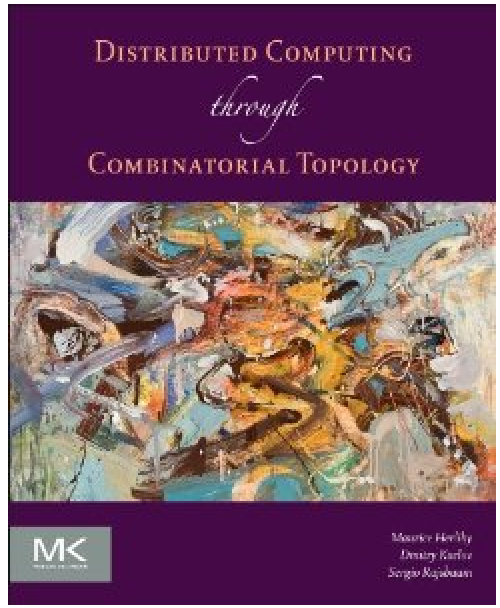
Wait-Free Computability for General Tasks



Companion slides for

Distributed Computing
Through Combinatorial Topology

Maurice Herlihy & Dmitry Kozlov & Sergio Rajsbaum

Road Map

Inherently colored tasks

Solvability for colored tasks

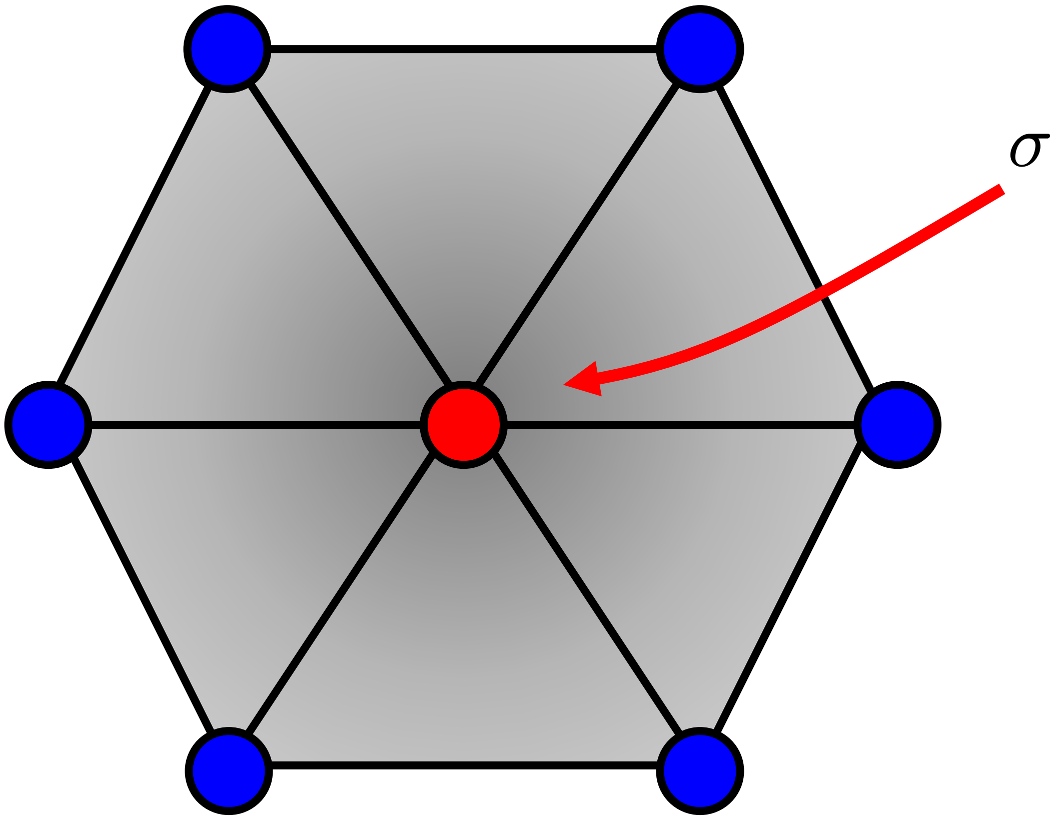
Protocol $\Rightarrow$ map

Map $\Rightarrow$ protocol

A Sufficient Topological Conditions

Review

Star

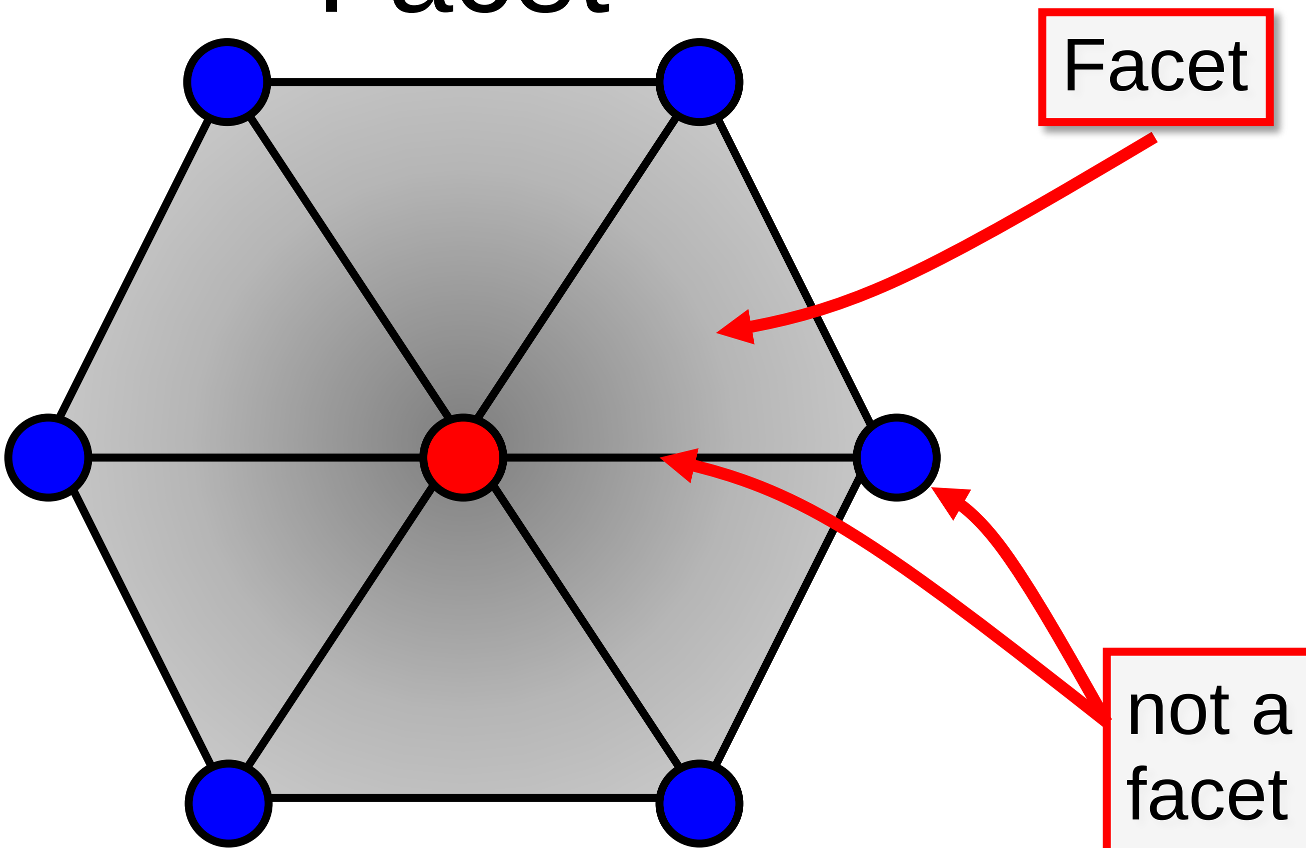


Star(σ,$\mathcal{K}$) is the complex of facets of $\mathcal{K}$ containing σ

Complex

Review

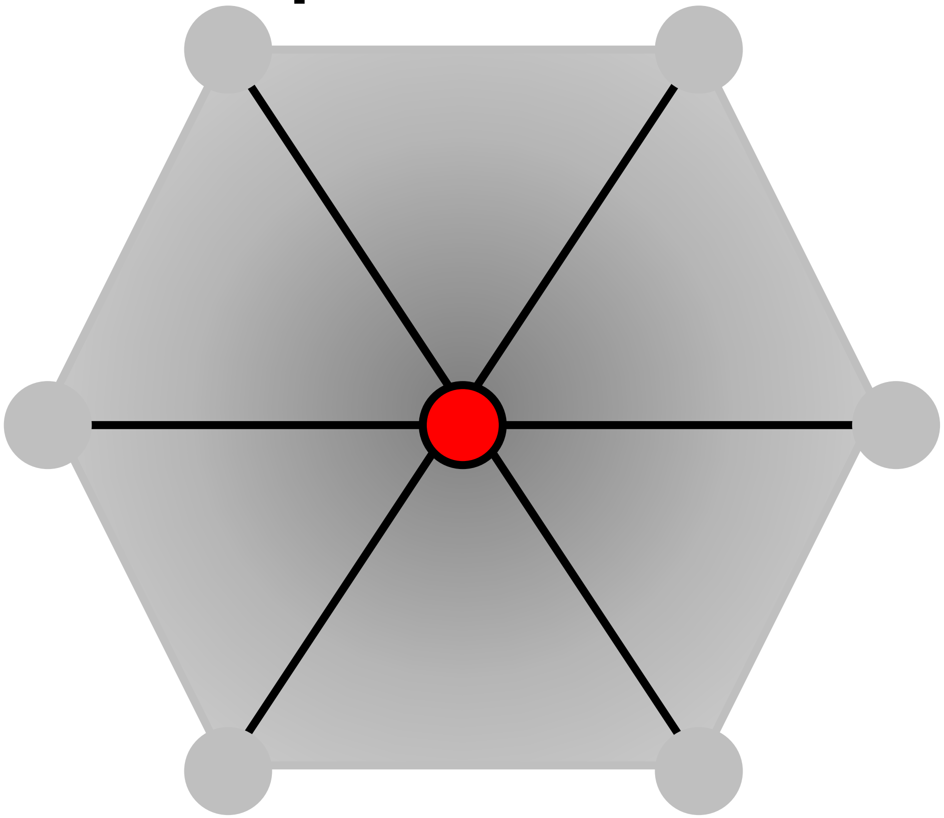
Facet



A *facet* is a simplex of maximal dimension

Review

Open Star

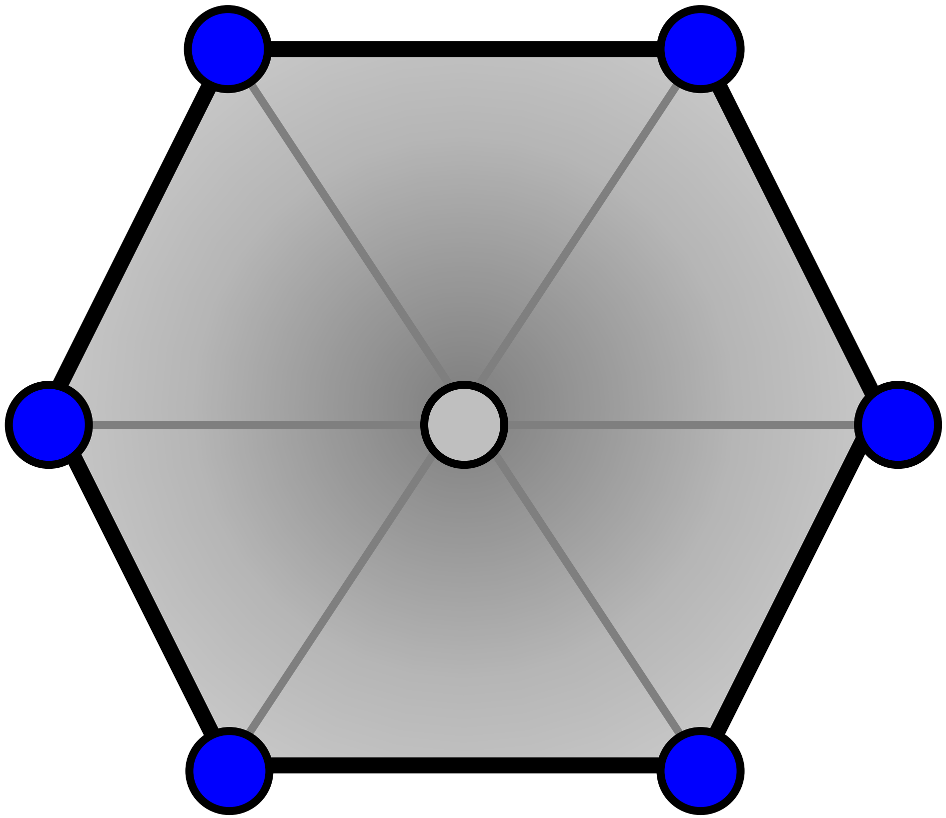


Star^o(σ,$\mathcal{K}$) union of interiors of simplexes containing σ

Point Set

Review

Link



Link(σ,$\mathcal{K}$) is the complex of simplices of Star(σ,$\mathcal{K}$) not containing σ

