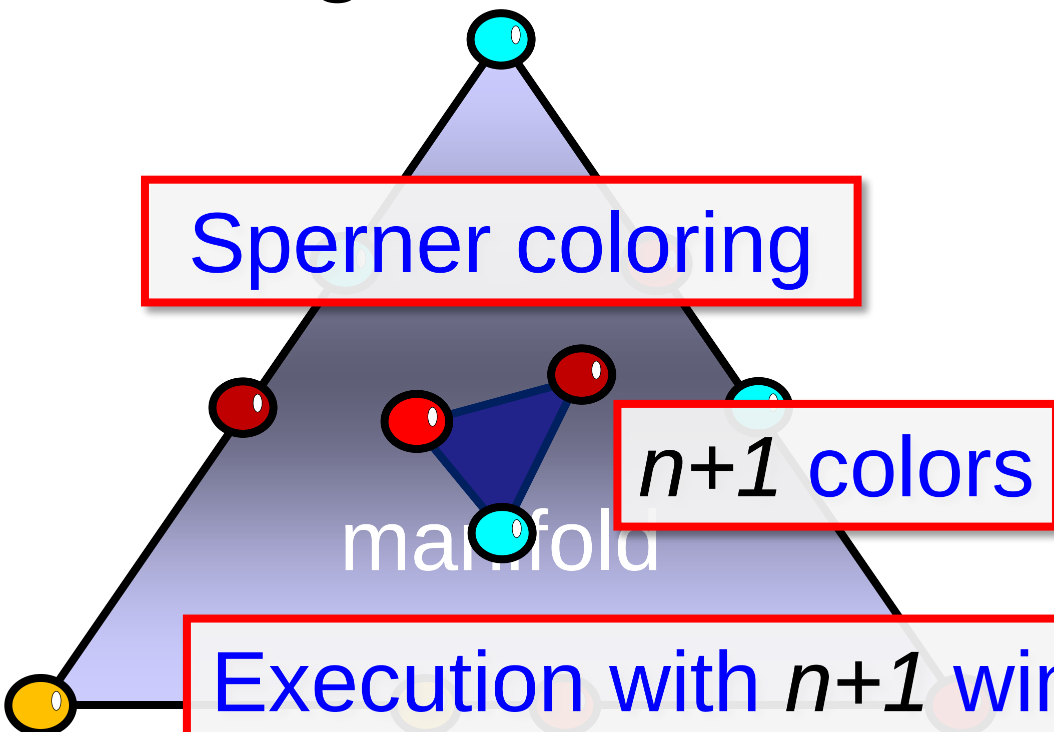




Manifold Task for *n*-Set Agreement



Manifold Task for *n*-Set Agreement

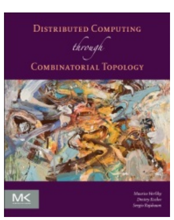


Sperner coloring

n+1 colors

Execution with *n*+1 winners

Contradiction: at most *n* can win



Road Map

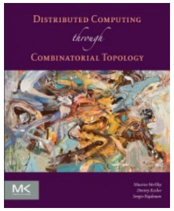
Manifolds

Immediate Snapshot Model

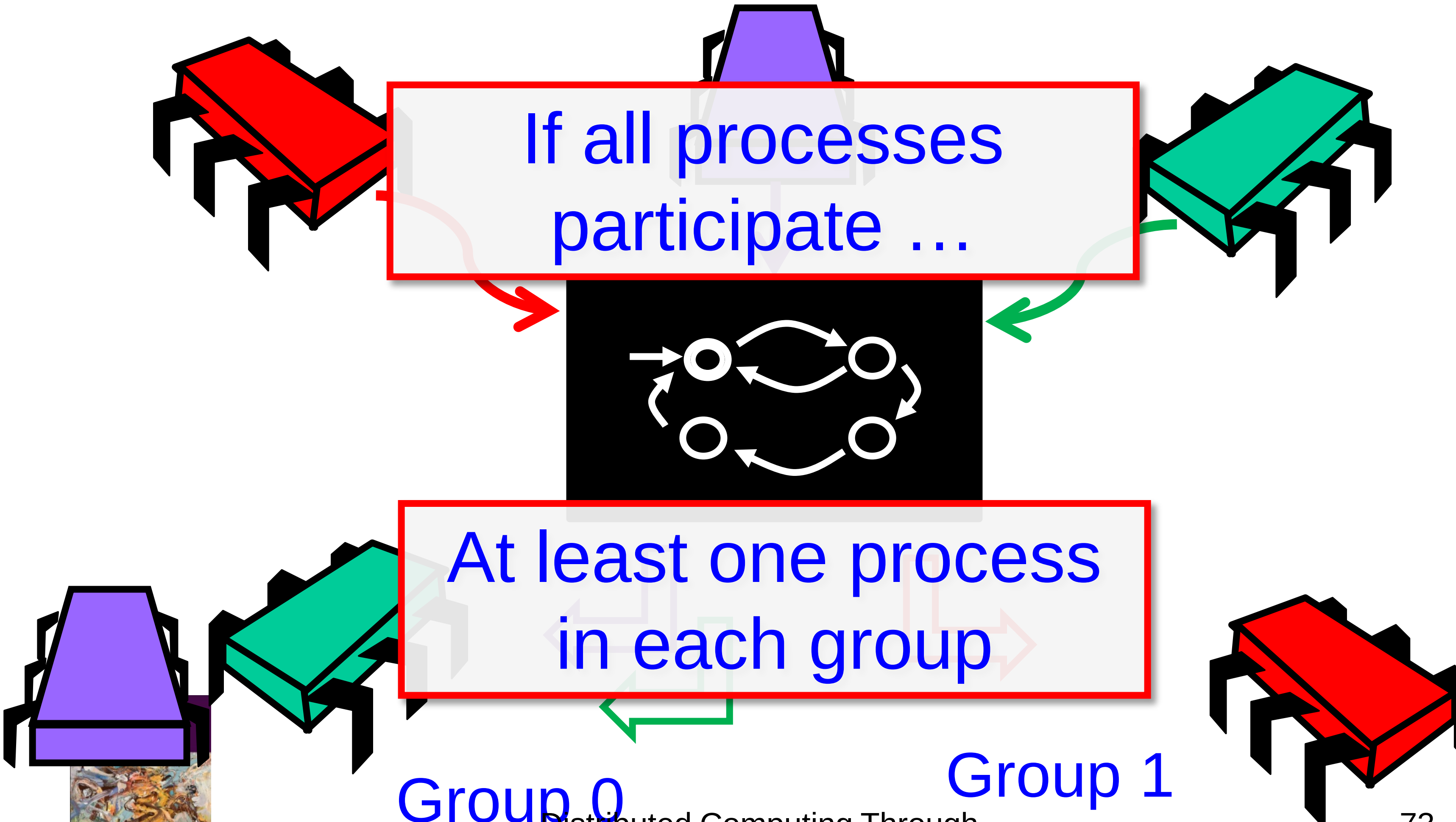
Sperner's Lemma and *k*-Set Agreement

Weak Symmetry-Breaking

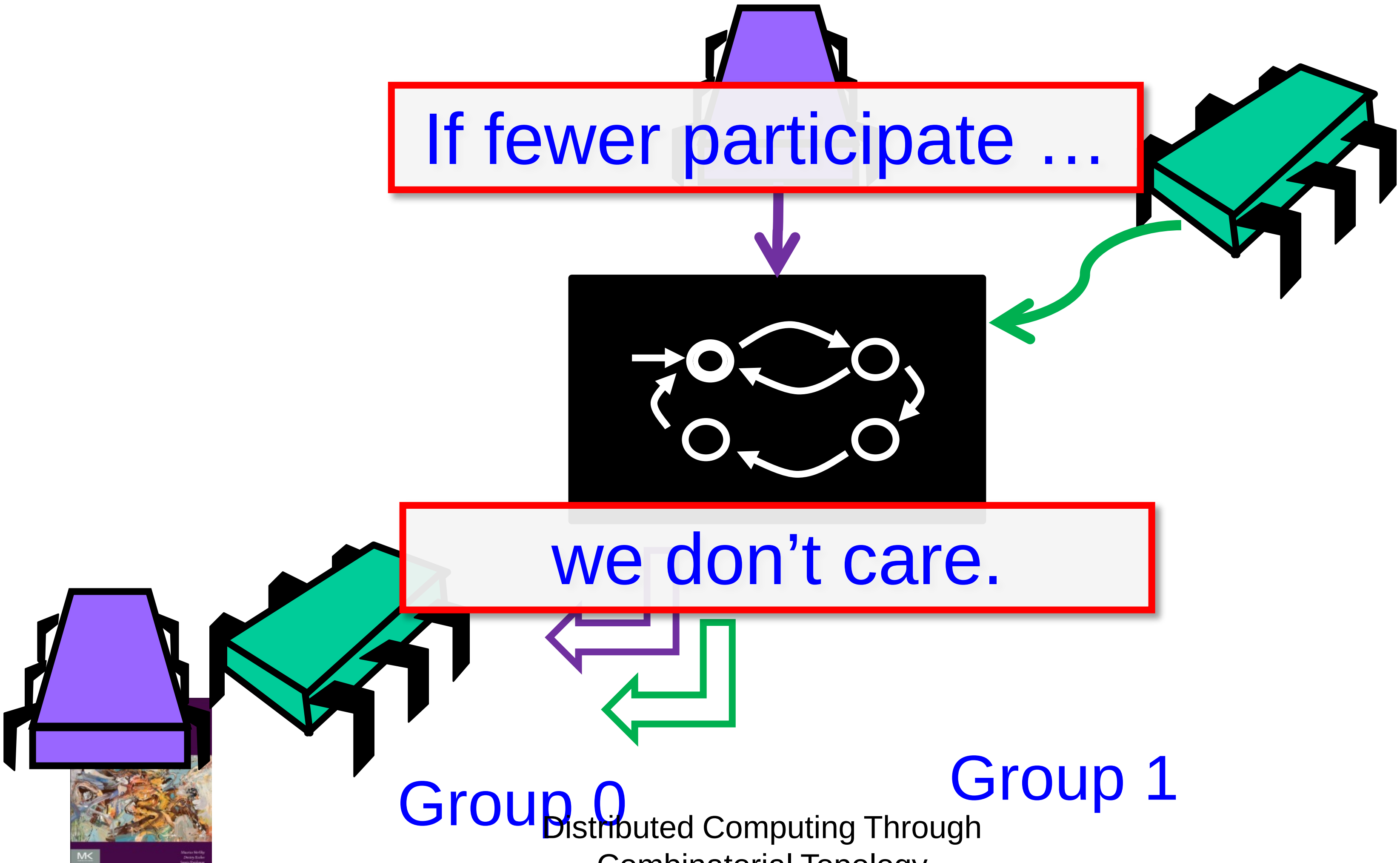
Separation results



Weak Symmetry-Breaking



Weak Symmetry-Breaking



Anonymous Protocols

Trivial solution: choose name parity

WSB protocol should be anonymous

Output depends on …

Input …

But not name

Interleaving …

Restriction on protocol, not task!

Road Map

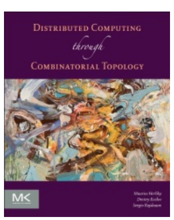
Manifolds

Immediate Snapshot Model

Sperner's Lemma and *k*-Set Agreement

Weak Symmetry-Breaking

Separation results

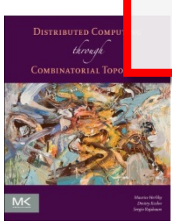


Next Step

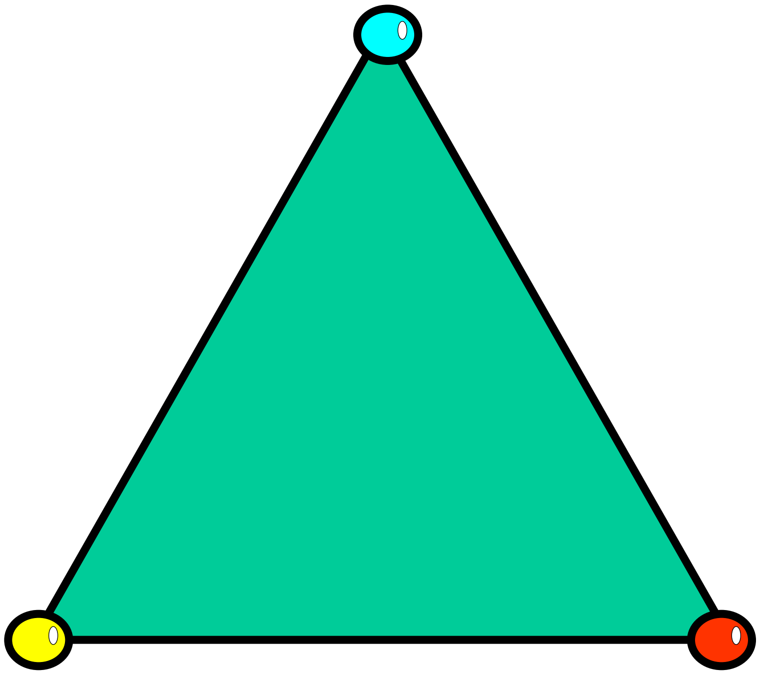
Construct manifold task that *solves* weak-symmetry-breaking

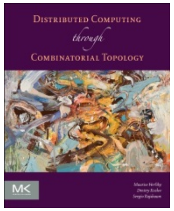
Because it is a manifold, it *cannot* solve *n*-set agreement