Complex

Review

A simplicial map ϕ is *rigid* if

$$\dim \phi(\sigma) = \dim \sigma.$$

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Solvability for colored tasks

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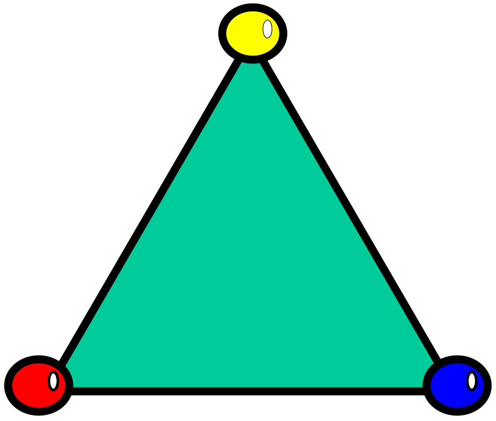
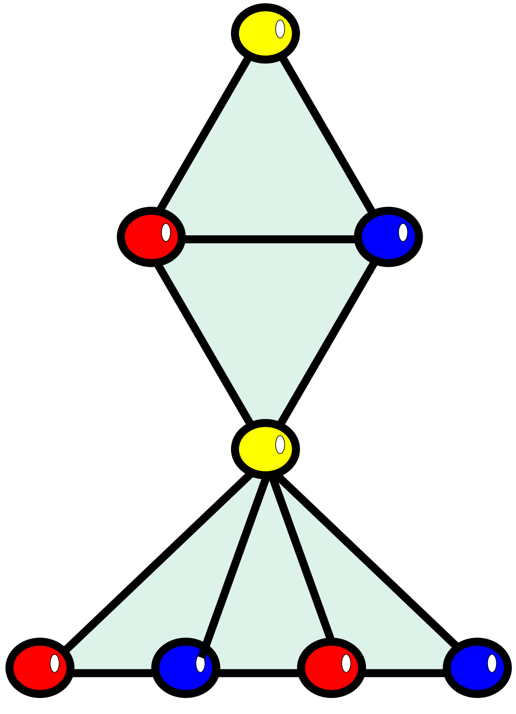
Map $\Rightarrow$ protocol

A Sufficient Topological Conditions

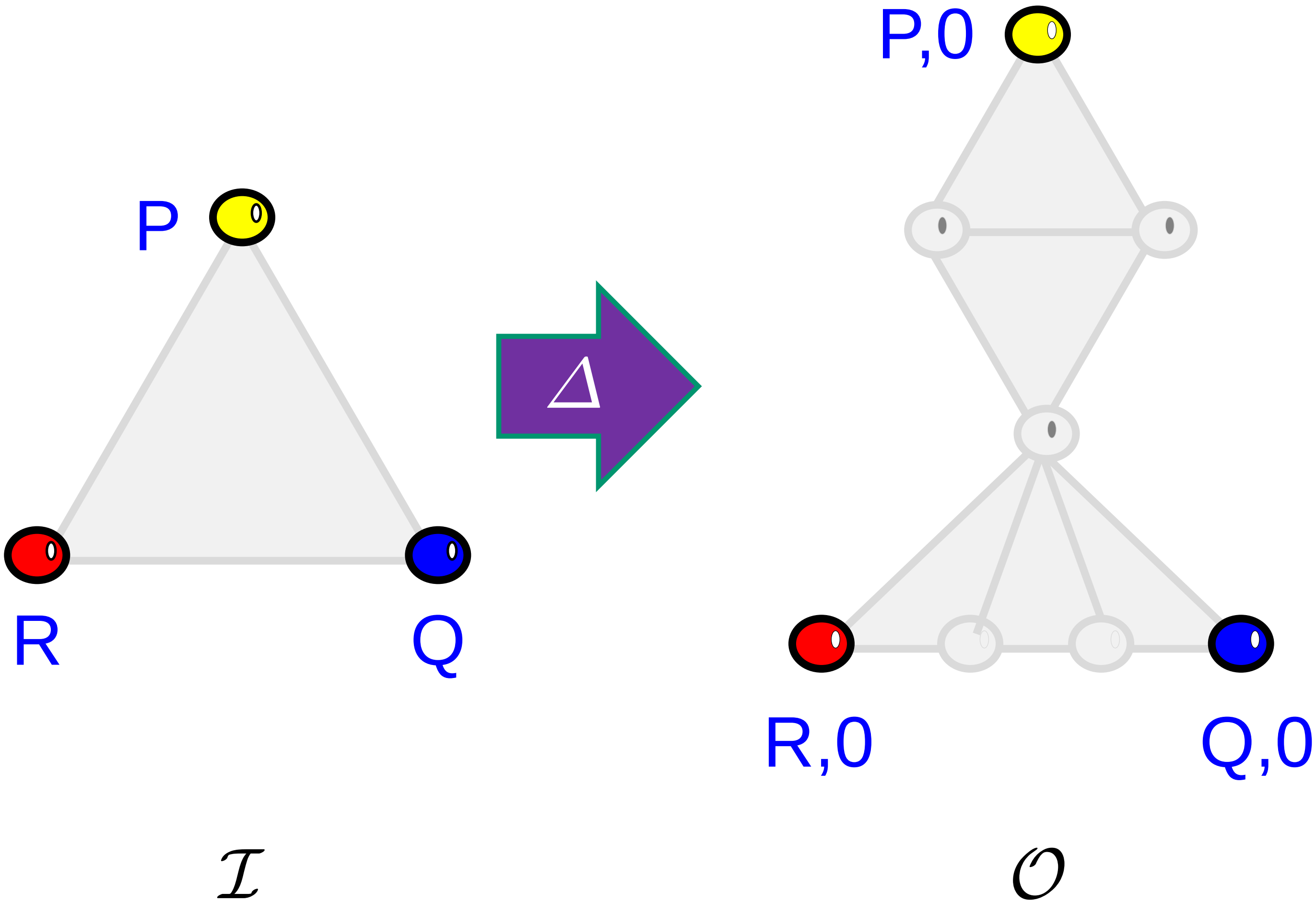
The Hourglass task





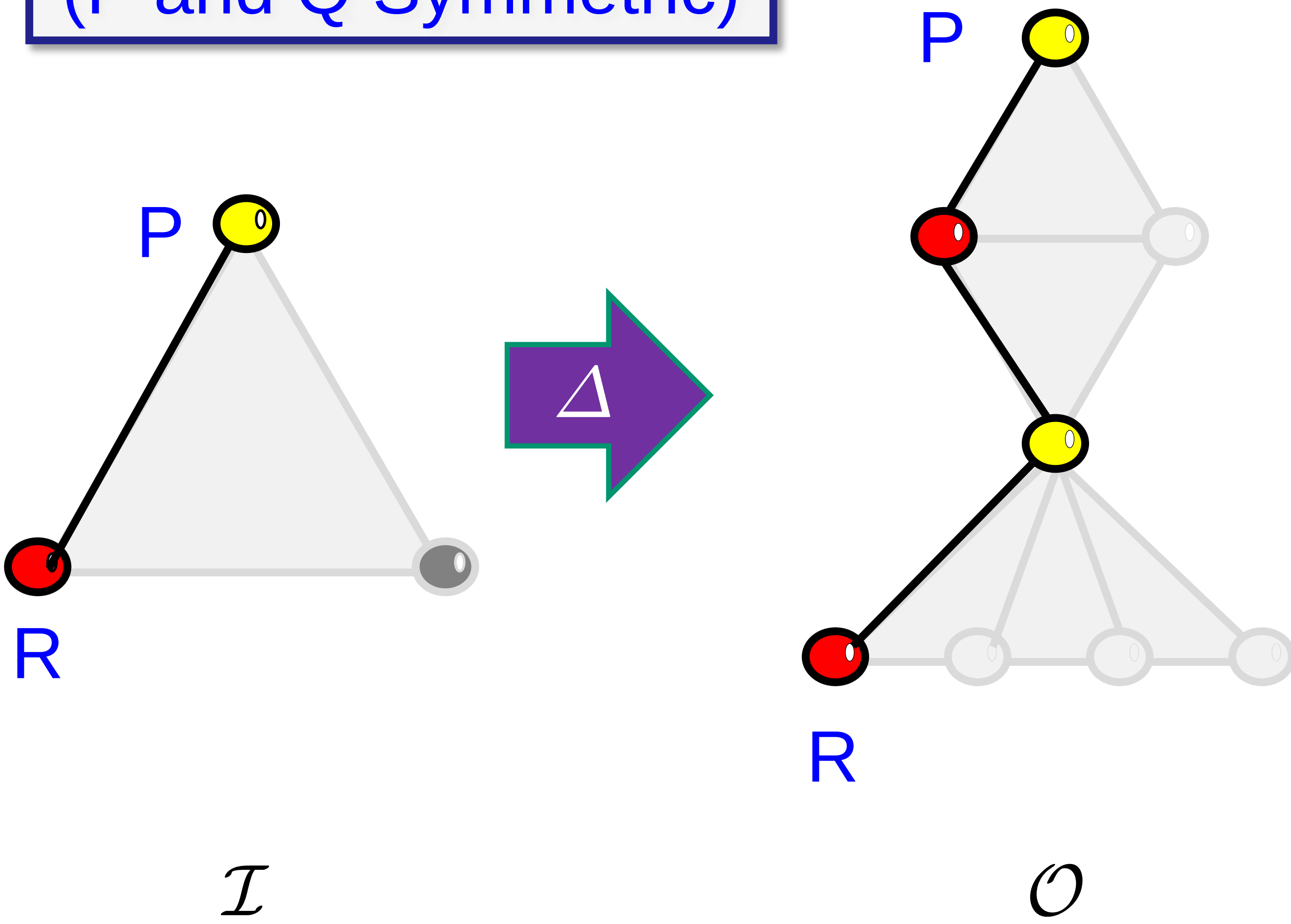
$\mathcal{I}$ $\mathcal{O}$

Single-Process Executions

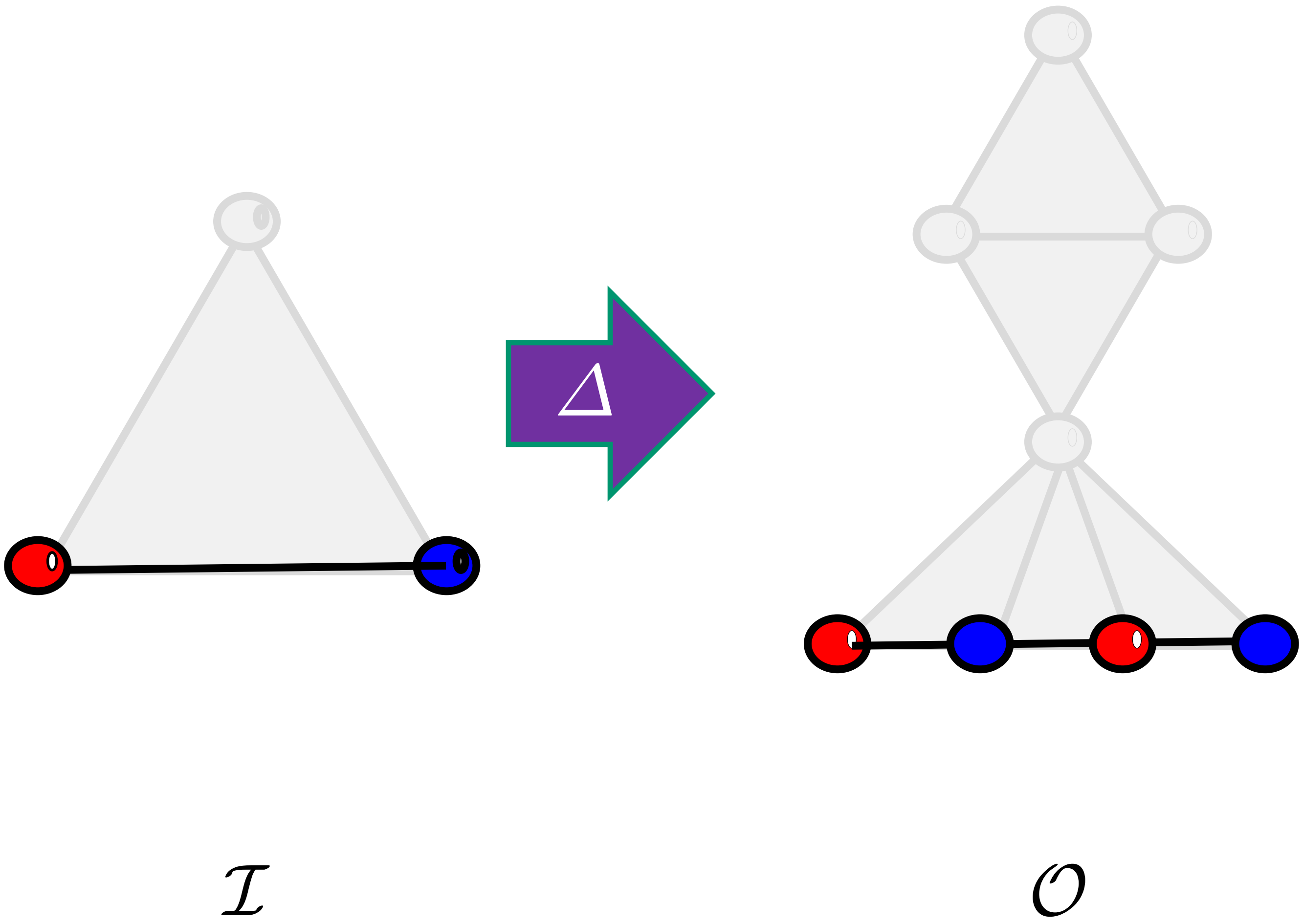


P and R only

(P and Q Symmetric)



Q and R only



Claim:

Hourglass satisfies conditions of fundamental theorem …

But has no wait-free immediate snapshot protocol!

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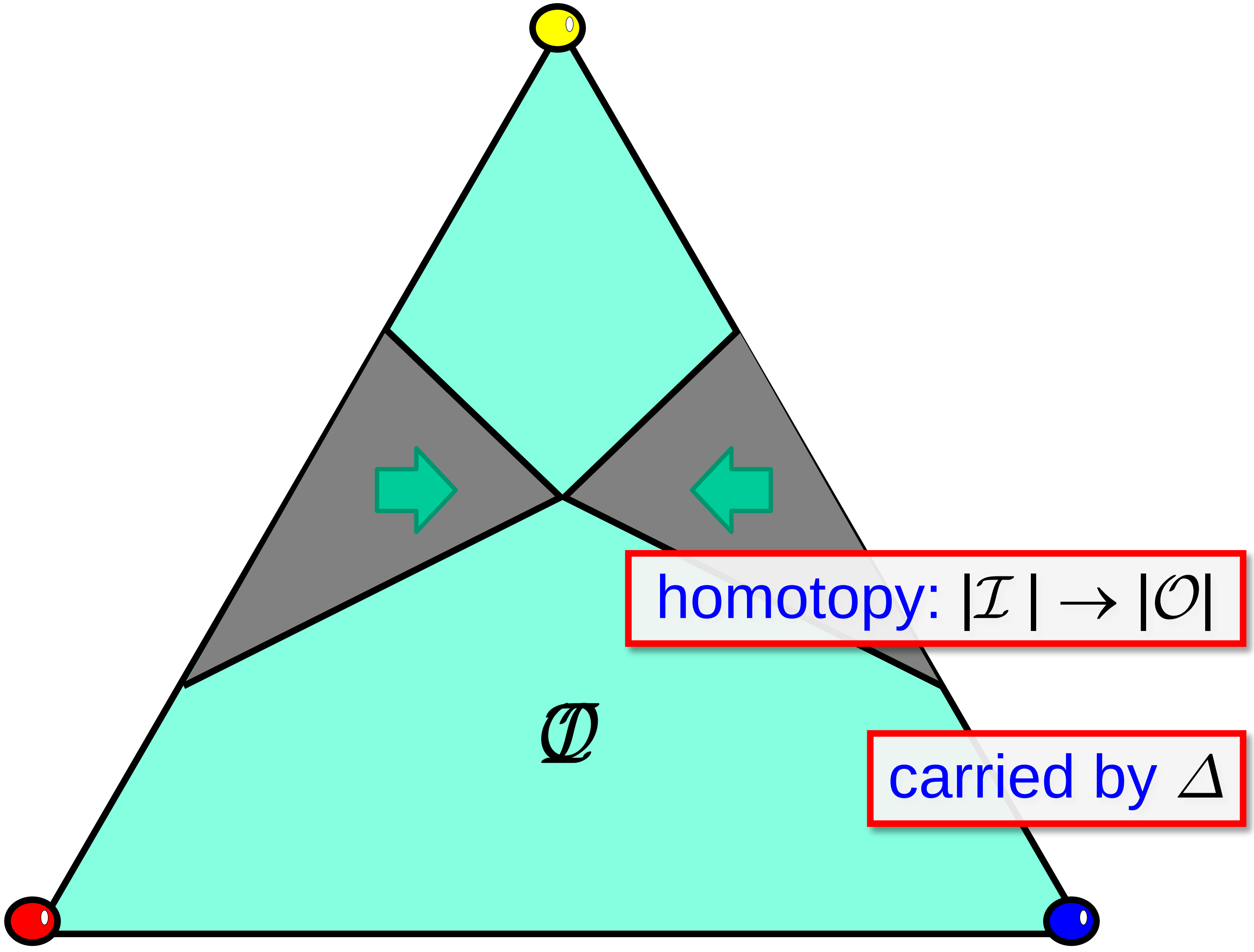

homotopy: $|\mathcal{I}| \to |\mathcal{O}|$
$\mathcal{D}$
carried by Δ

Claim:

Hourglass satisfies conditions of fundamental theorem …

But has no wait-free immediate snapshot protocol!

Claim:

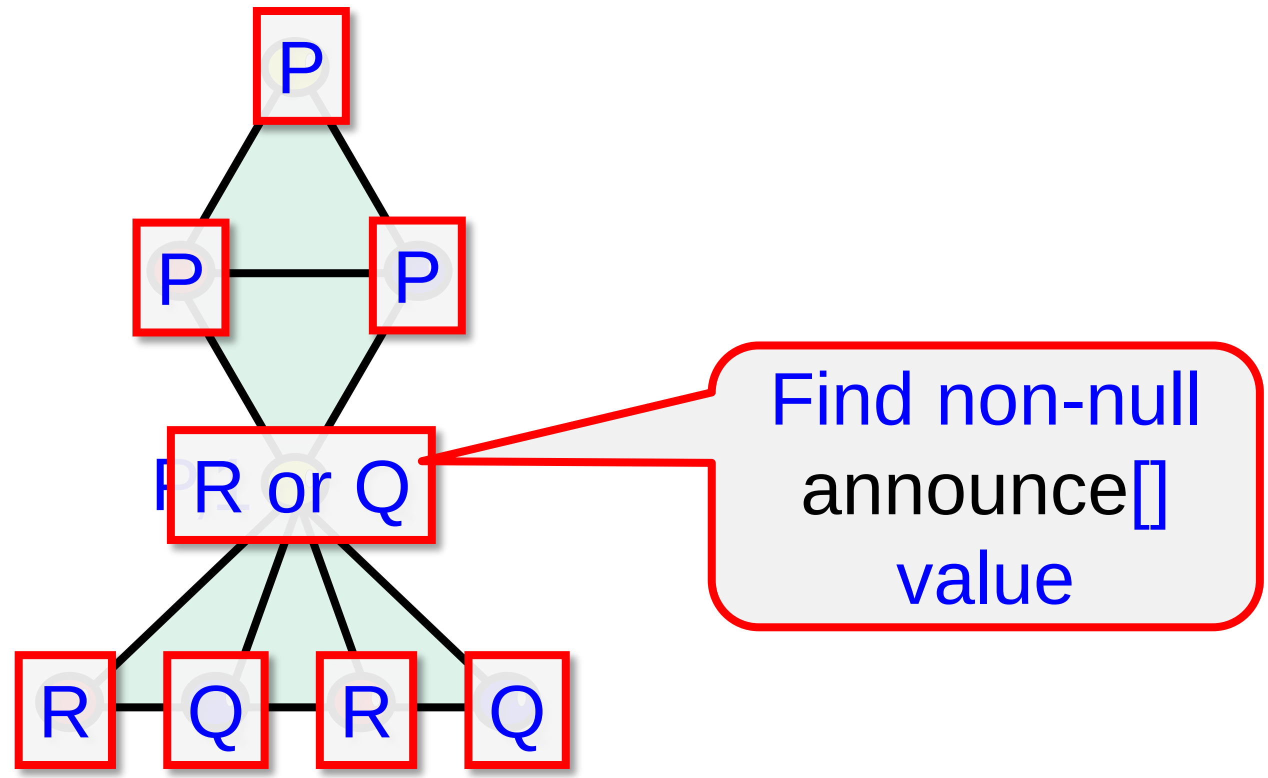
The Hourglass task solves 2-set agreement …

Which has no wait-free read-write protocol.

Protocol:

Write input value to announce array …

Run Hourglass task …



Look in announce[] array …

What Went Wrong?

Theorem

A colorless $(\mathcal{I},\mathcal{O},\Delta)$ has a wait-free immediate snapshot protocol iff there is a continuous map …

f: $|\mathcal{I}| \rightarrow |\mathcal{O}|$...

carried by Δ

One Direction is OK

Theorem

If $(\mathcal{I},\mathcal{O},\Delta)$ has a wait-free read-write protocol …

then there is a continuous map …

f: $|\mathcal{I}| \rightarrow |\mathcal{O}|$...

carried by Δ

The Other Direction Fails

Theorem?

If there is a continuous map …

f: $|\mathcal{I}| \rightarrow |\mathcal{O}|$…

carried by Δ …

then does $(\mathcal{I},\mathcal{O},\Delta)$ have a wait-free IS protocol?

Review

$f\colon |\mathcal{I}| \to |\mathcal{O}|$...

Simplicial approximation

$\phi\colon \mathrm{Bary}^N\, \mathcal{I} \to \mathcal{O}$

Repeated snapshot

Protocol

Review

f: $|\mathcal{I}| \to |\mathcal{O}|$...

ϕ: $\text{Bary}^N\ \mathcal{I} \to \mathcal{O}$

Protocol

Simplicial approximation

Not color-preserving

Repeated snapshot

Another process's output?

Road Map

Inherently colored tasks

Solvability for colored tasks

Protocol $\Rightarrow$ map

Map $\Rightarrow$ protocol

A Sufficient Topological Conditions

Theorem

A colorless $(\mathcal{I},\mathcal{O},\Delta)$ has a wait-free immediate snapshot protocol iff there is a continuous map …

f: $|\mathcal{I}| \to |\mathcal{O}|$…

carried by Δ

How can we adapt this theorem to colored tasks?