Separation: *n*-set agreement is *harder* than WSB

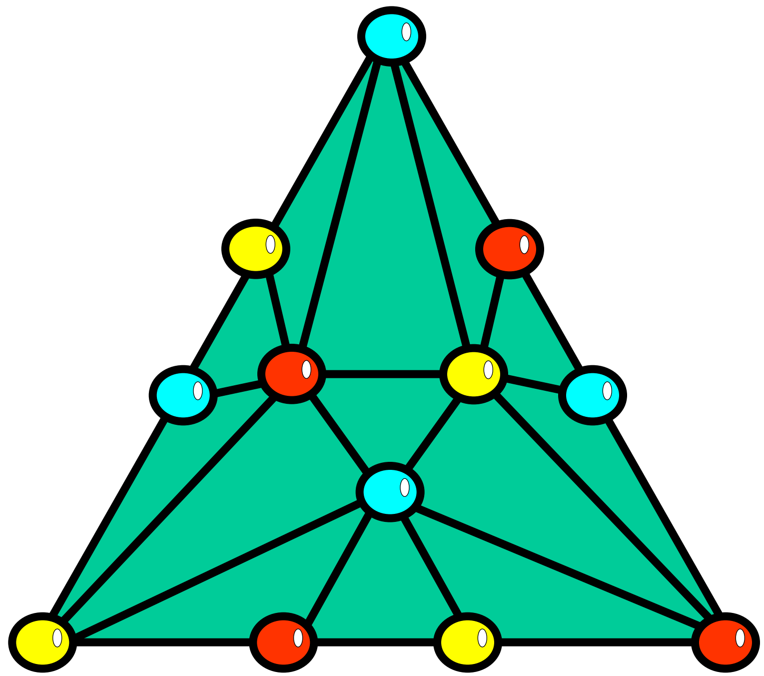


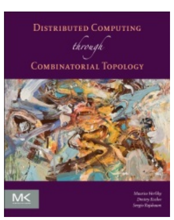
A Simplex



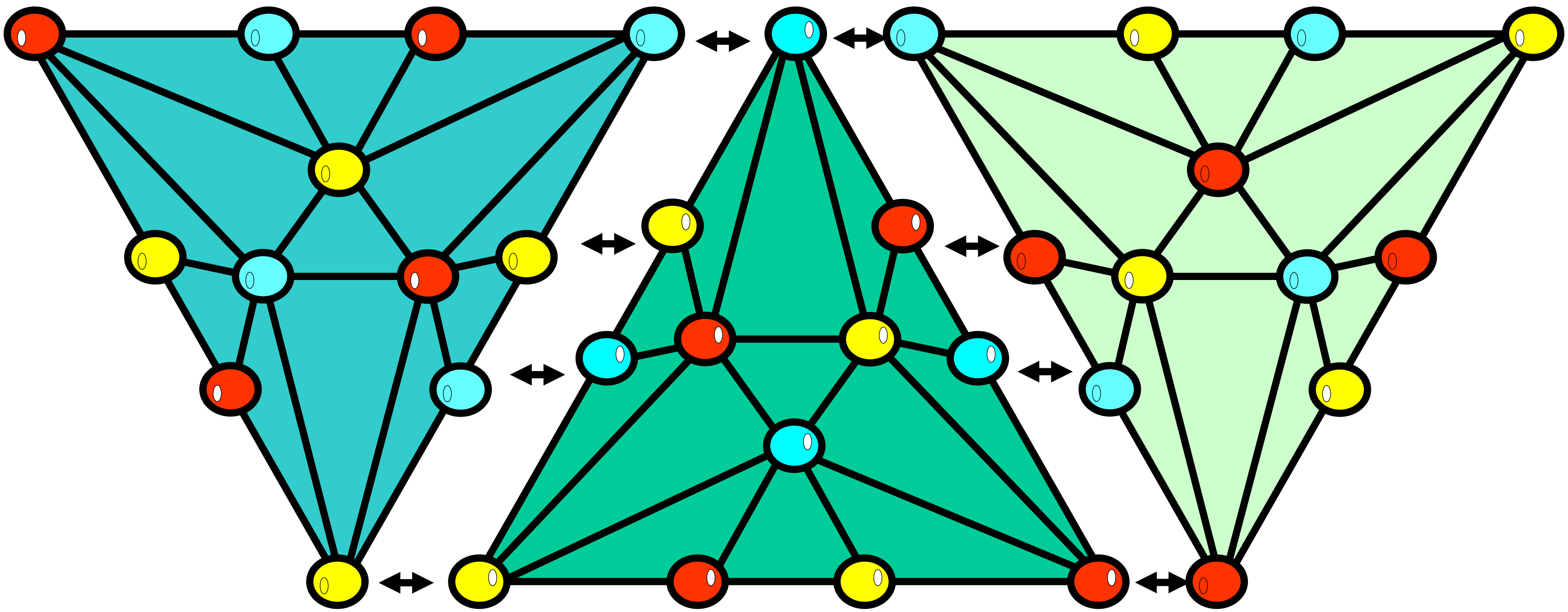

Distributed Computing
through
Combinatorial Topology
MK

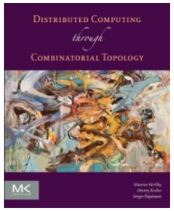
Standard Chromatic Subdivision



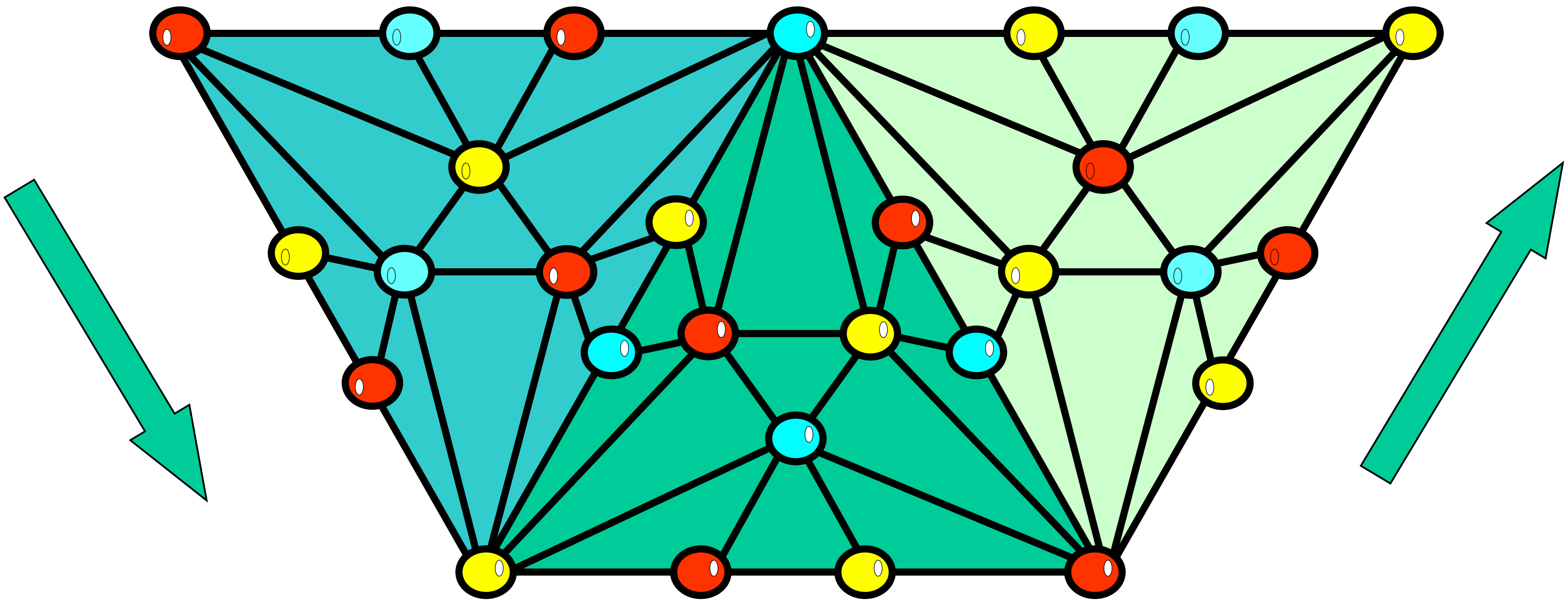


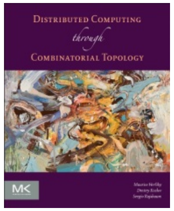
Glue Three Copies Together



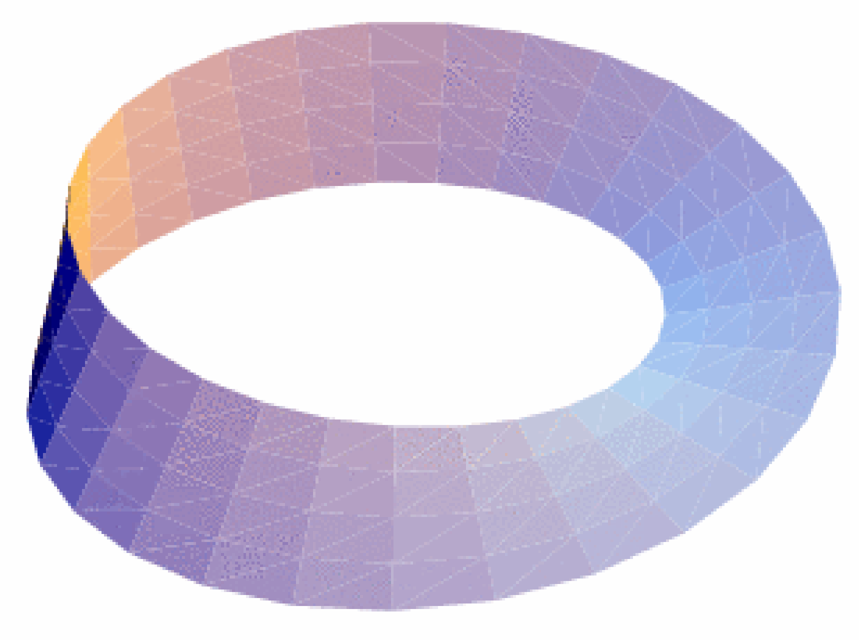


Glue Opposite Edges

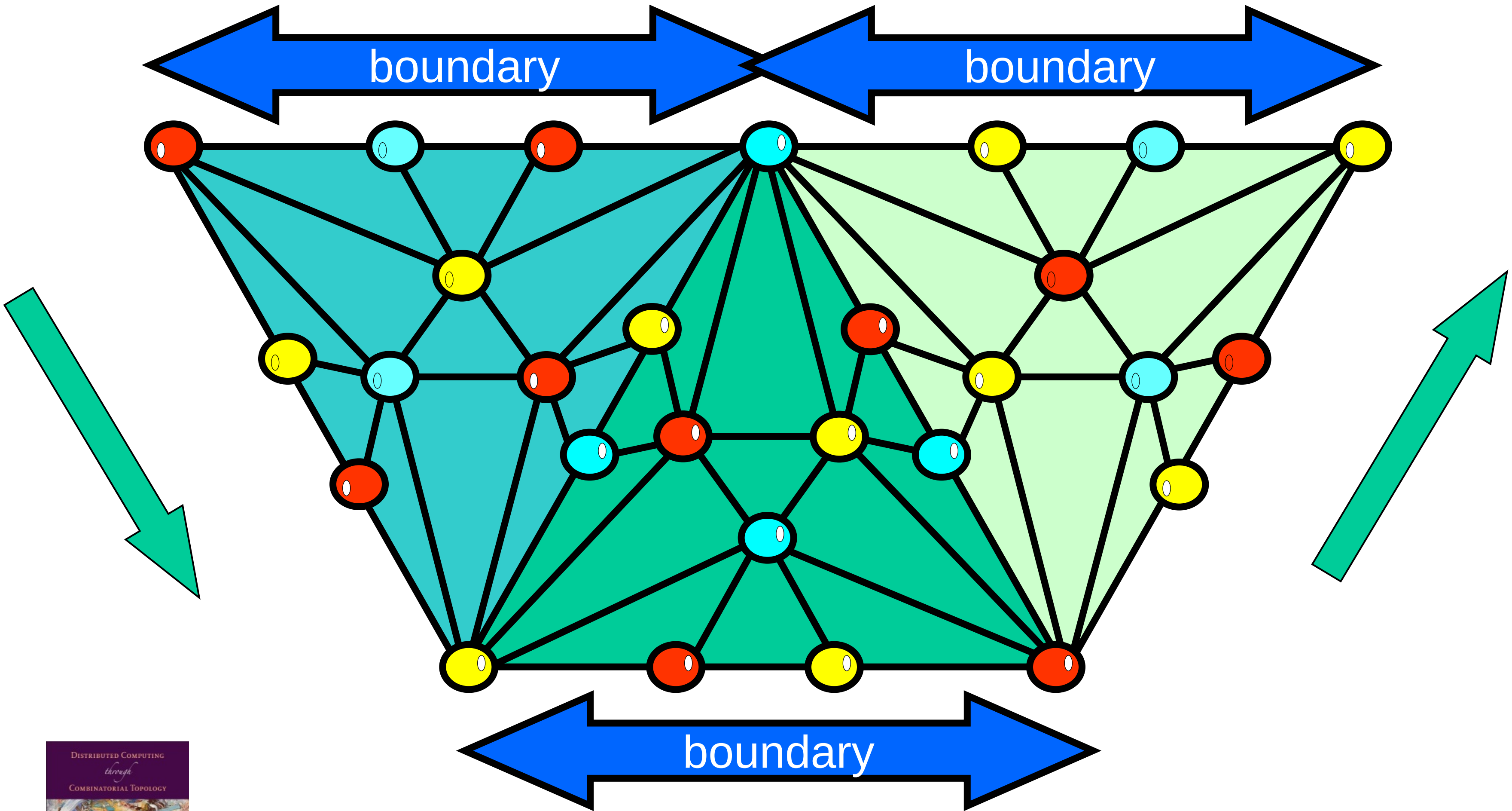