Fundamental Theorem for Colored Tasks

Theorem

$(\mathcal{I},\mathcal{O},\Delta)$ has a wait-free read-write protocol iff …

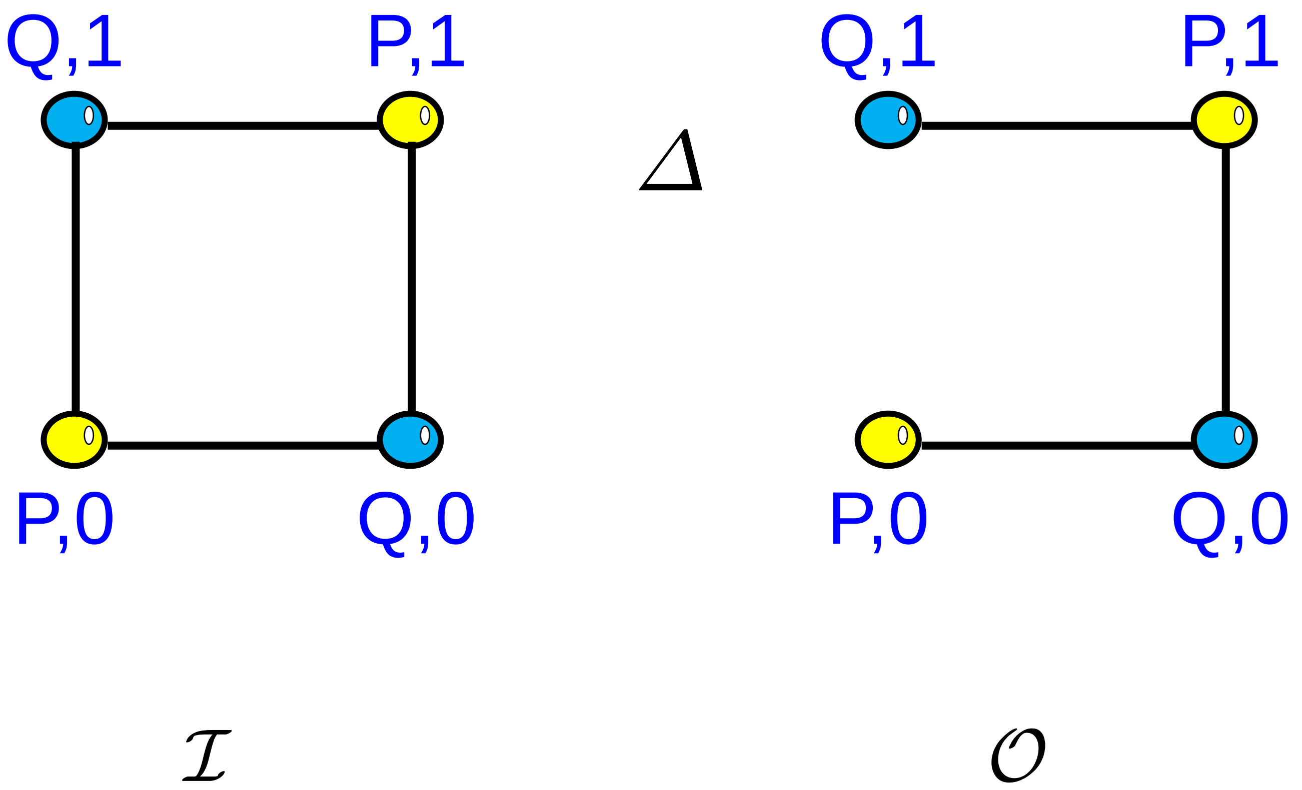
$\mathcal{I}$ has a *chromatic* subdivision Div $\mathcal{I}$ …