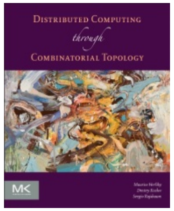


The Moebius Task

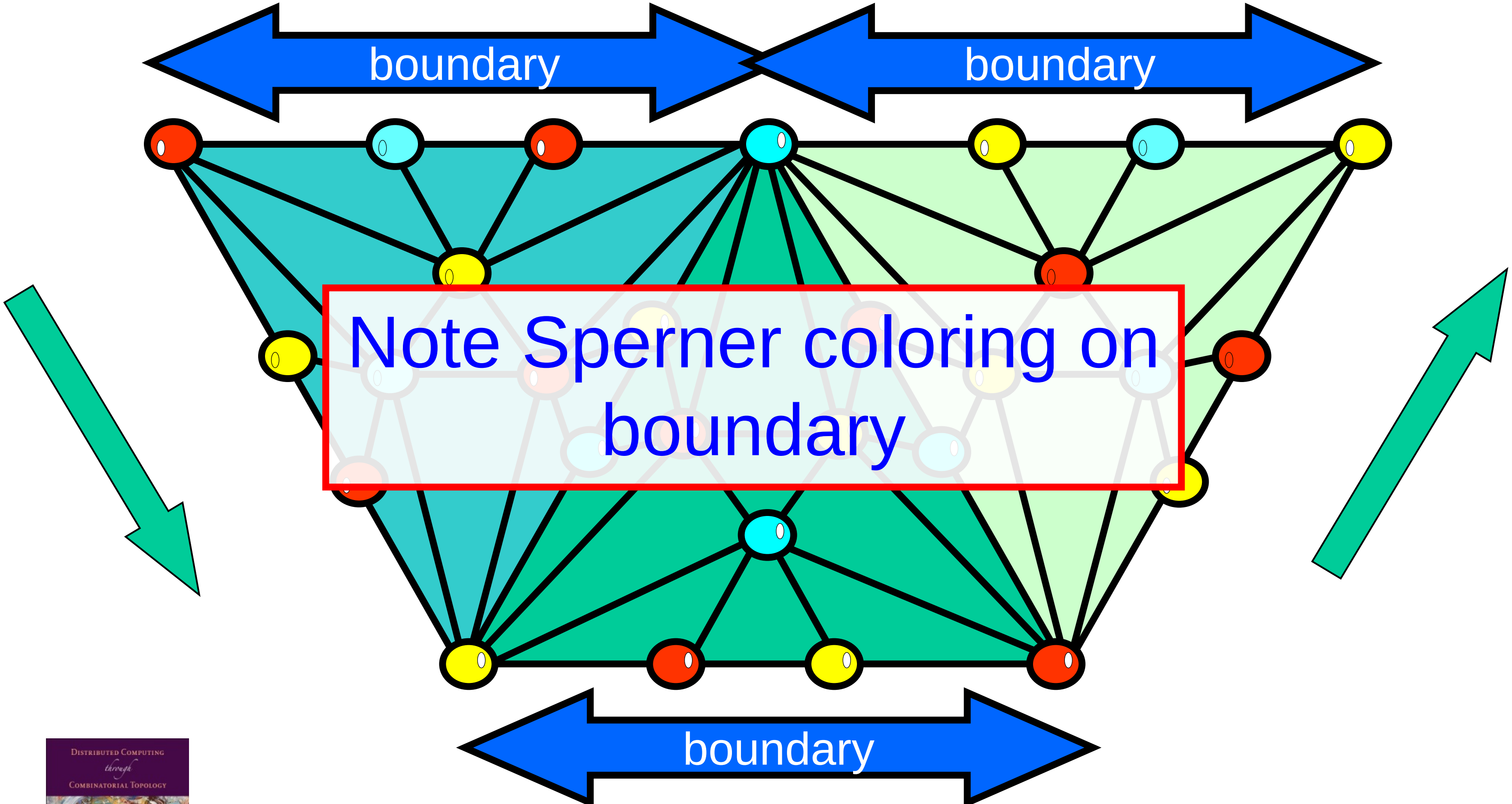


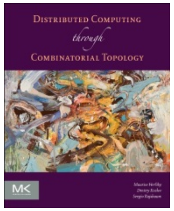
Defines a Manifold Task





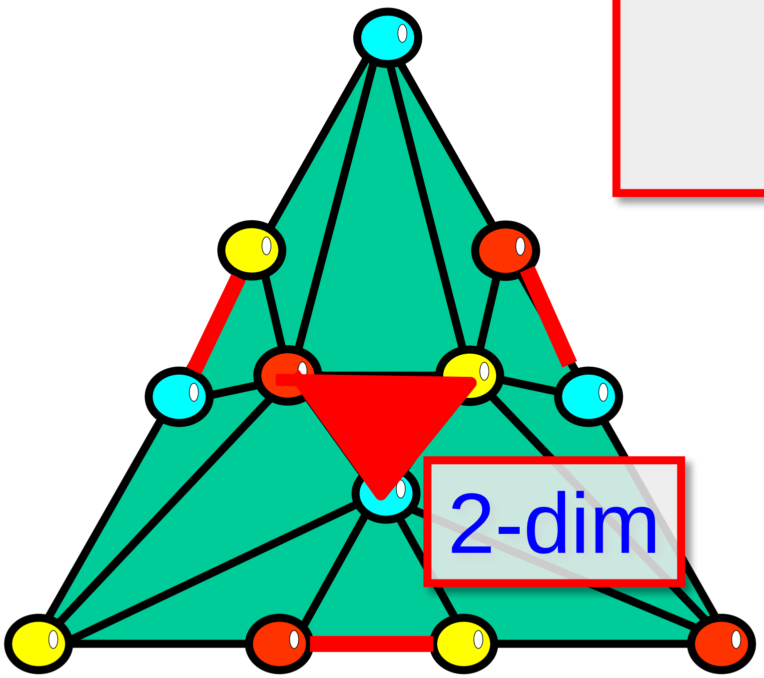
Manifold Task



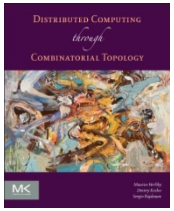


Terminology

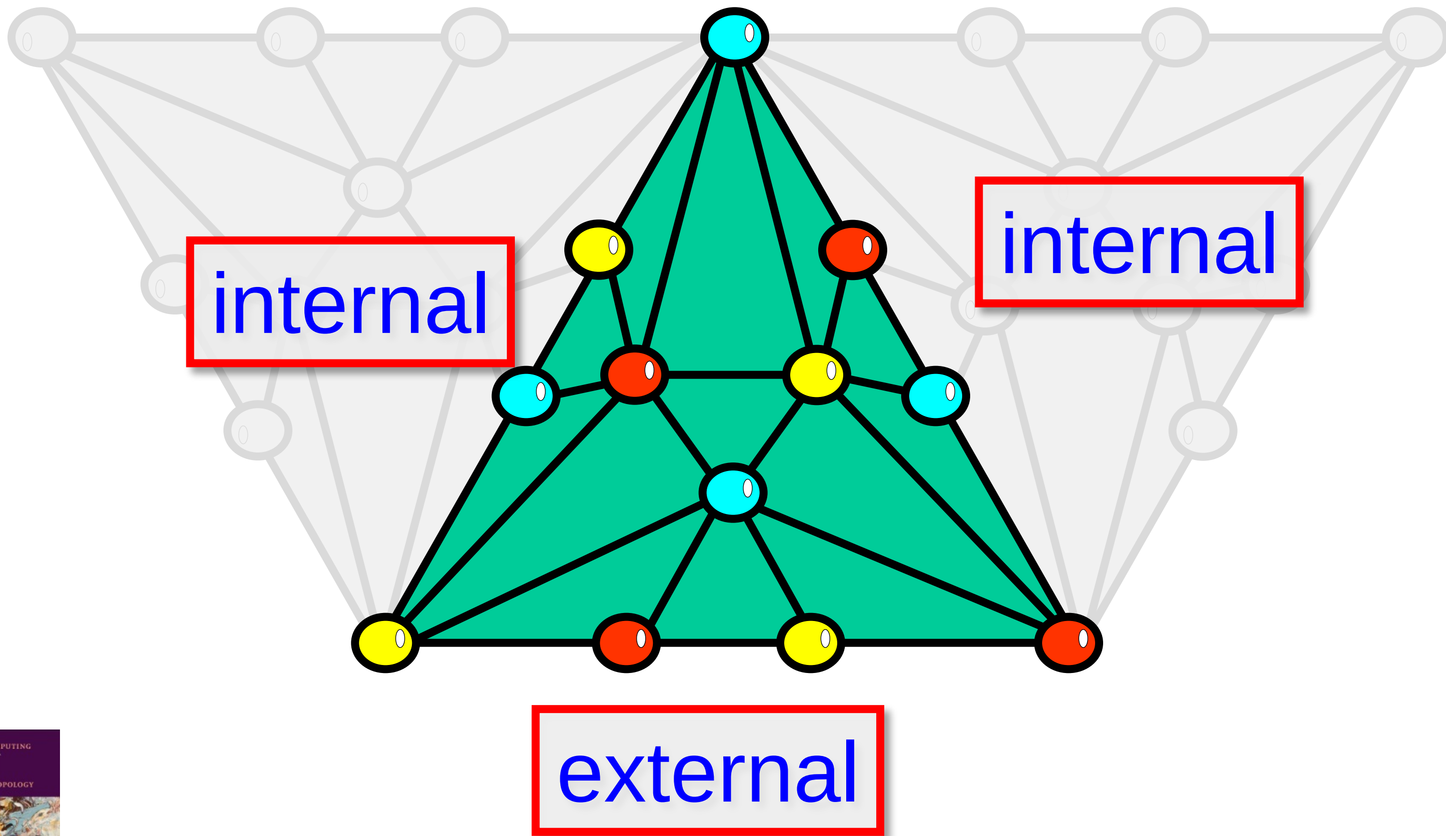
Each face has a central simplex

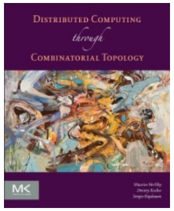


1-dim



Subdivided Faces



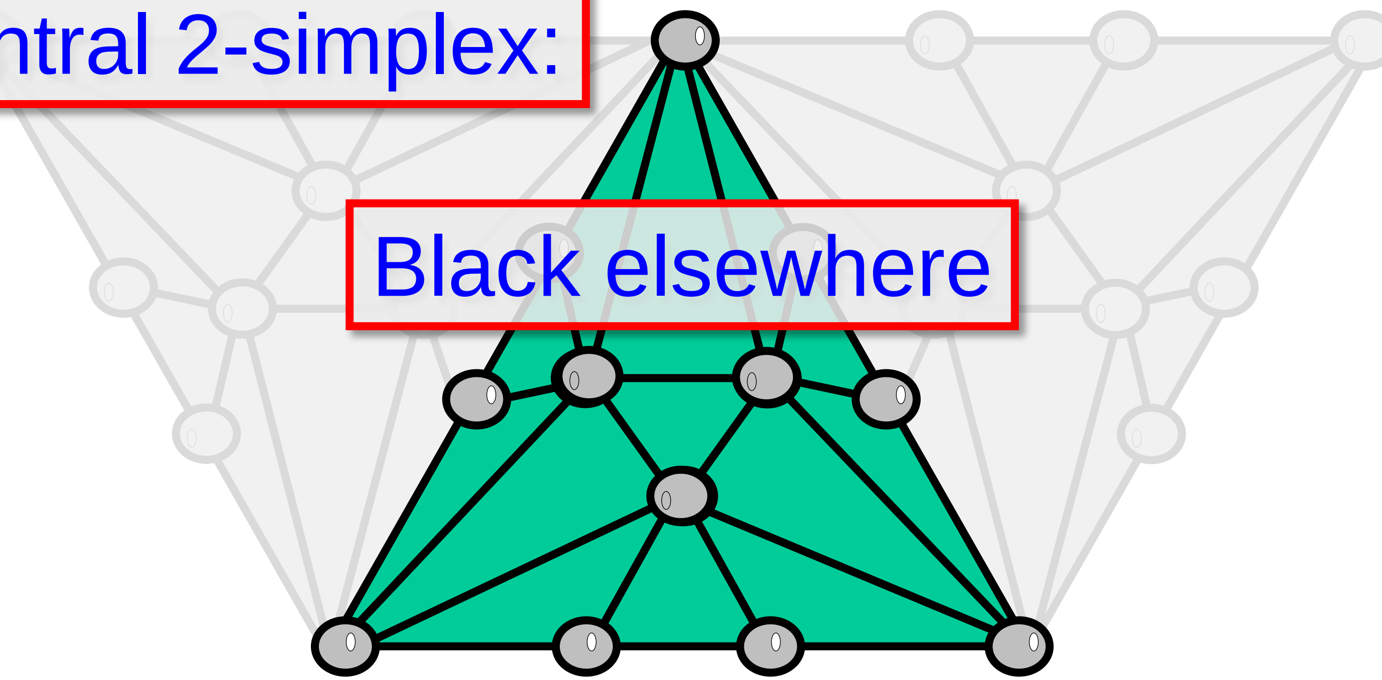


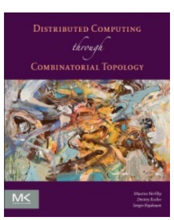
Black-and-White Coloring (I)

Central 2-simplex:

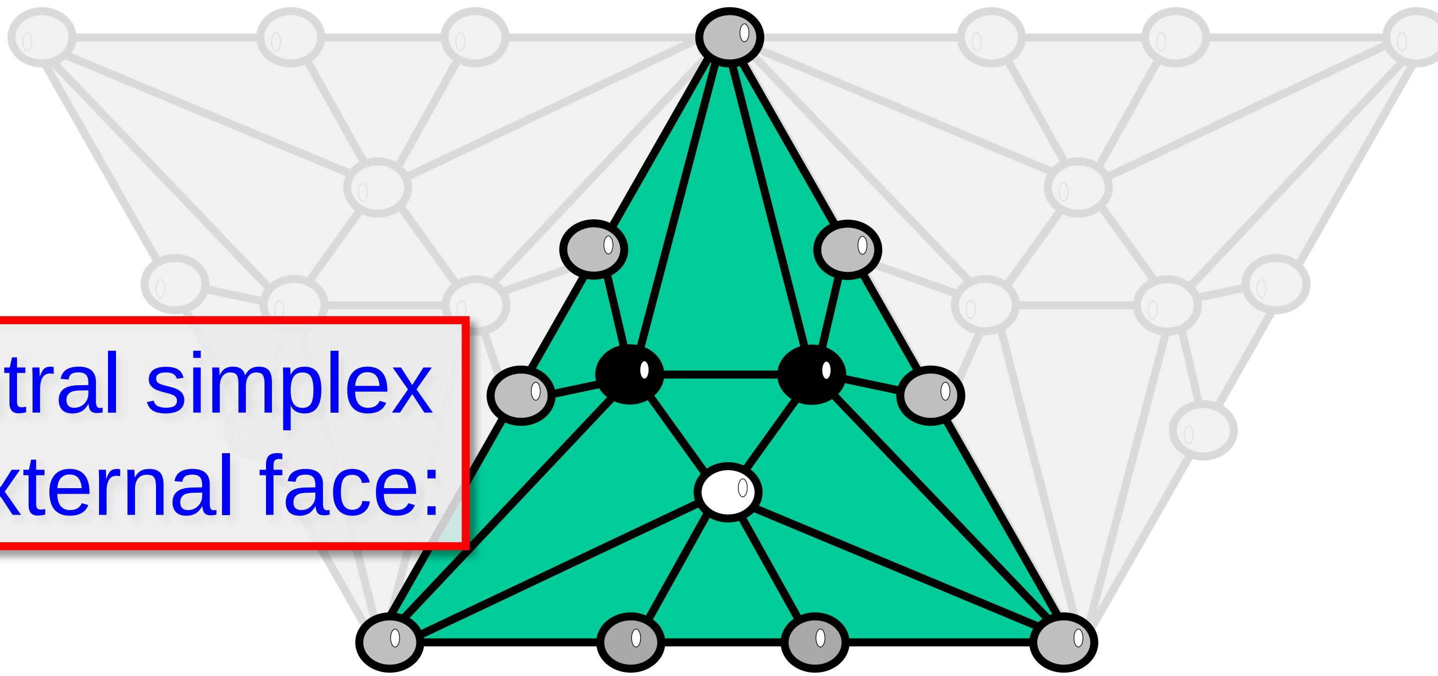
Black elsewhere

White near external face



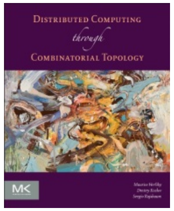


Black-and-White Coloring (II)

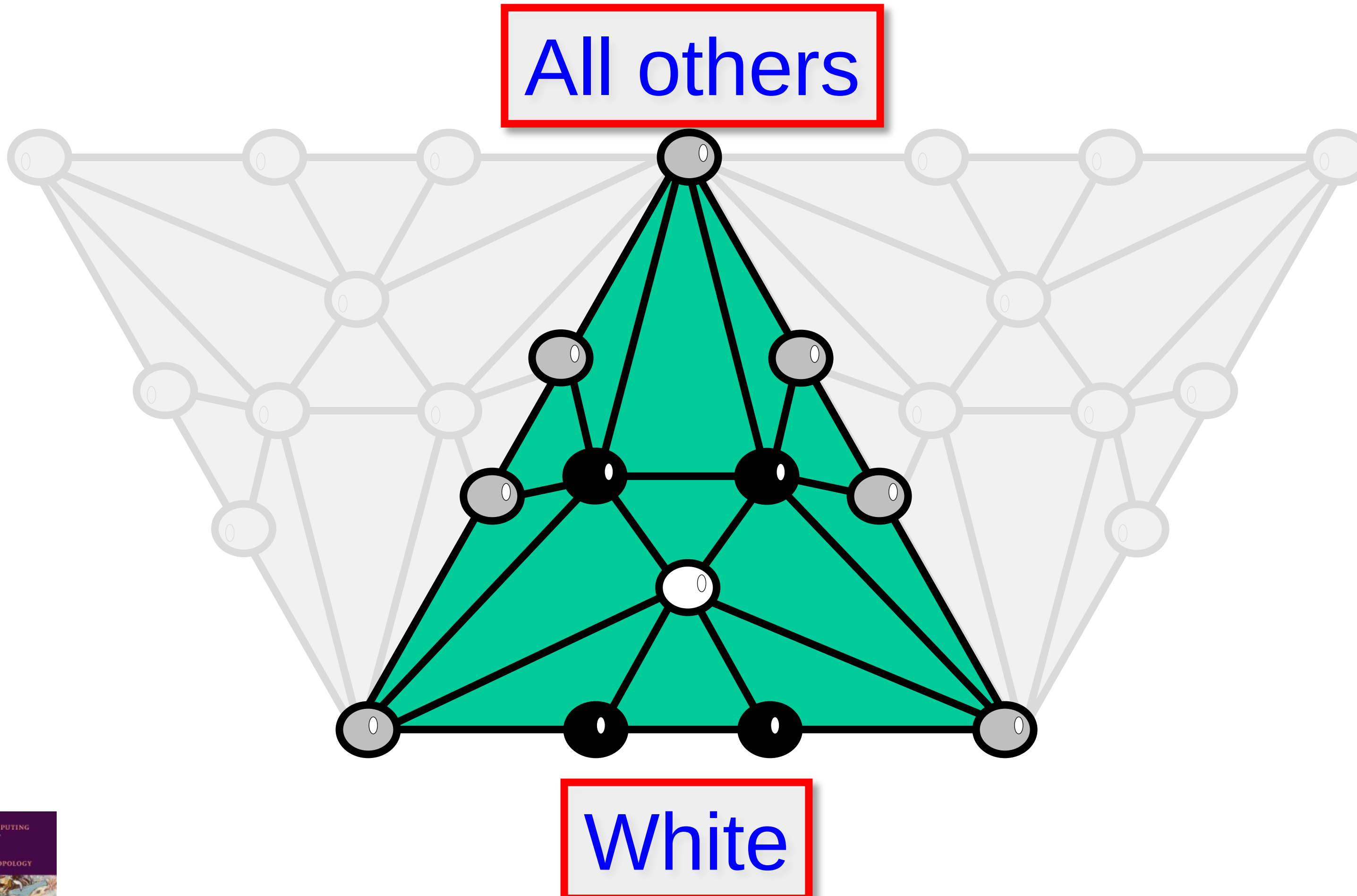


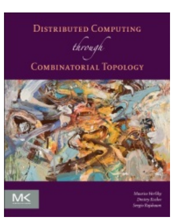
Central simplex of external face:

Black

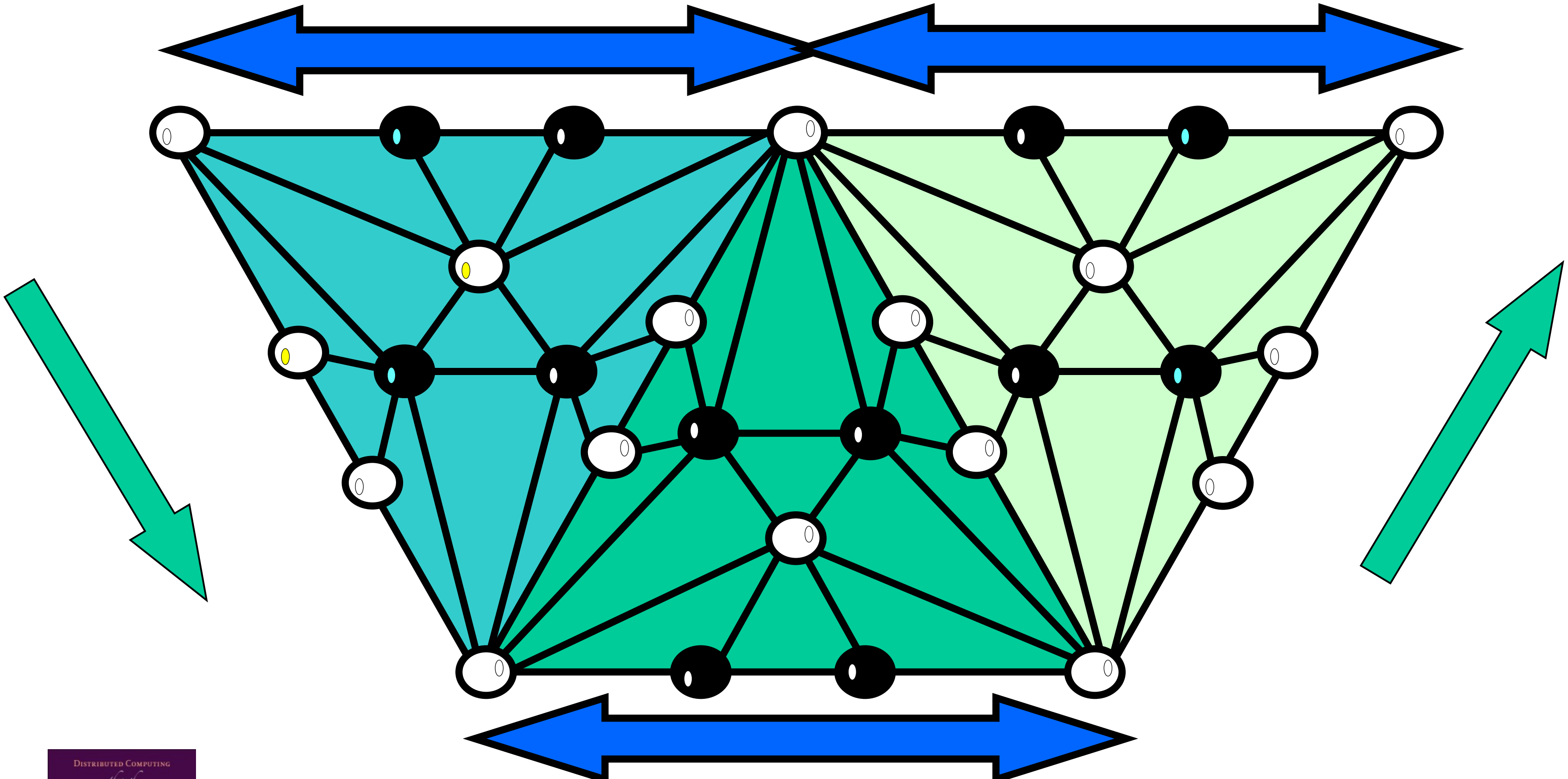


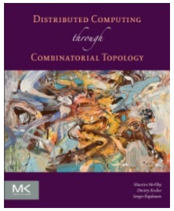
Black-and-White Coloring (III)



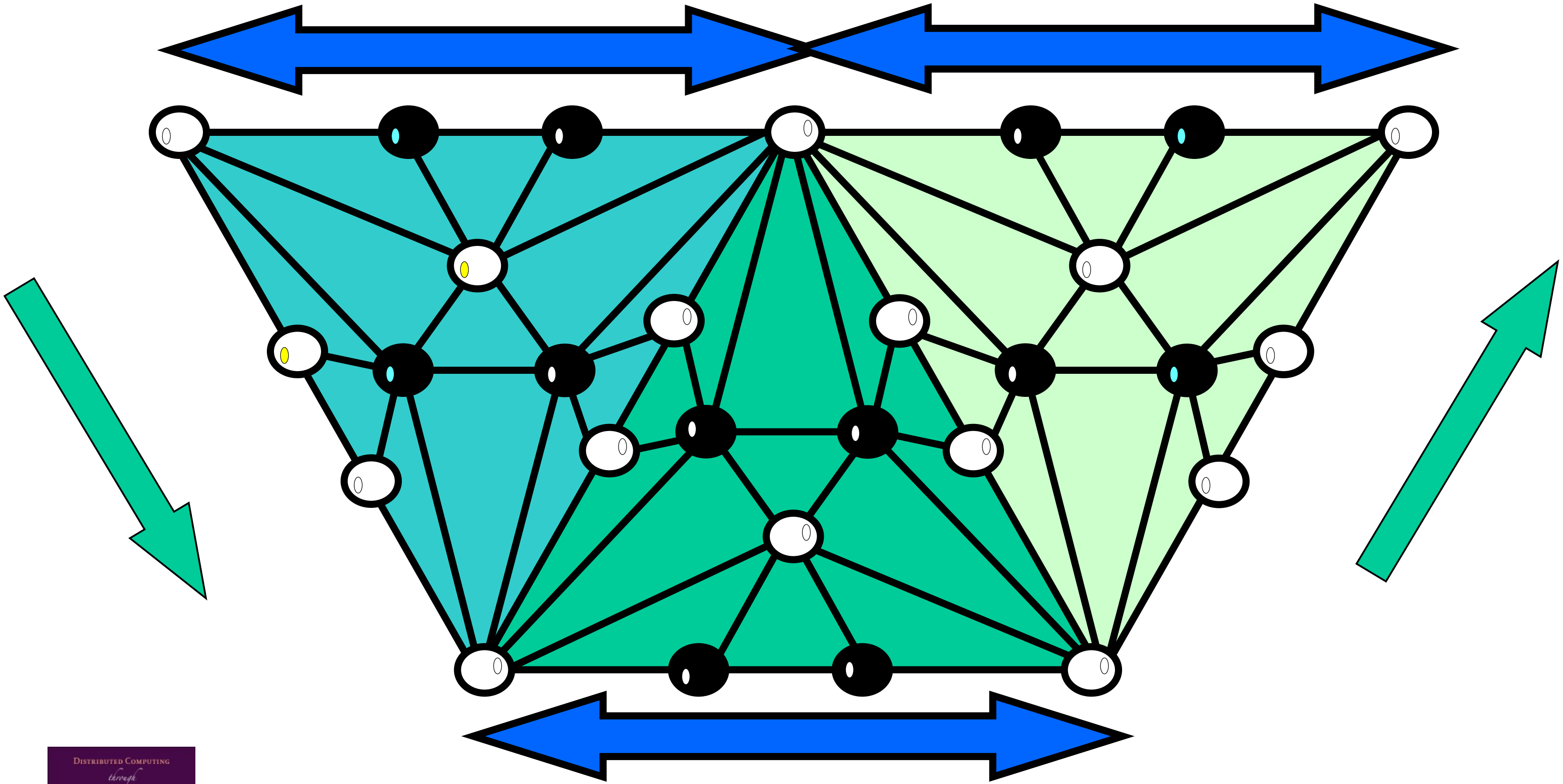


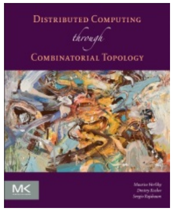
Moebius Solves WSB



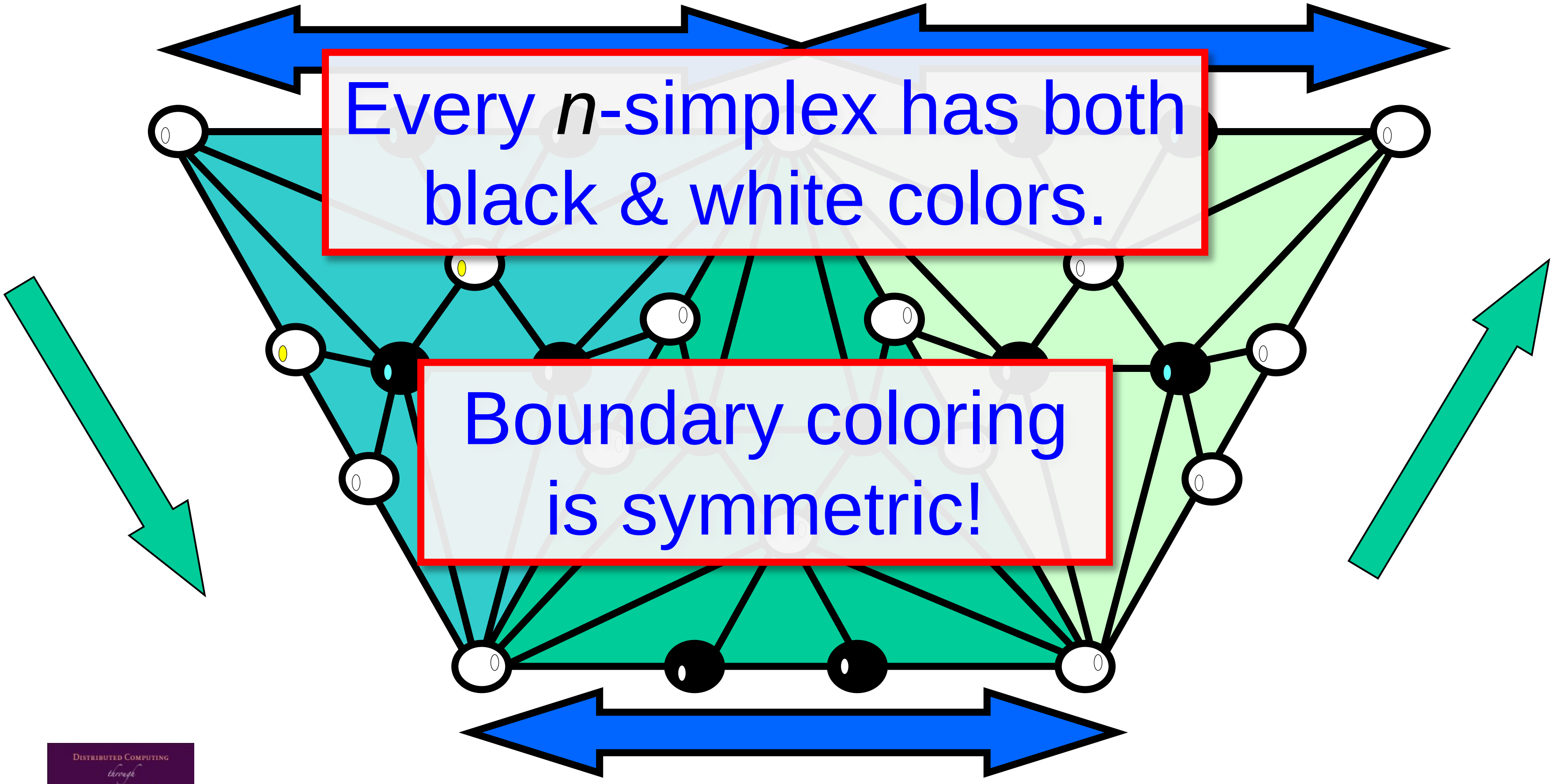


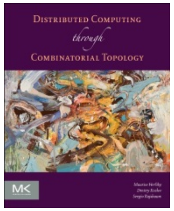
Moebius Solves WSB





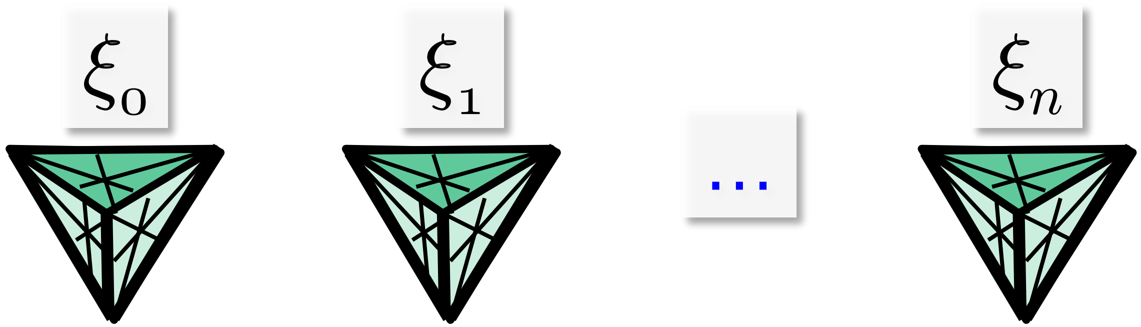
Moebius Solves WSB





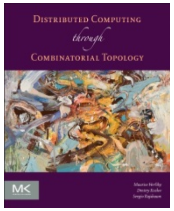
General Construction

$n = 2N$

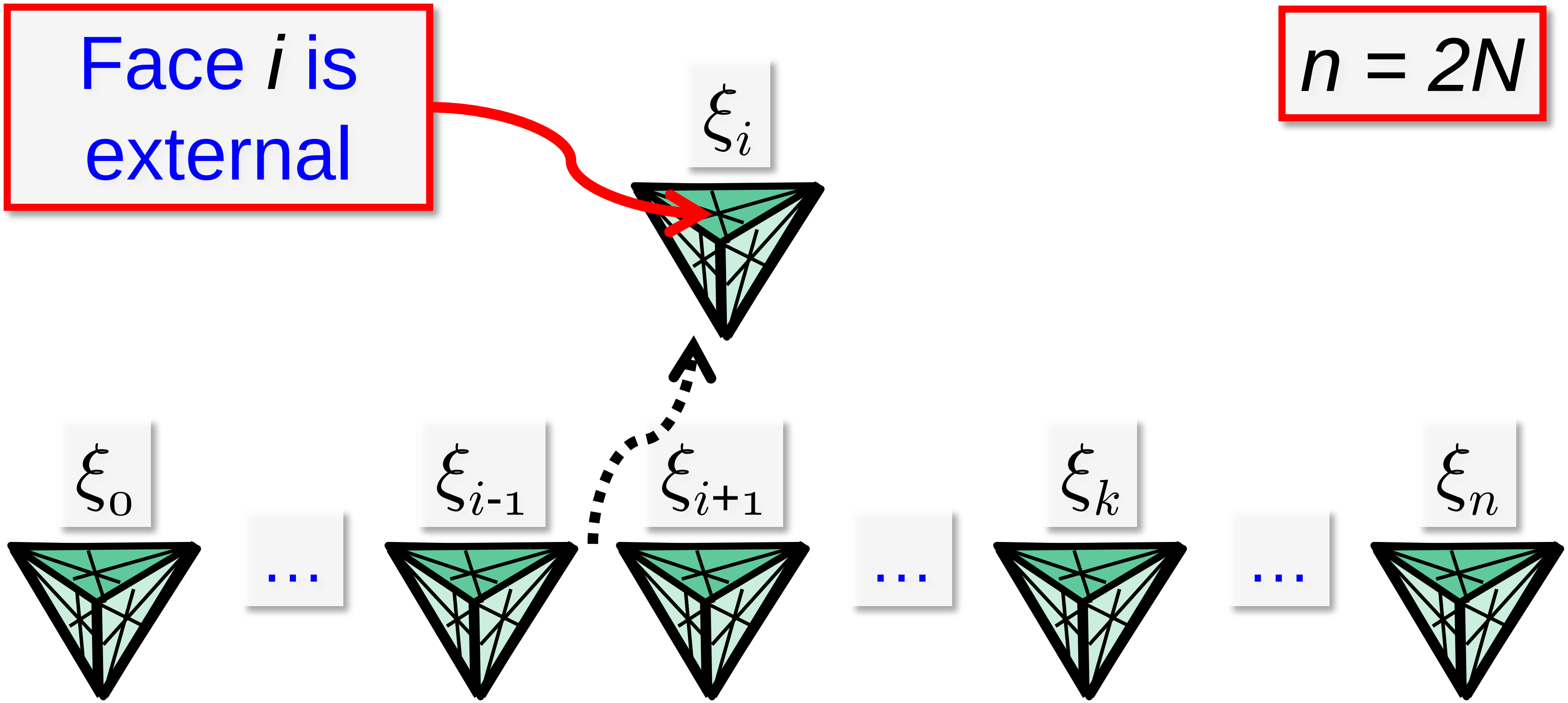


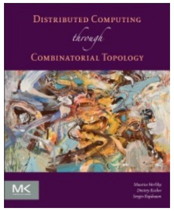
(picture not exact!)