& *color-preserving* simplicial map

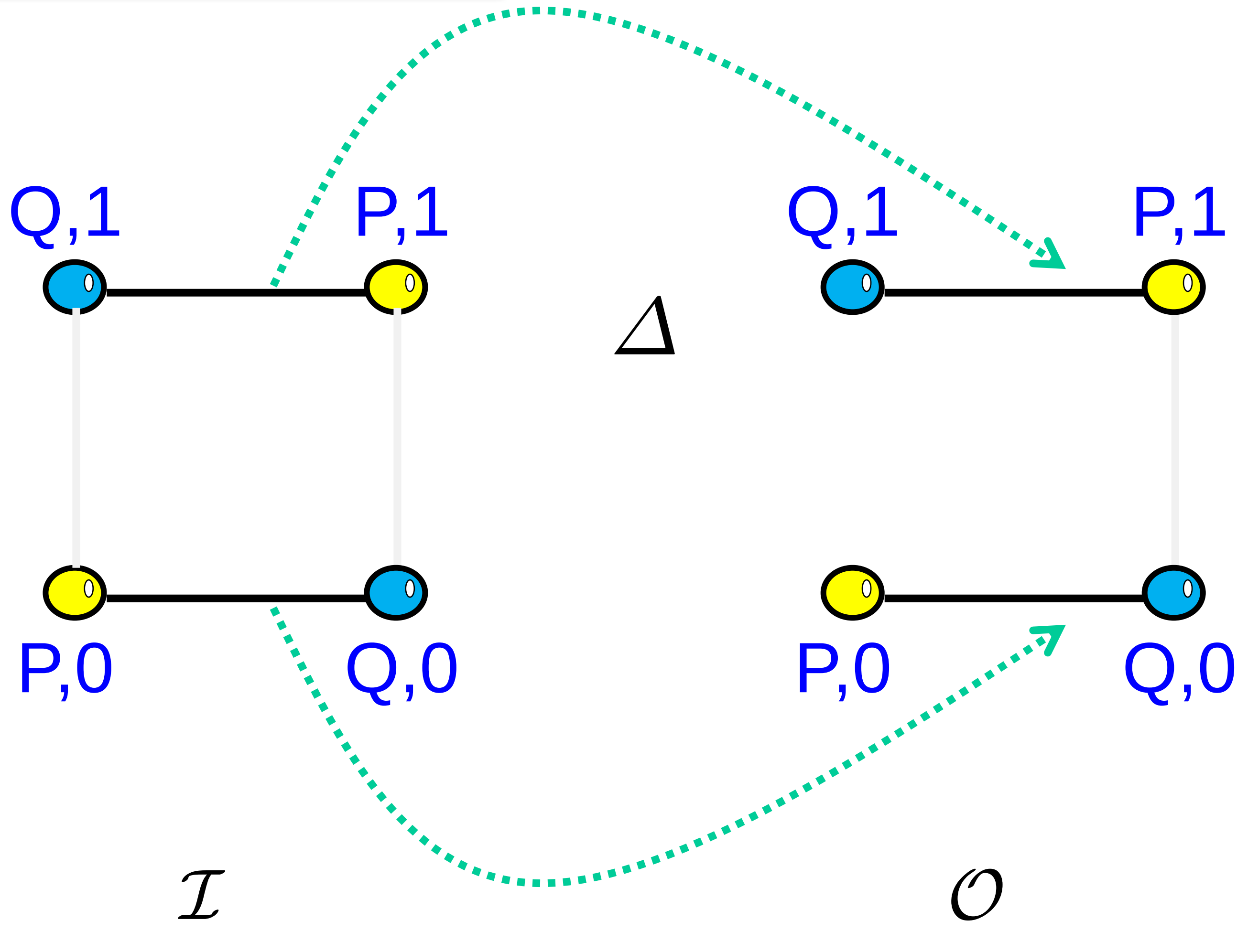
ϕ: Div $\mathcal{I} \to \mathcal{O}$…

carried by Δ

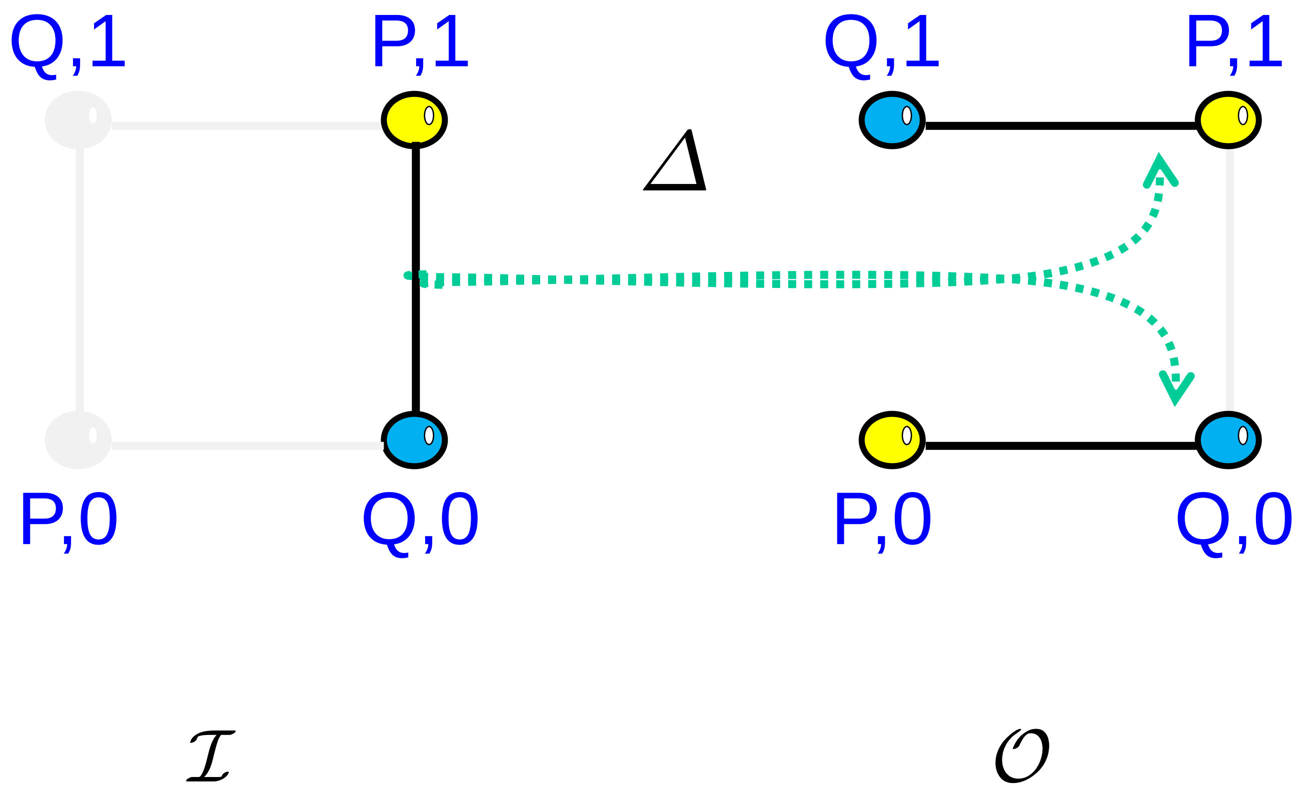
Quasi-Consensus



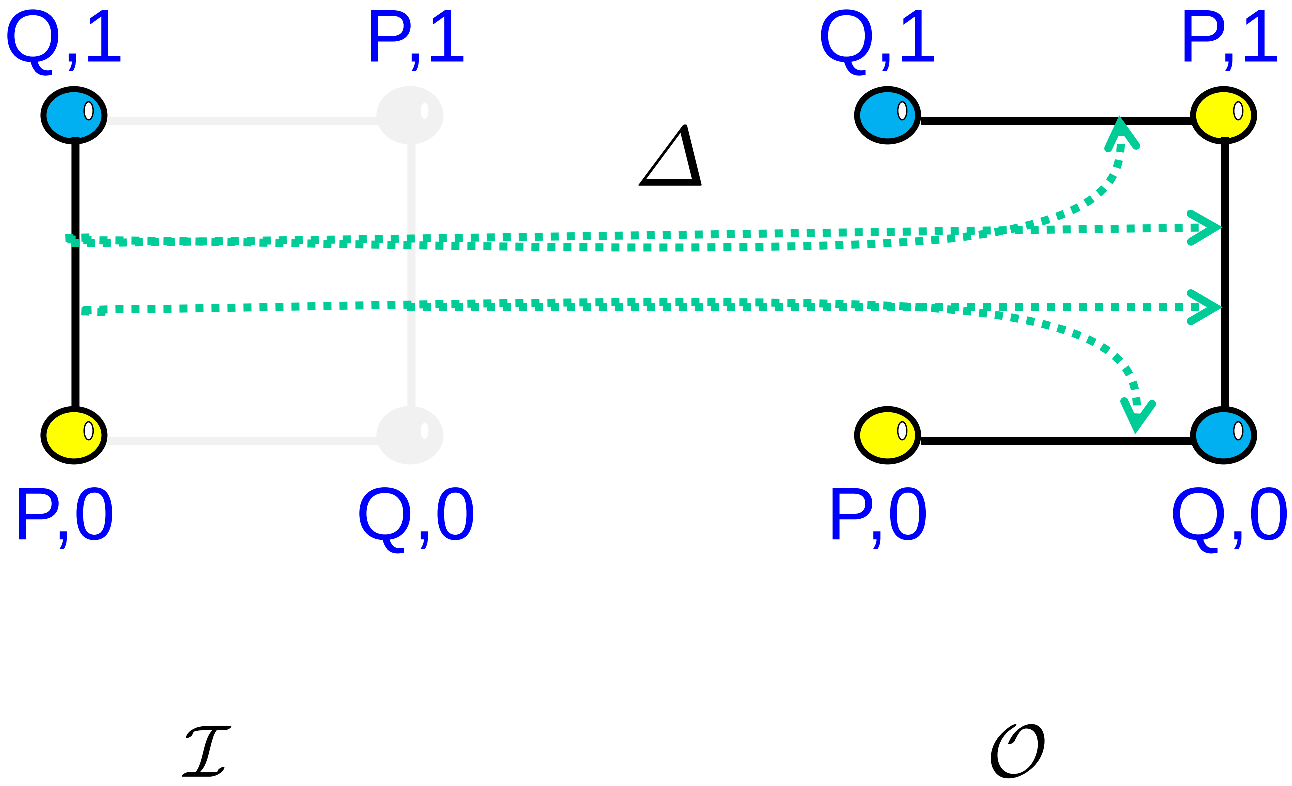
Quasi-Consensus



Quasi-Consensus

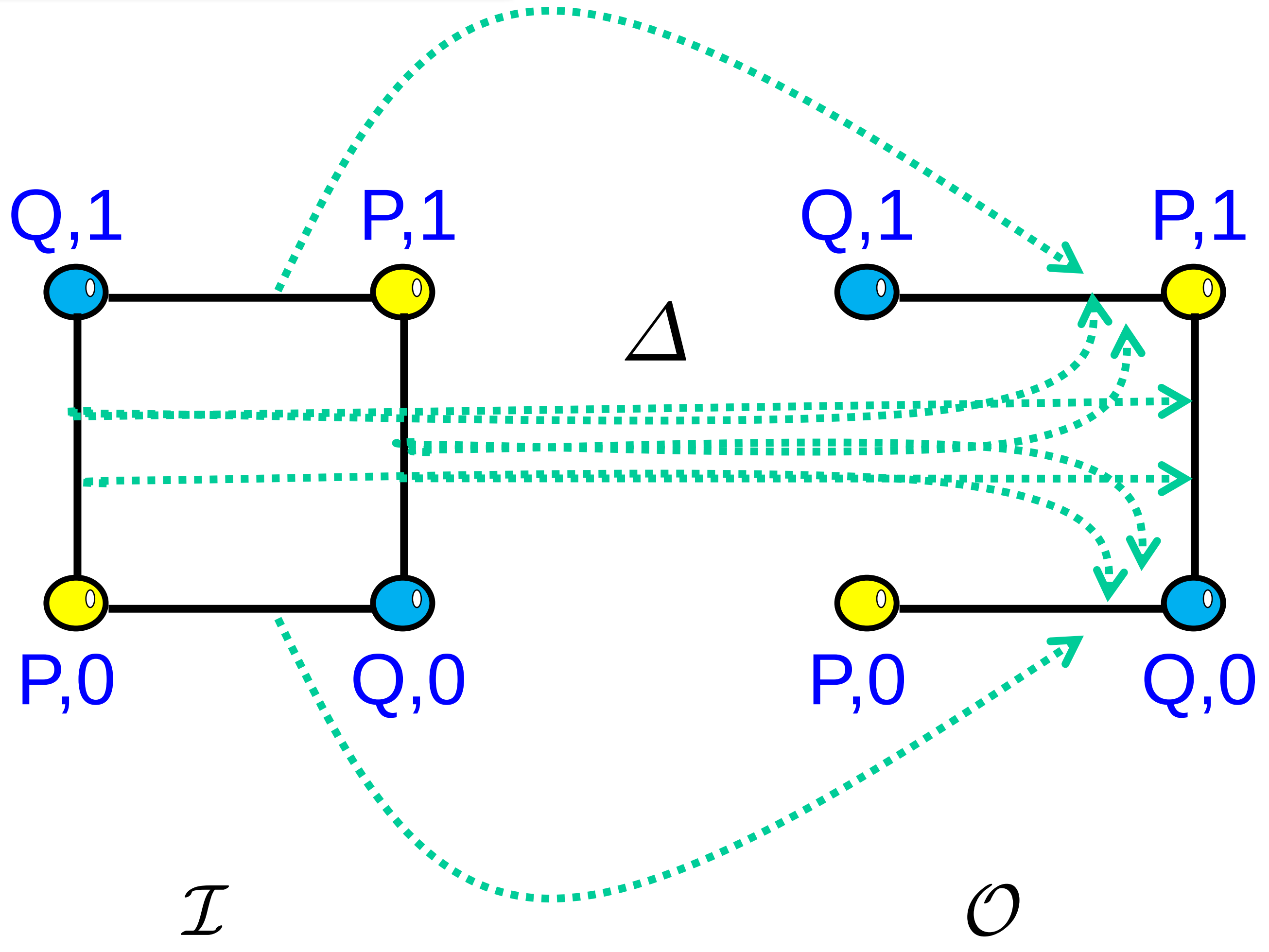


Quasi-Consensus

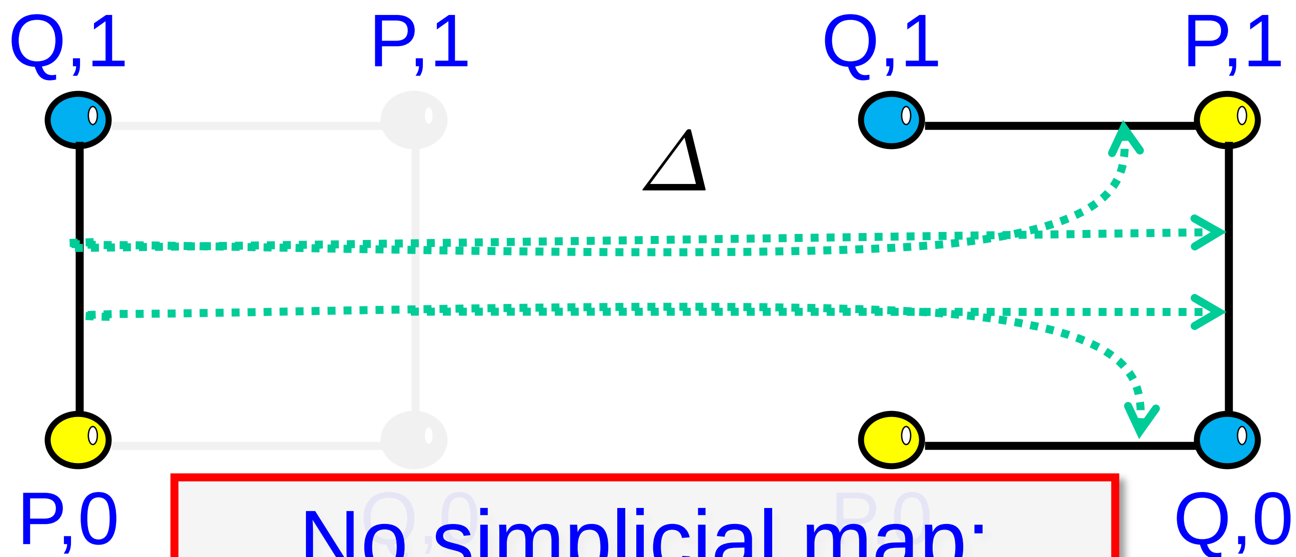


Not a colorless task!

Quasi-Consensus



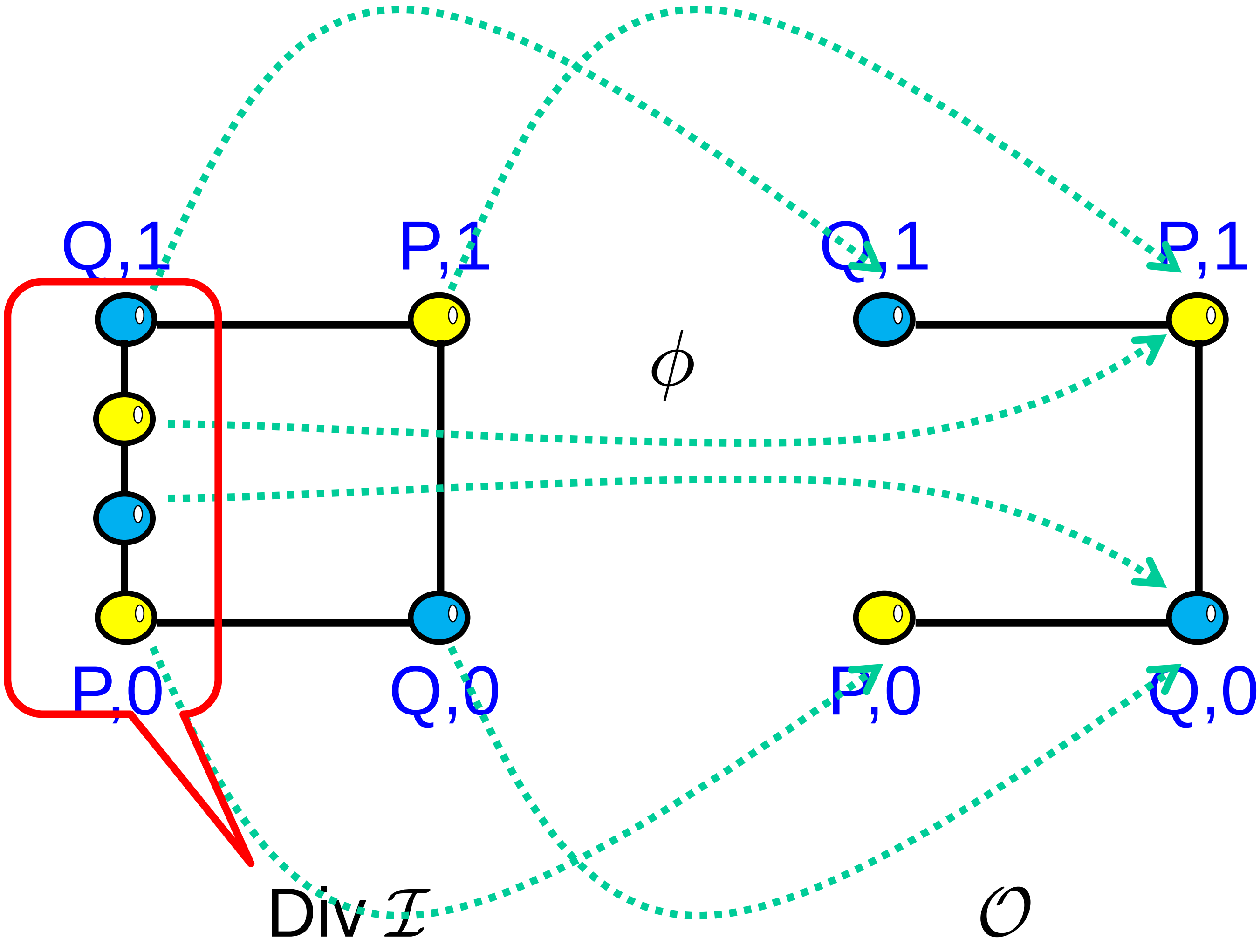
Quasi-Consensus



No simplicial map:

$$\mathcal{I} \to \mathcal{O}$$

carried by Δ



Q,1
P,1
Q,1
P,1
ϕ
P,0
Q,0
P,0
Q,0
Div $\mathcal{I}$
$\mathcal{O}$

```
// code for P
T decide(T input) {
  announce[P] = input;
  if (input == 1)
    return 1;
  else if (announce[Q] != 1)
    return 0
  else
    return 1
}
```

Code is asymmetric!

```
// code for Q
T decide(T input) {
  announce[P] = input;
  if (input == 0)
    return 0;
  else if (announce[P] != 0)
    return 1
  else
    return  0
}
```