$n+1$ "copies" $\xi_0, \ldots \xi_n$ of $\mathcal{IS}(\sigma)$

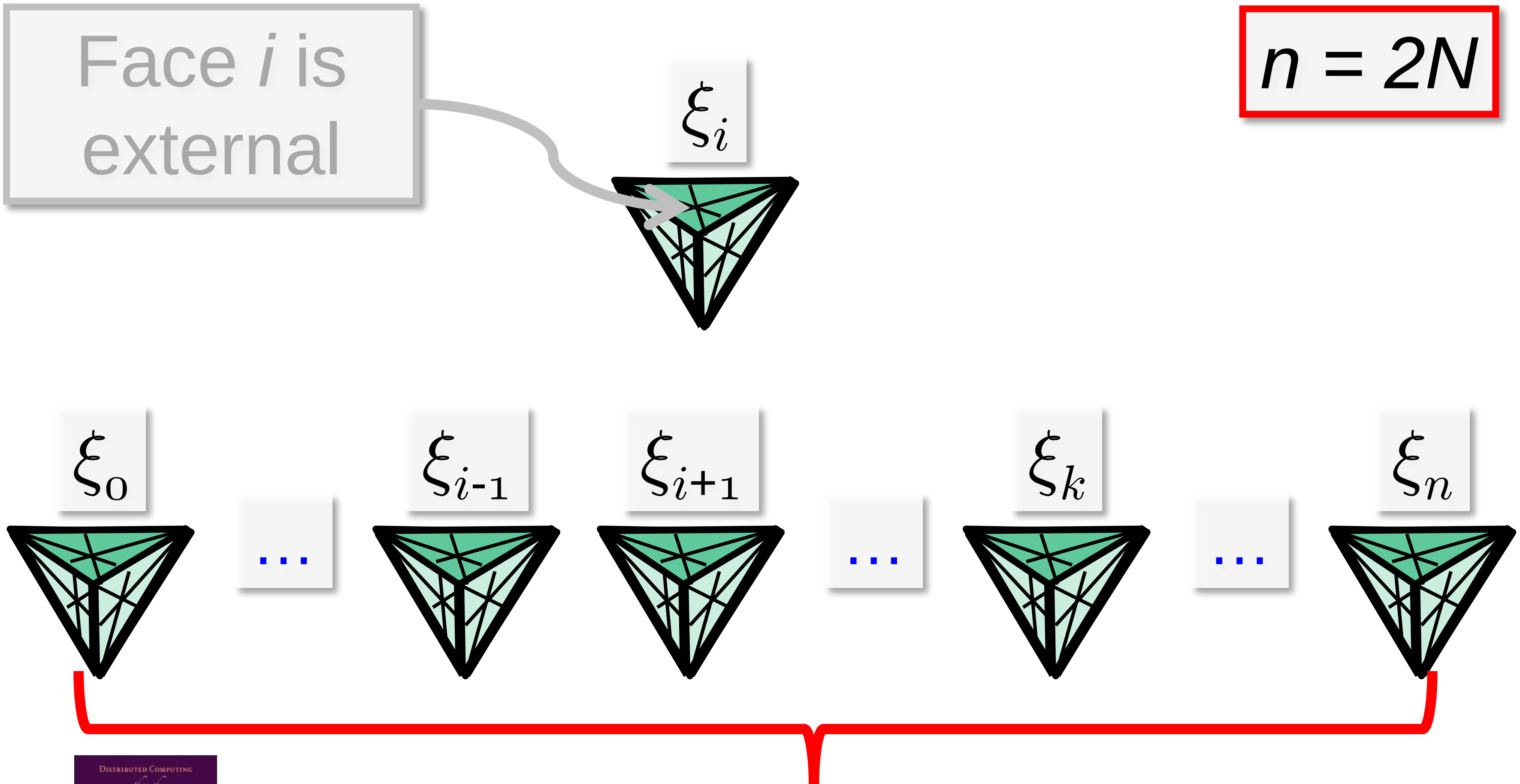


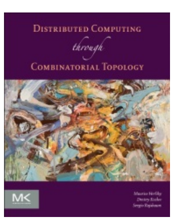
General Construction





General Construction



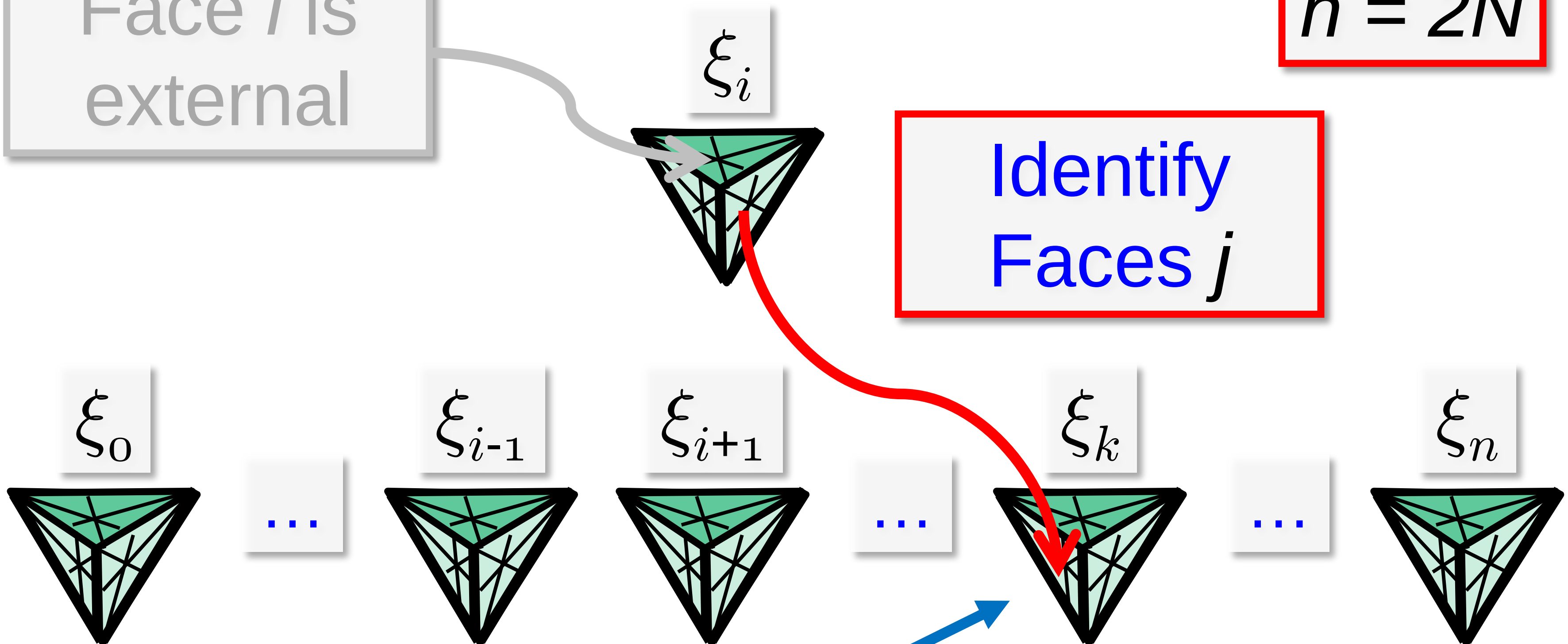


General Construction

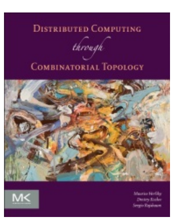
Face *i* is external

n = 2*N*

Identify Faces *j*



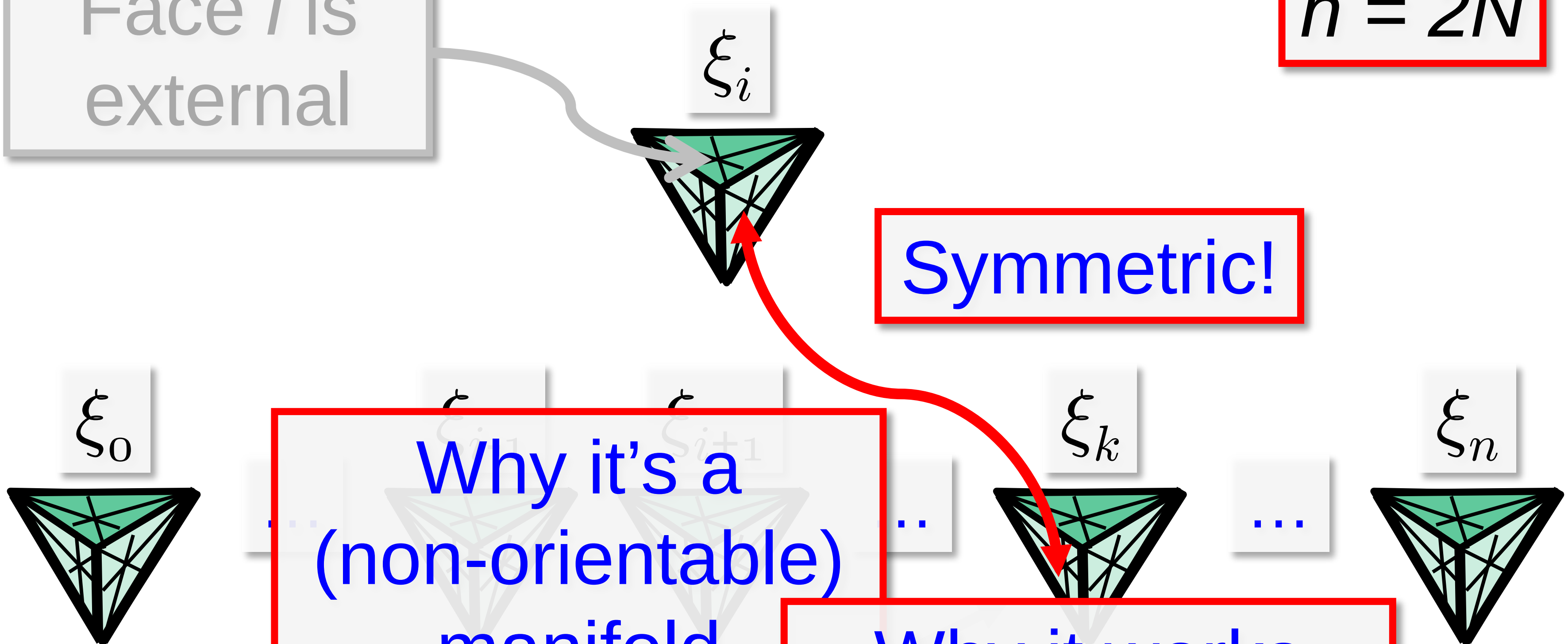
Rank *j*+*N* out of 2*N*



General Construction

Face i is external

$n = 2N$

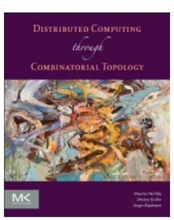


Symmetric!

Why it's a (non-orientable) manifold

Why it works only for even dimensions

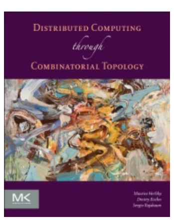
Rank $j+N$ out of $2N$



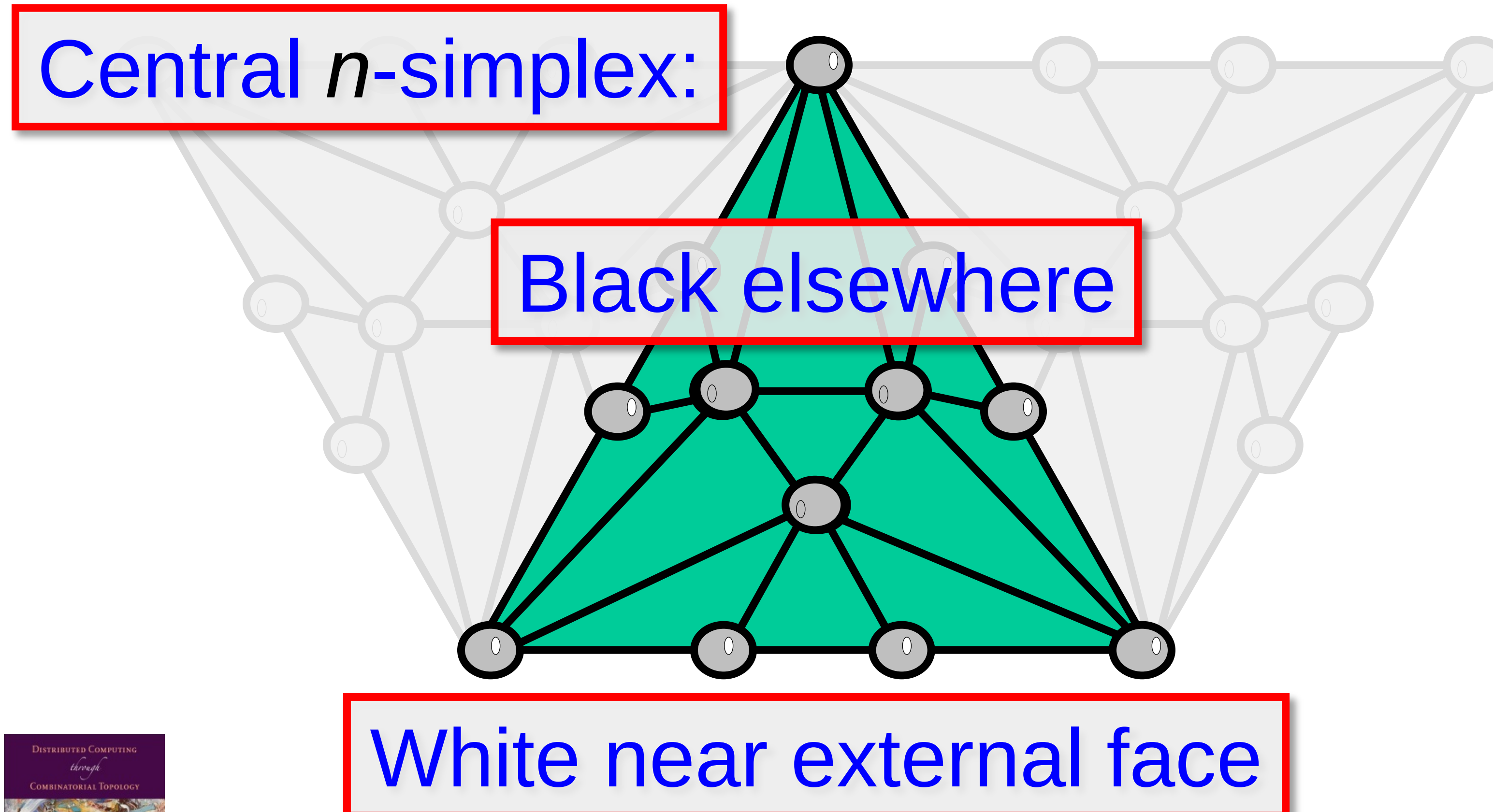
Open Problem

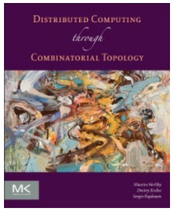
Generalize to odd dimensions …

or find counterexample.

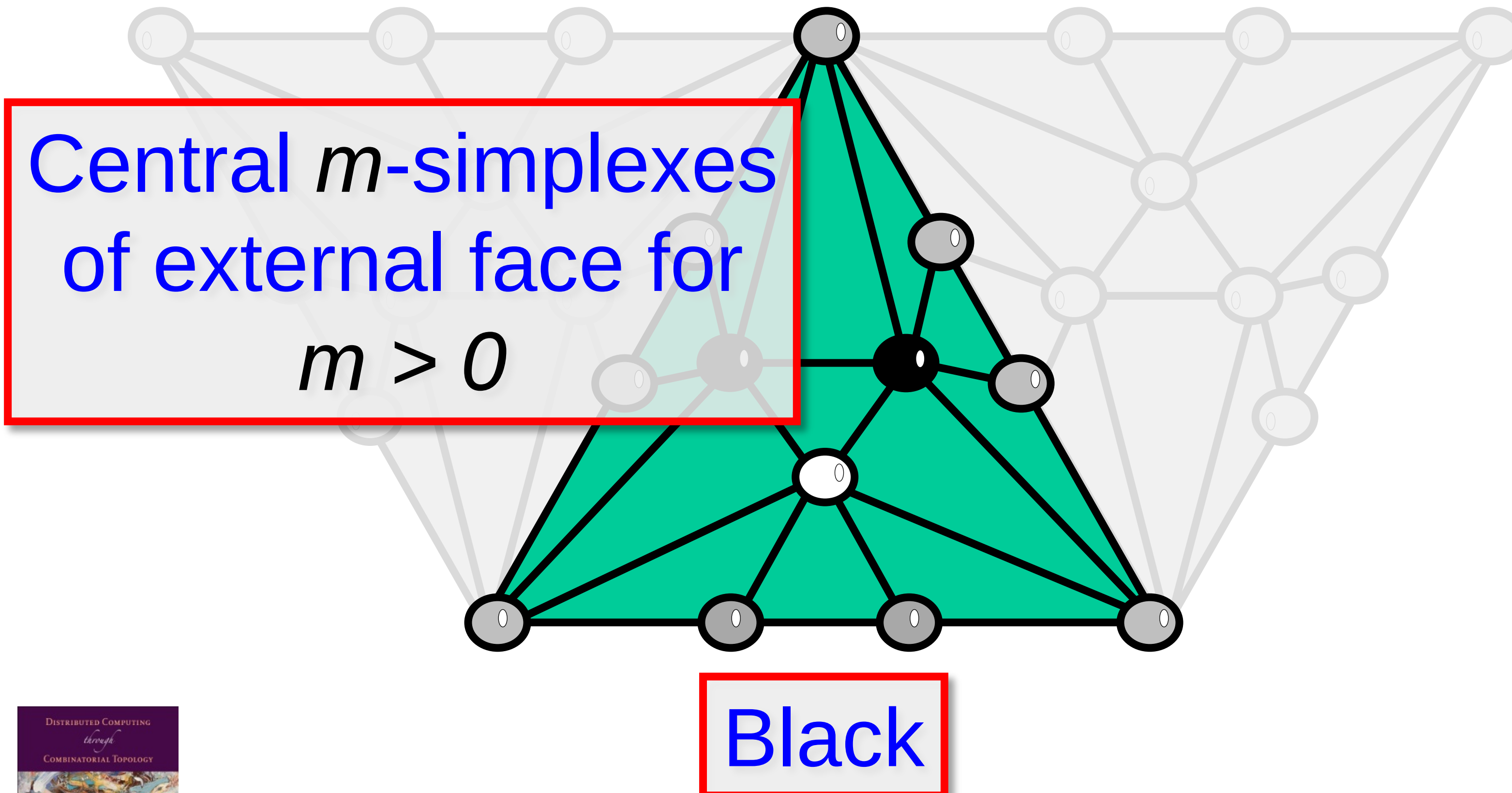


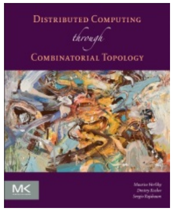
Black-and-White Coloring (I)



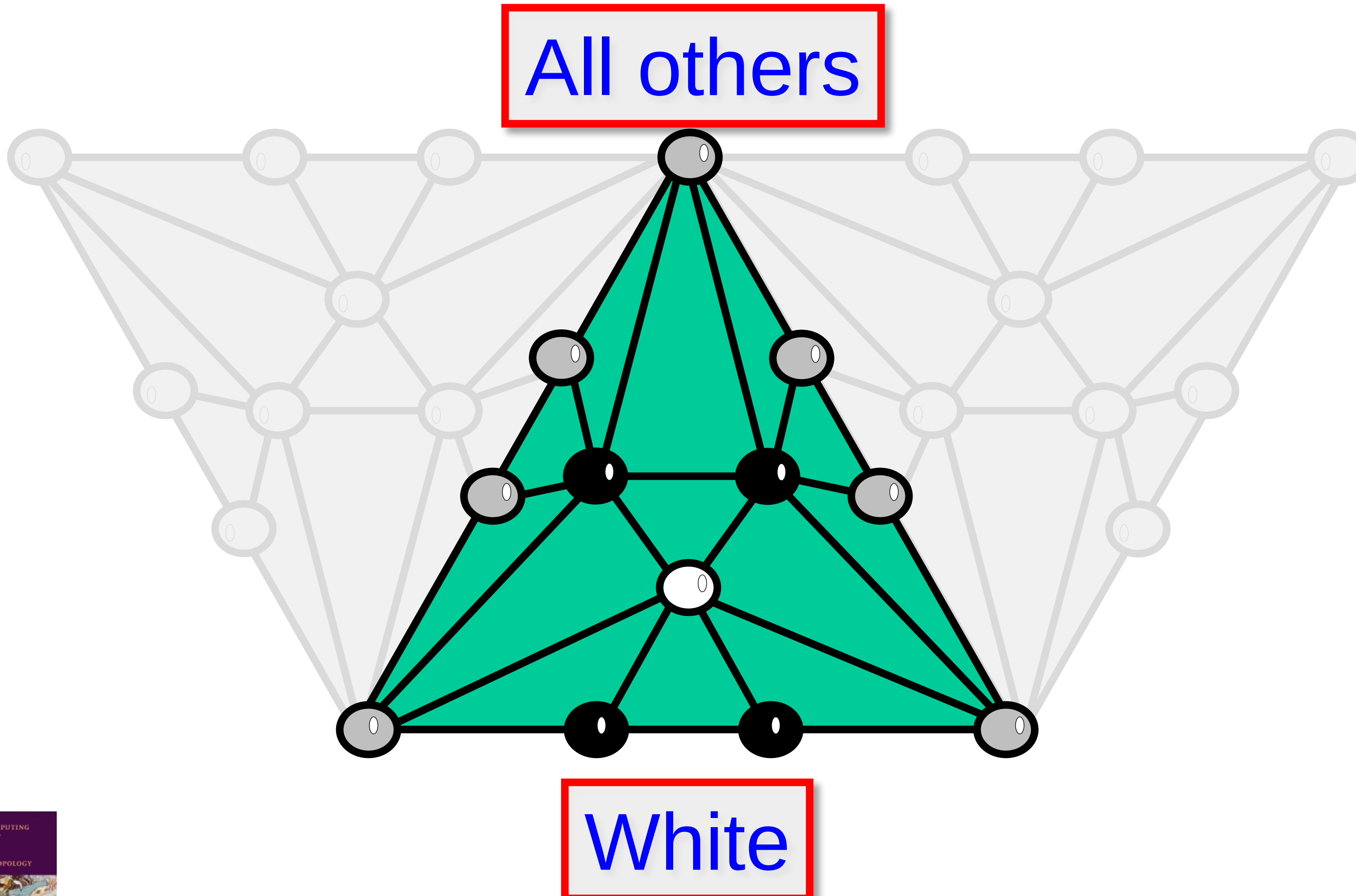


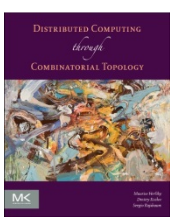
Black-and-White Coloring (II)





Black-and-White Coloring (III)