Road Map

Inherently colored tasks

Solvability for colored tasks

Protocol $\Rightarrow$ map

Map $\Rightarrow$ protocol

A Sufficient Topological Conditions

Protocol $\Rightarrow$ Map

protocol

decision map

$$\mathcal{I} \xrightarrow{\Xi} \Xi(\mathcal{I}) \xrightarrow{\delta} \mathcal{O}$$

input complex

protocol complex

output complex

carried by Δ

Protocol $\Rightarrow$ Map

$$\Xi(\mathcal{I}) \xrightarrow{\delta} \mathcal{O}$$

subdivision of input complex

carried by Δ

Road Map

Inherently colored tasks

Solvability for colored tasks

Protocol $\Rightarrow$ map

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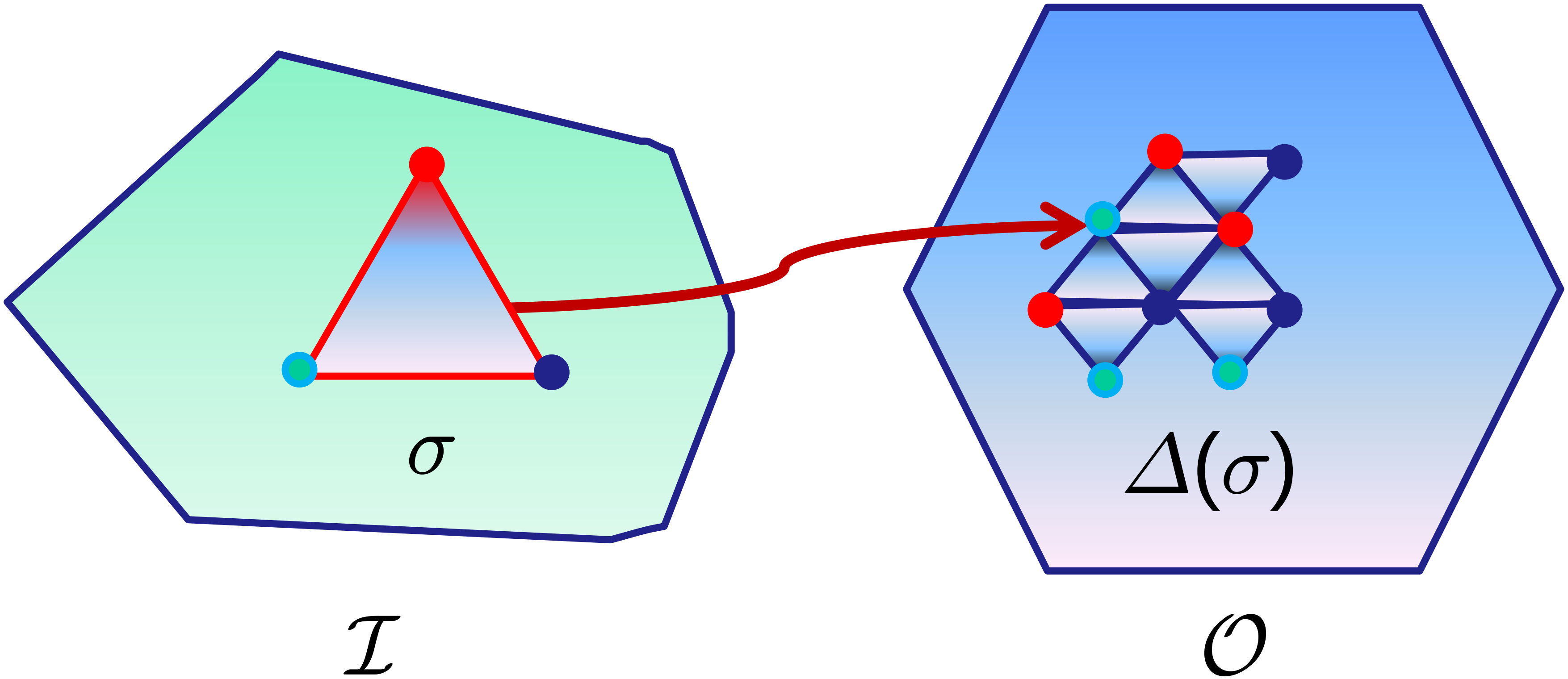
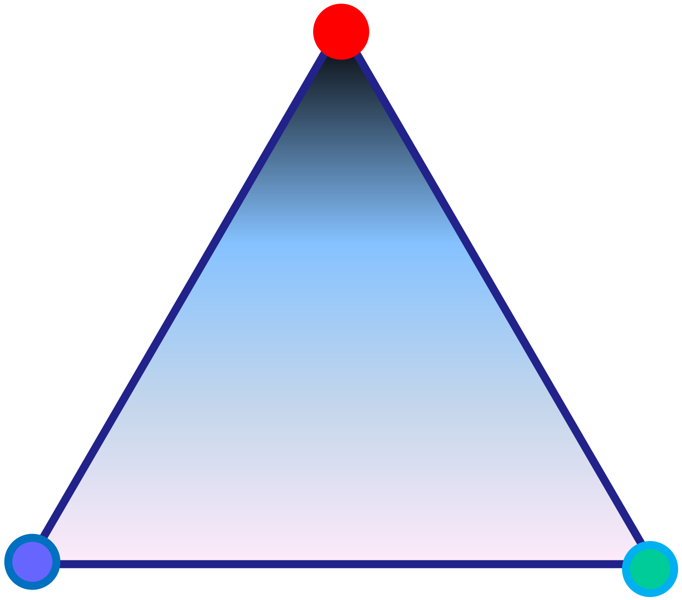
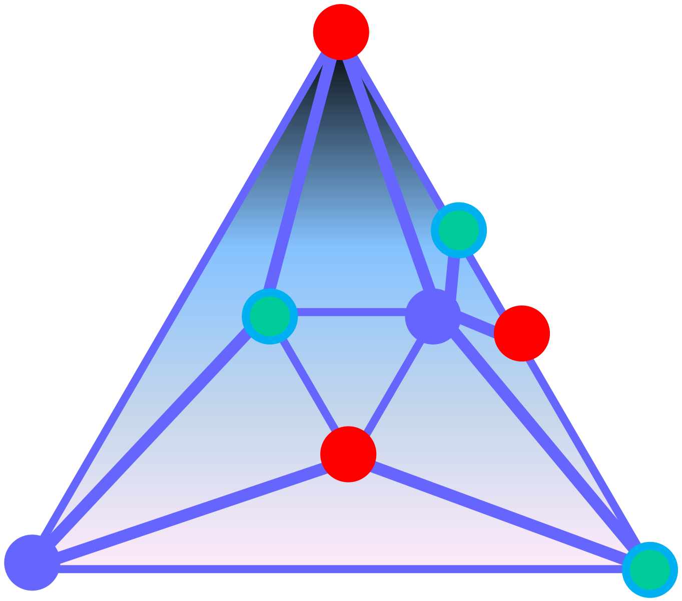
A Sufficient Topological Conditions