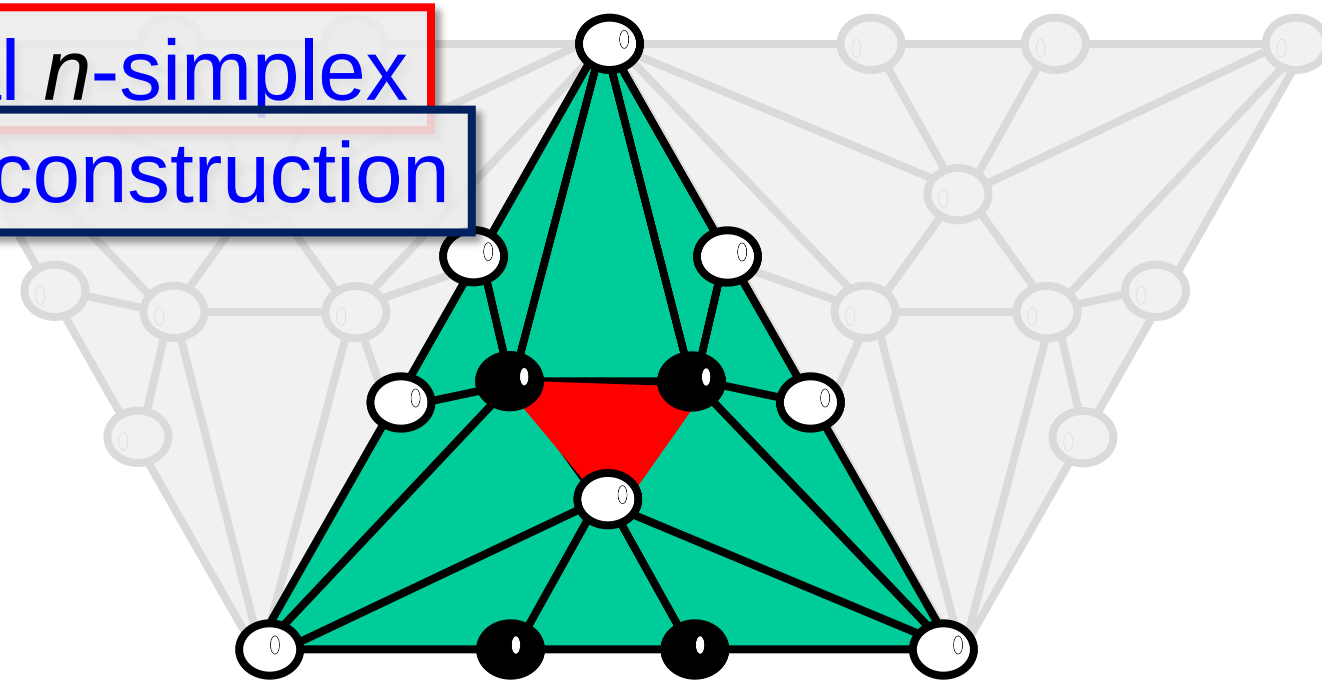


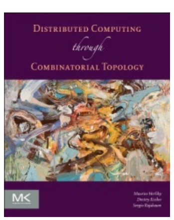


No Monochromatic *n*-simplexes

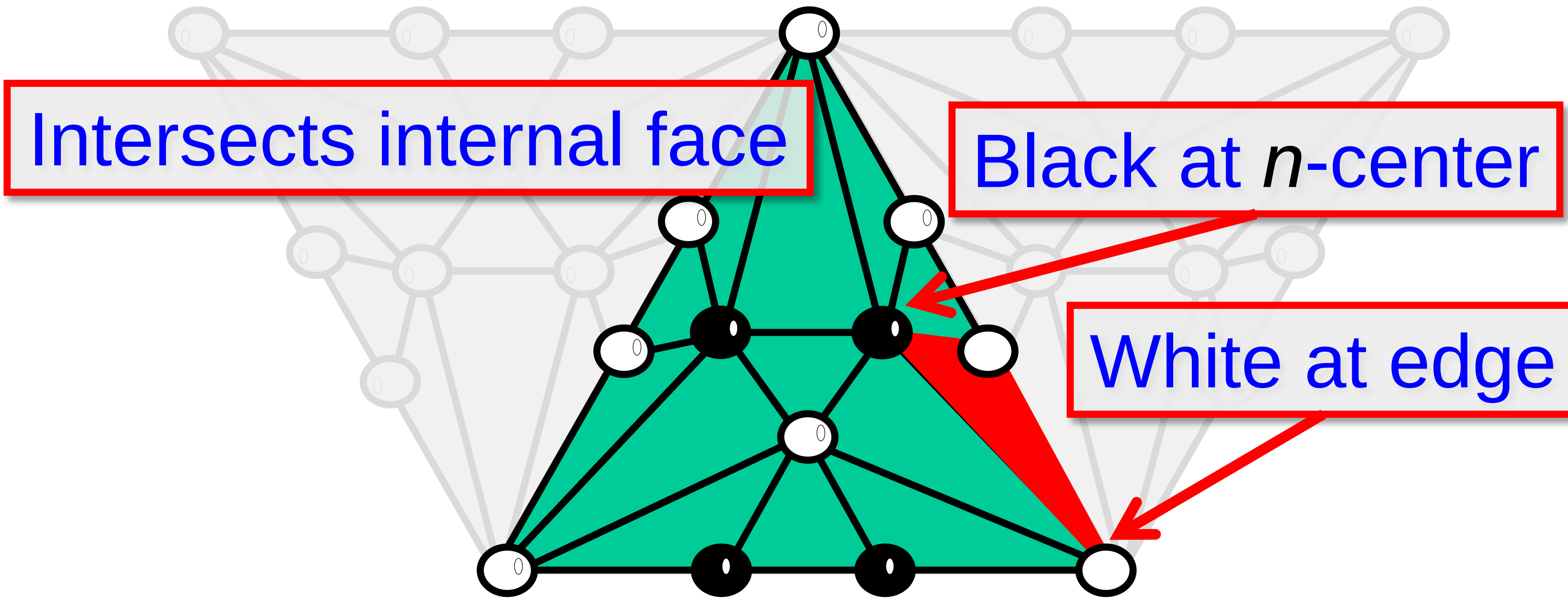
Central *n*-simplex

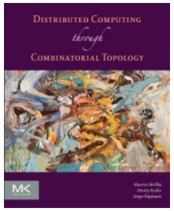
By construction



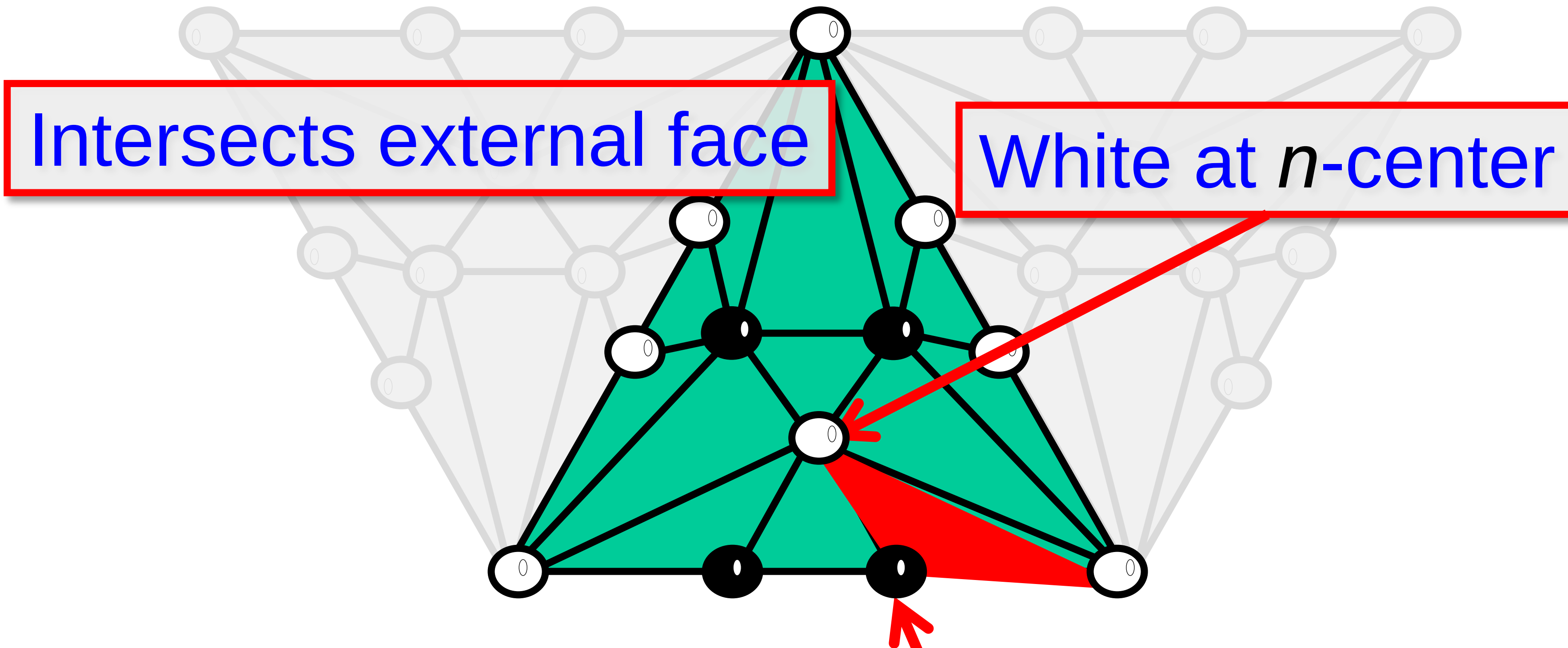


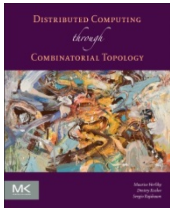
No Monochromatic *n*-simplexes





No Monochromatic *n*-simplexes





Progress

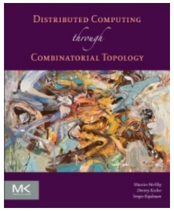
Weak Symmetry-Breaking

Moebius Task

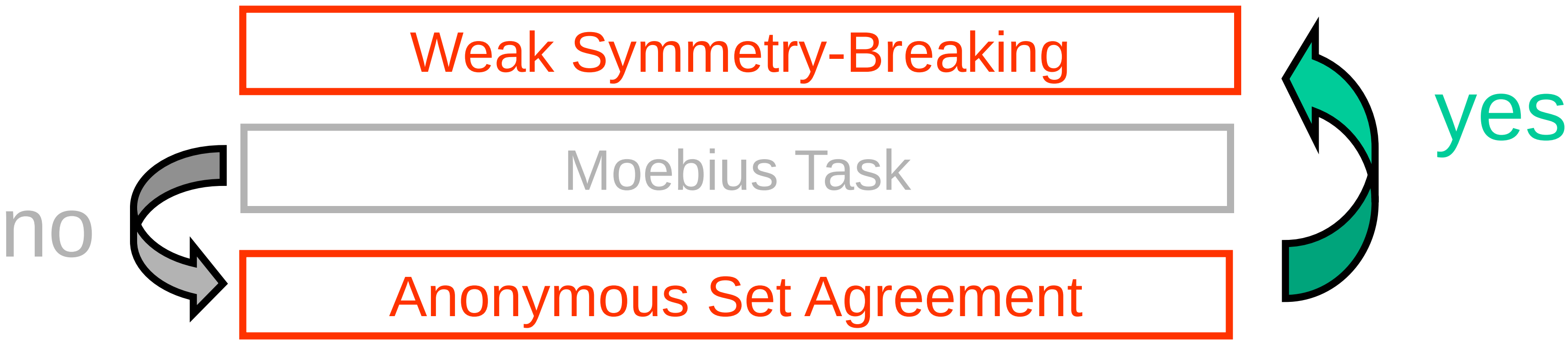
Set Agreement

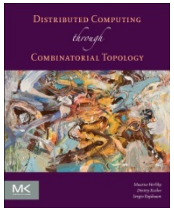
yes

no

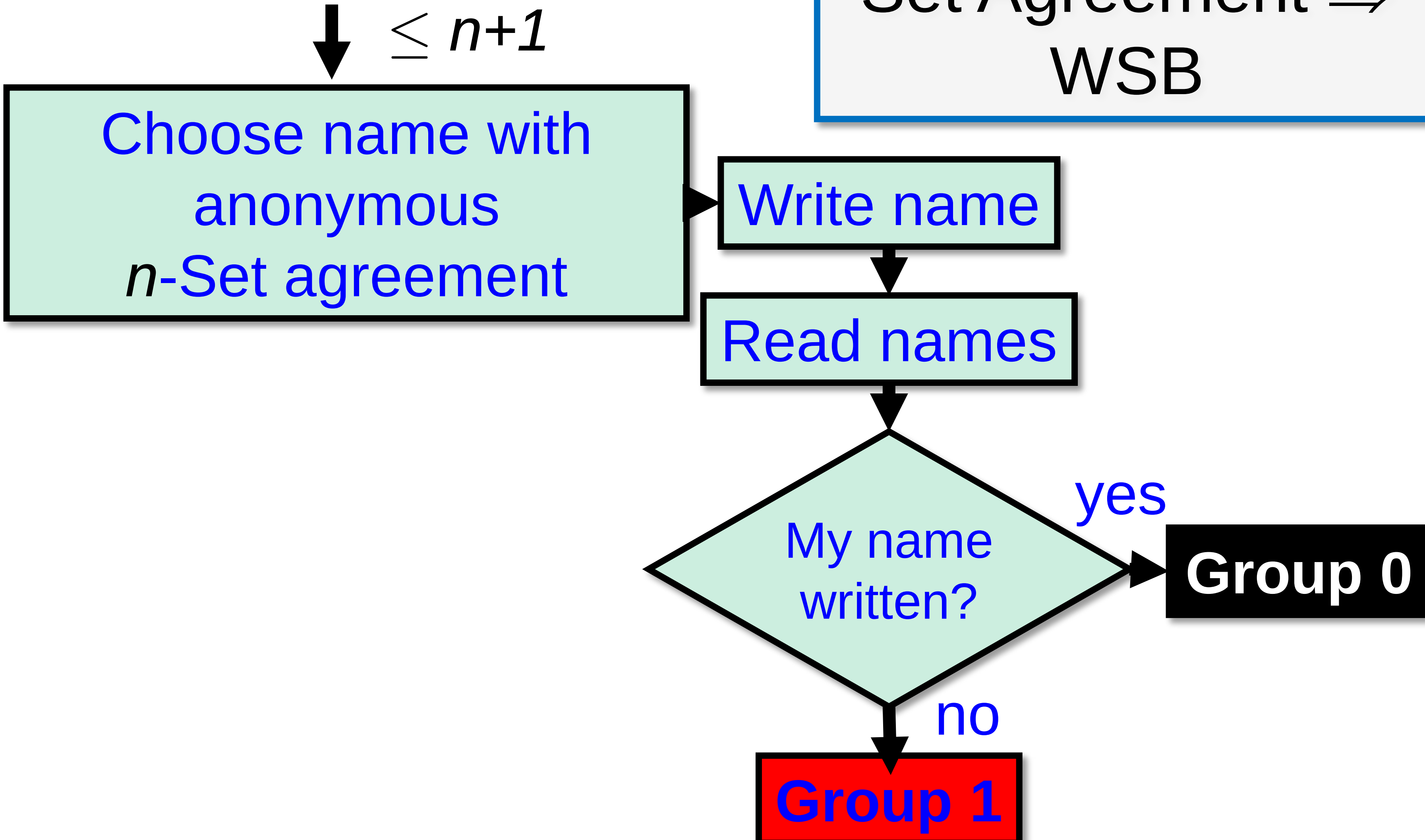


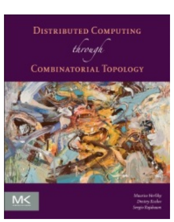
Next Step


Weak Symmetry-Breaking
Moebius Task
Anonymous Set Agreement
yes
no

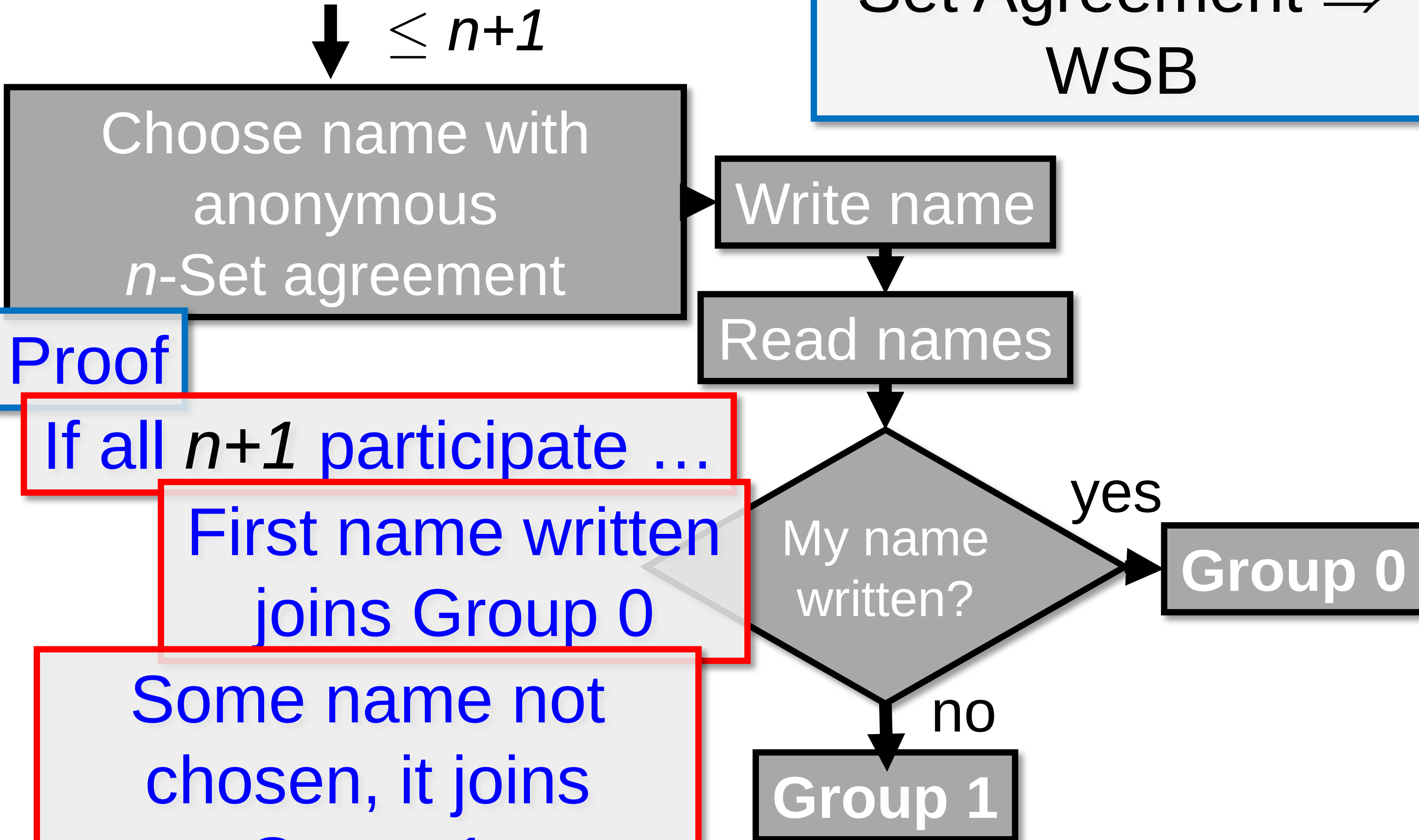

Distributed Computing
through
Combinatorial Topology
MK

Set Agreement ⇒ WSB





Set Agreement ⇒ WSB

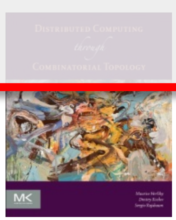


Proof

If all $n+1$ participate …

First name written joins Group 0

Some name not chosen, it joins Group 1



Set Agreement ⇒ WSB

$\leq n+1$

Choose name with anonymous *n*-Set agreement

Write name

Read names

yes

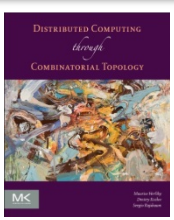
Group 0

Proof

Protocol is anonymous …

Because we use anonymous set agreement "black box"

(*Any* set agreement protocol can be made anonymous)

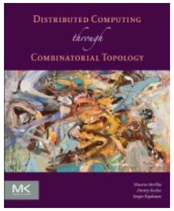


Conclusions

Some tasks harder than others …

n-set agreement solves weak-symmetry breaking

But not vice-versa

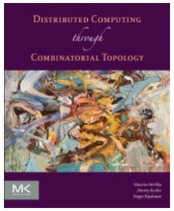


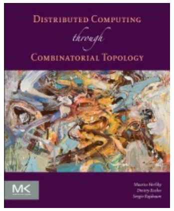
Remarks

Combinatorial and algorithmic arguments complement one another

Combinatorial: what we can't do

Algorithmic: what we can do




Distributed Computing
through
Combinatorial Topology
MK