Task $(\mathcal{I},\mathcal{O},\Delta)$

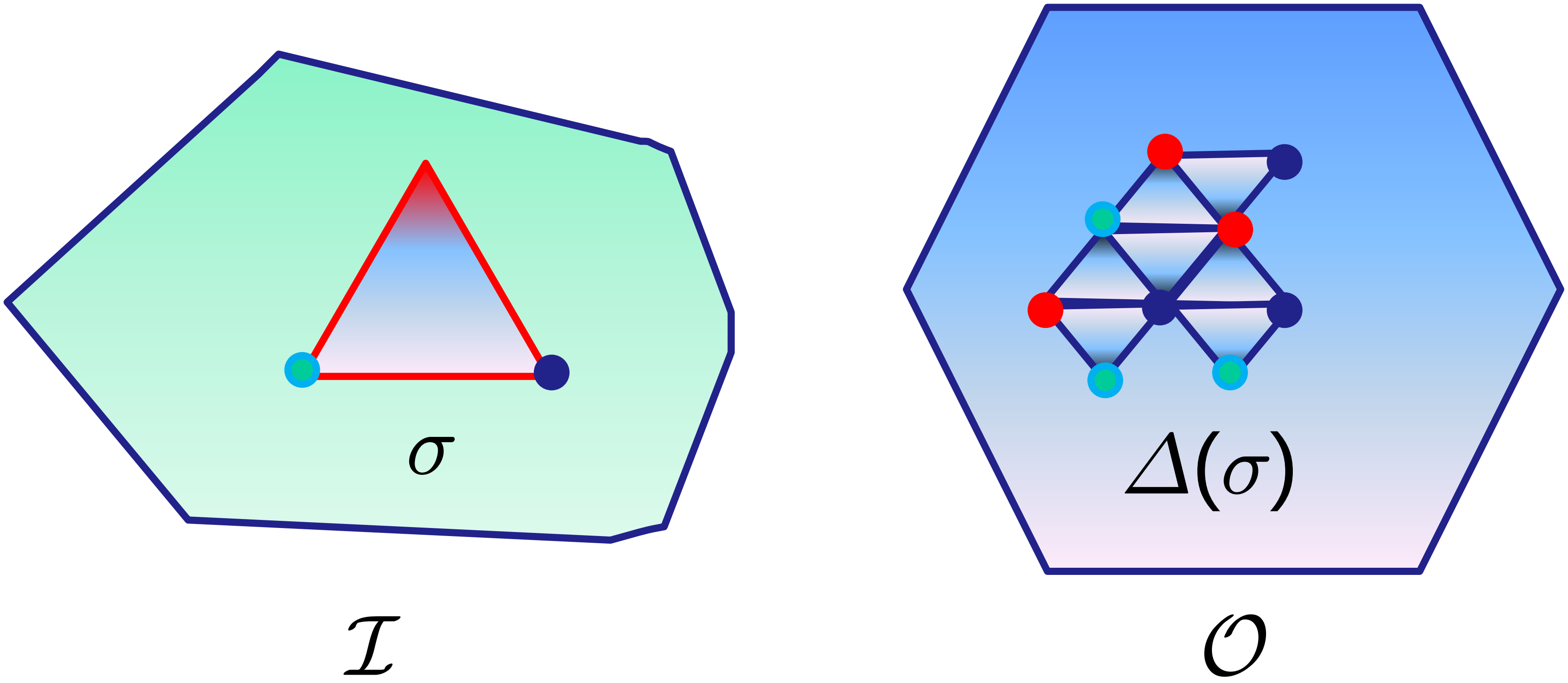




chromatic simplex σ

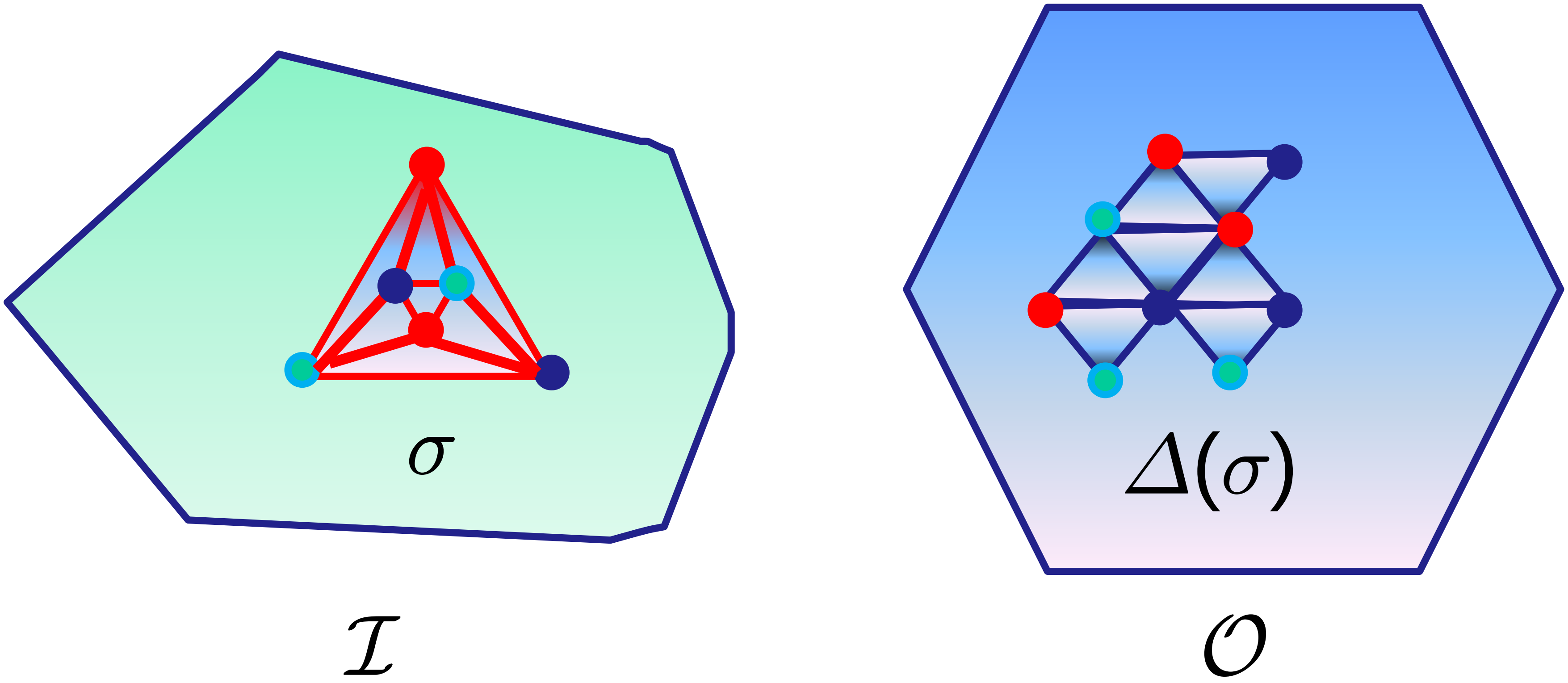


chromatic subdivision Div σ


σ
$\Delta(\sigma)$
$\mathcal{I}$
$\mathcal{O}$

Theorem says …

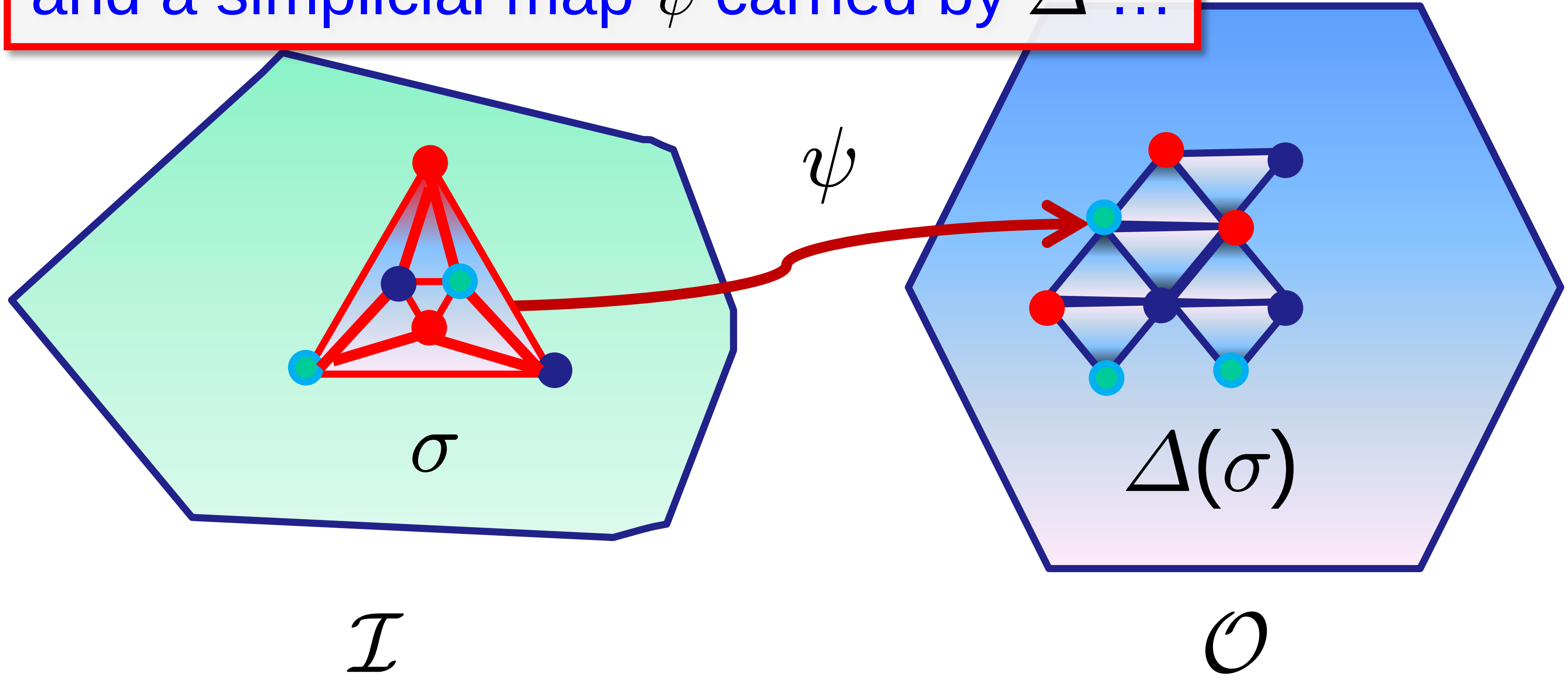
If there is a chromatic subdivision …



Theorem says …

If there is a chromatic subdivision …

and a simplicial map ψ carried by Δ …



Theorem says …

If there is a chromatic subdivision …

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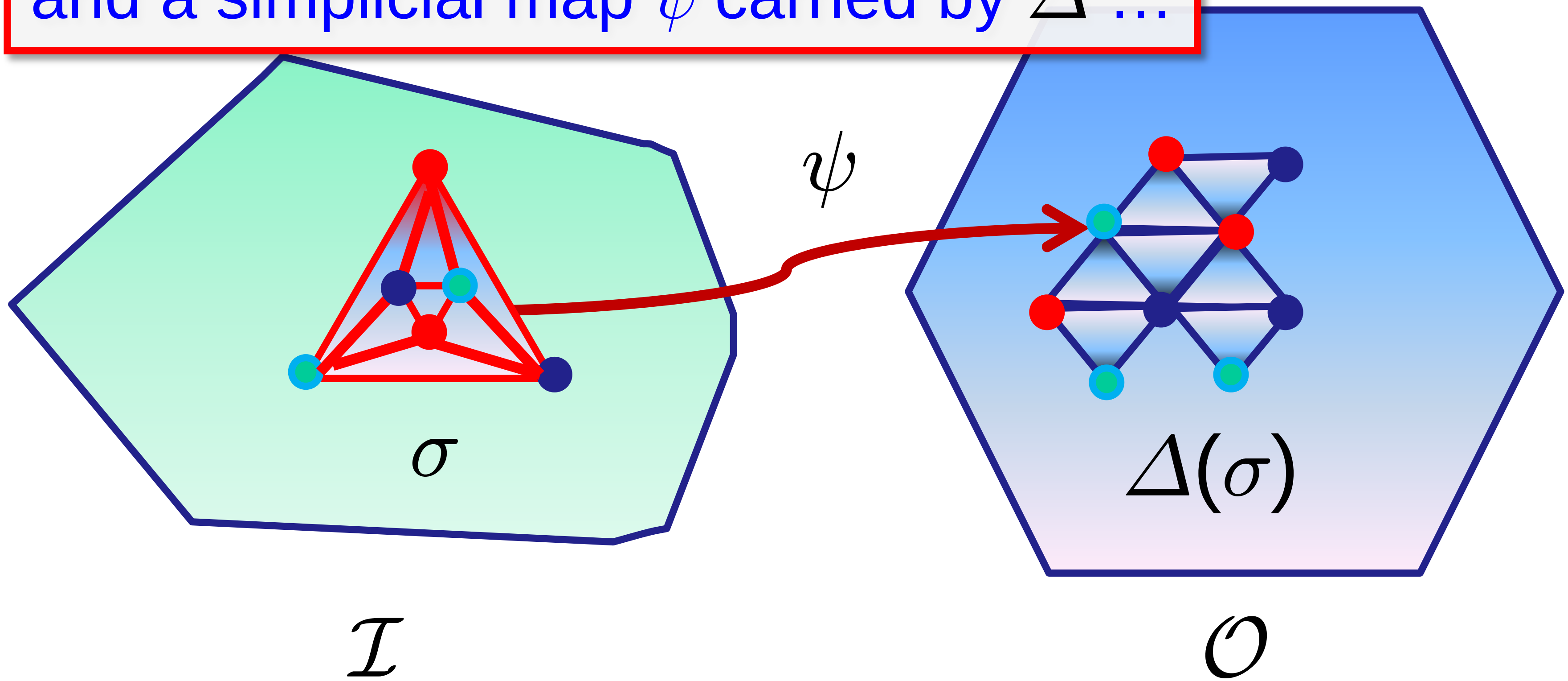
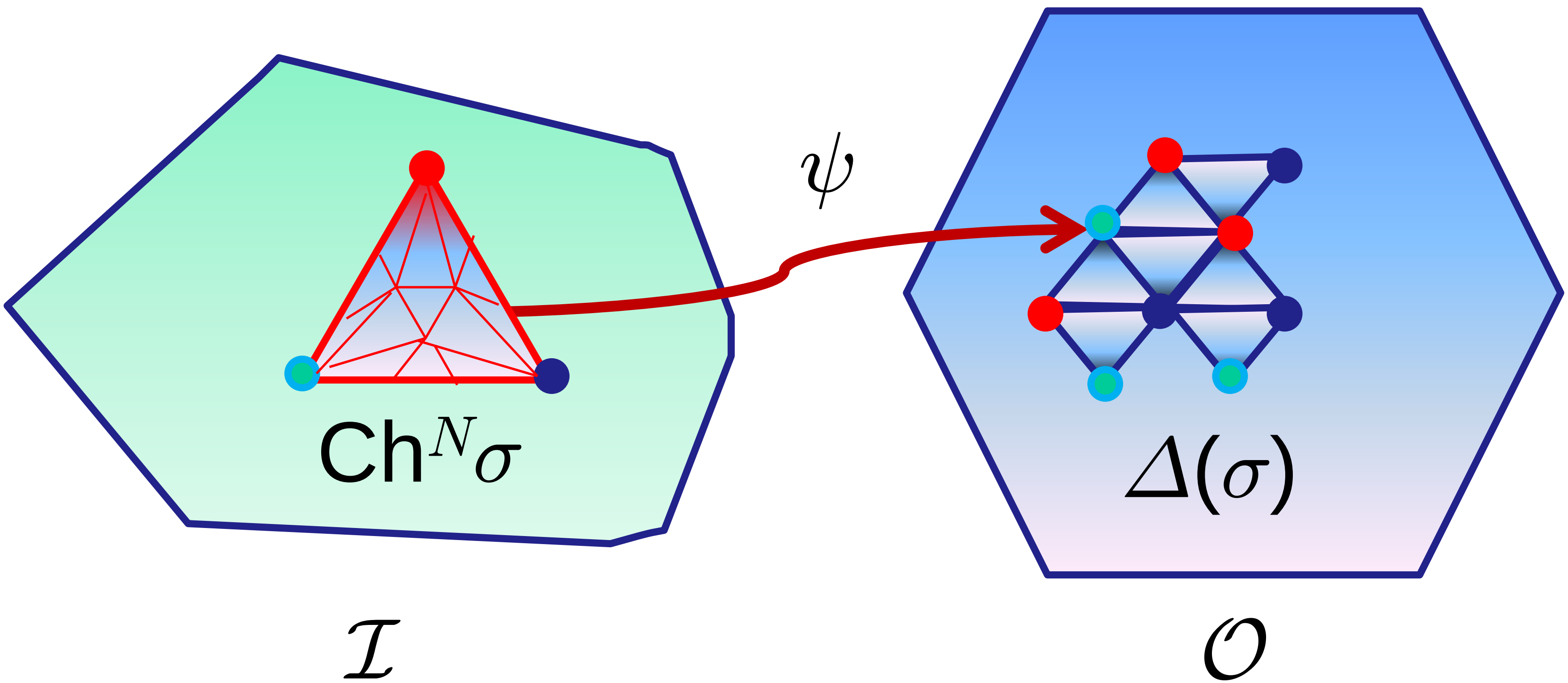


… then there is a wait-free IS protocol!

Let's start with something easier …

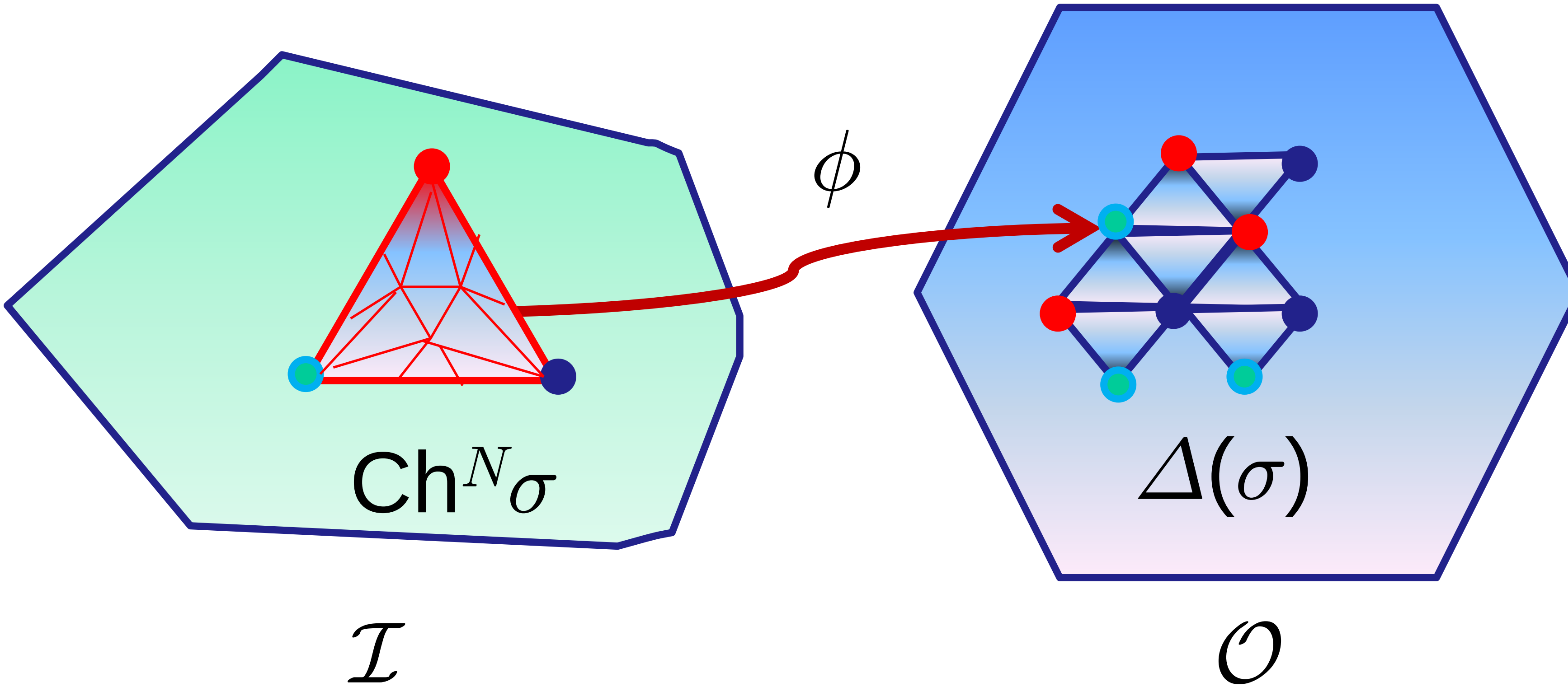
Let's start with a special case …

If there is a simplicial map ψ: $\mathrm{Ch}^N \sigma \rightarrow \Delta(\sigma)$ …



Let's start with something easier …

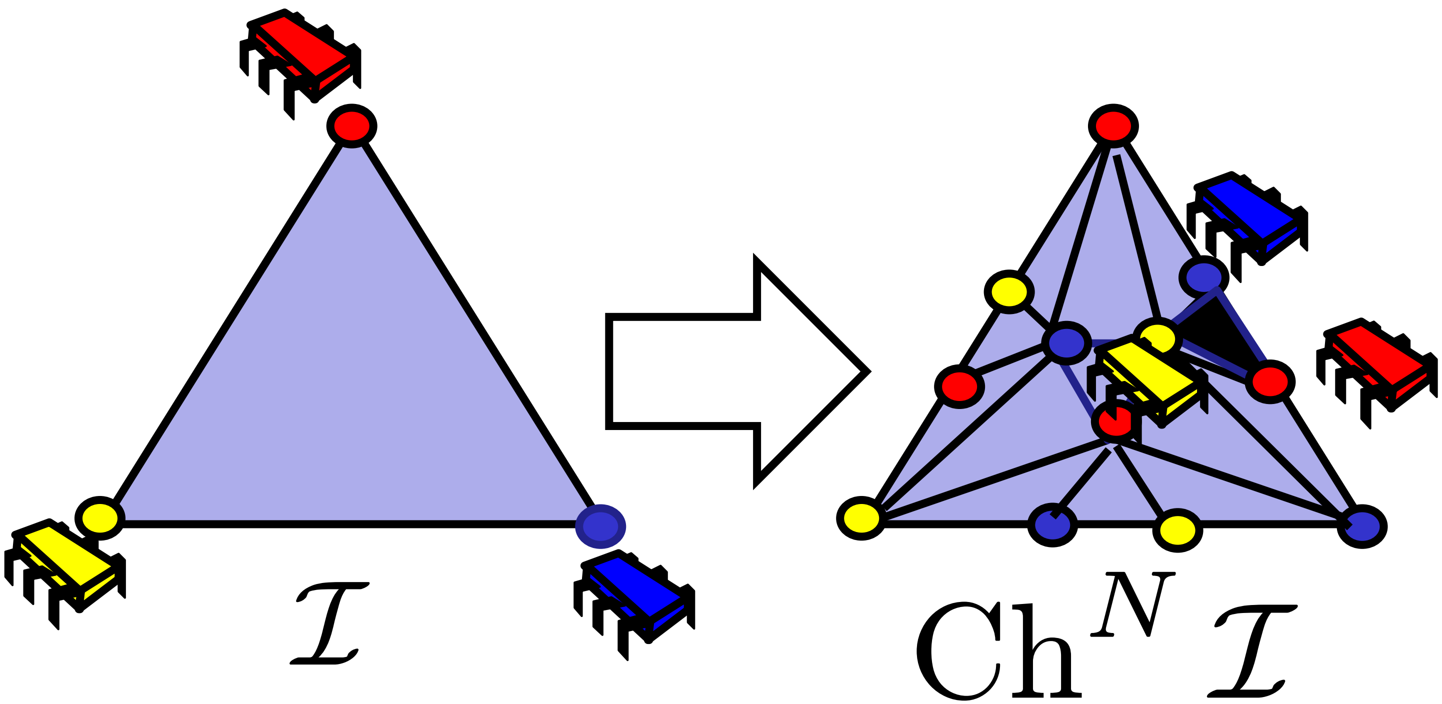
If there is a simplicial map ϕ: $\mathrm{Ch}^{\mathrm{N}}\,\sigma \rightarrow \Delta(\sigma)$ …



… then there is a wait-free IS protocol!

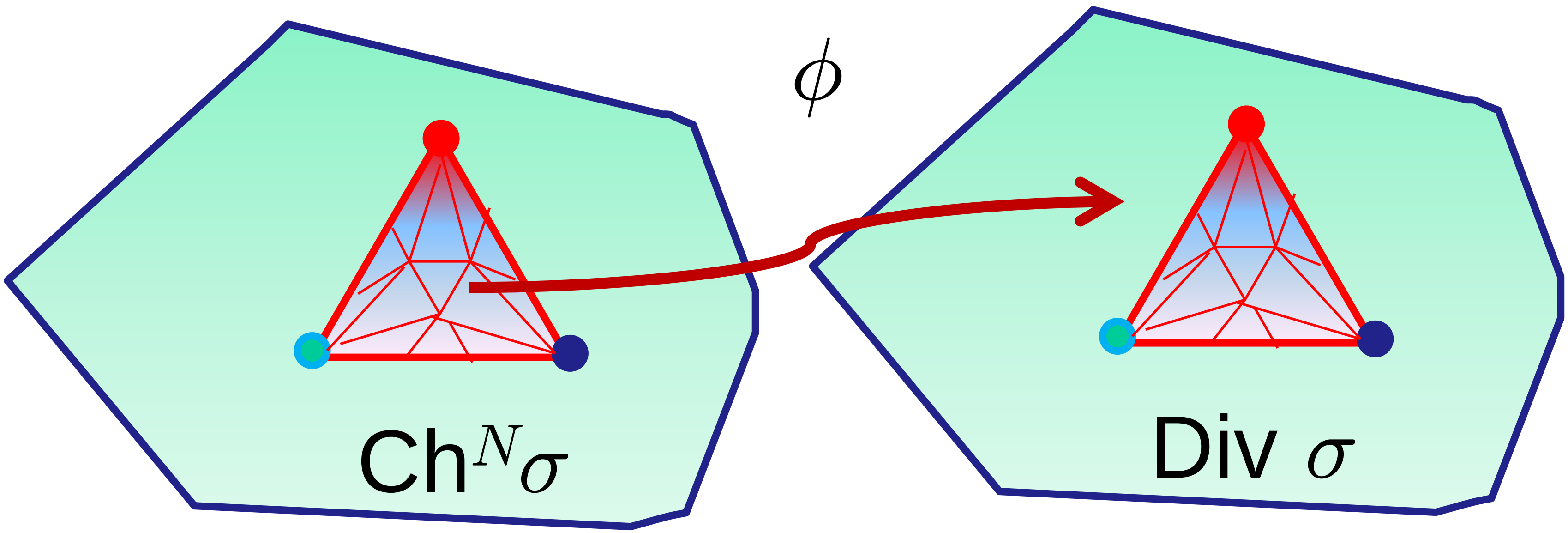
Protocol

Iterated immediate snapshot

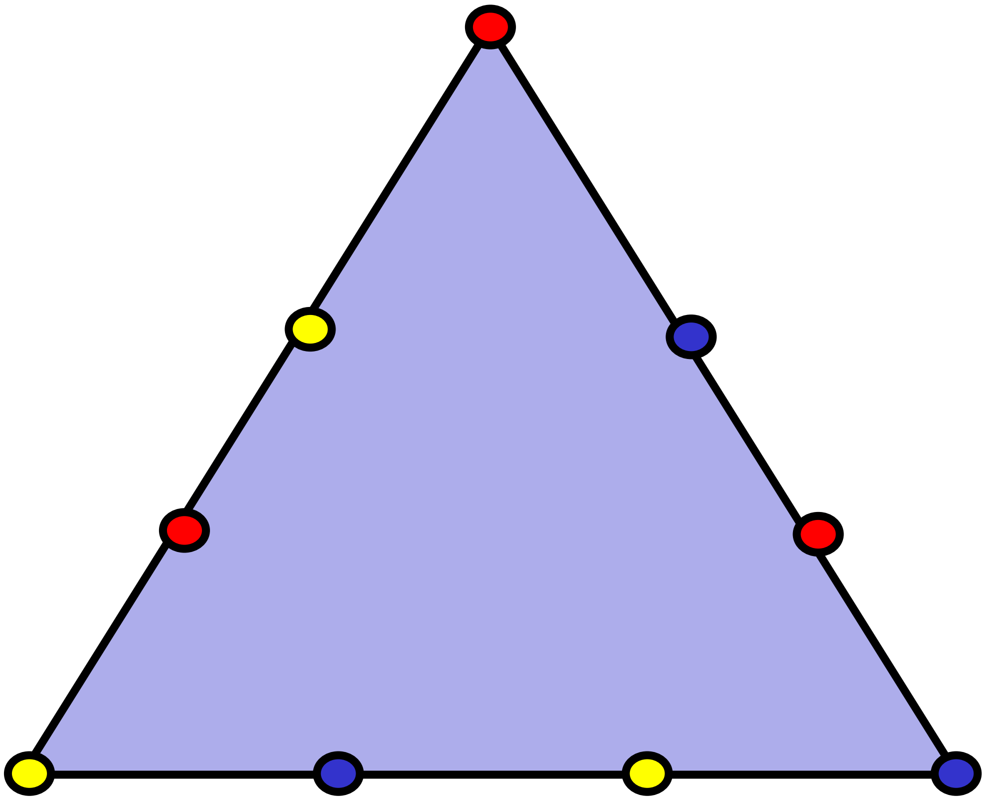


For any chromatic subdivision Div σ …

If there is a color and carrier-preserving simplicial map ϕ: $\mathrm{Ch}^N \sigma \rightarrow \mathrm{Div}\ \sigma$ …

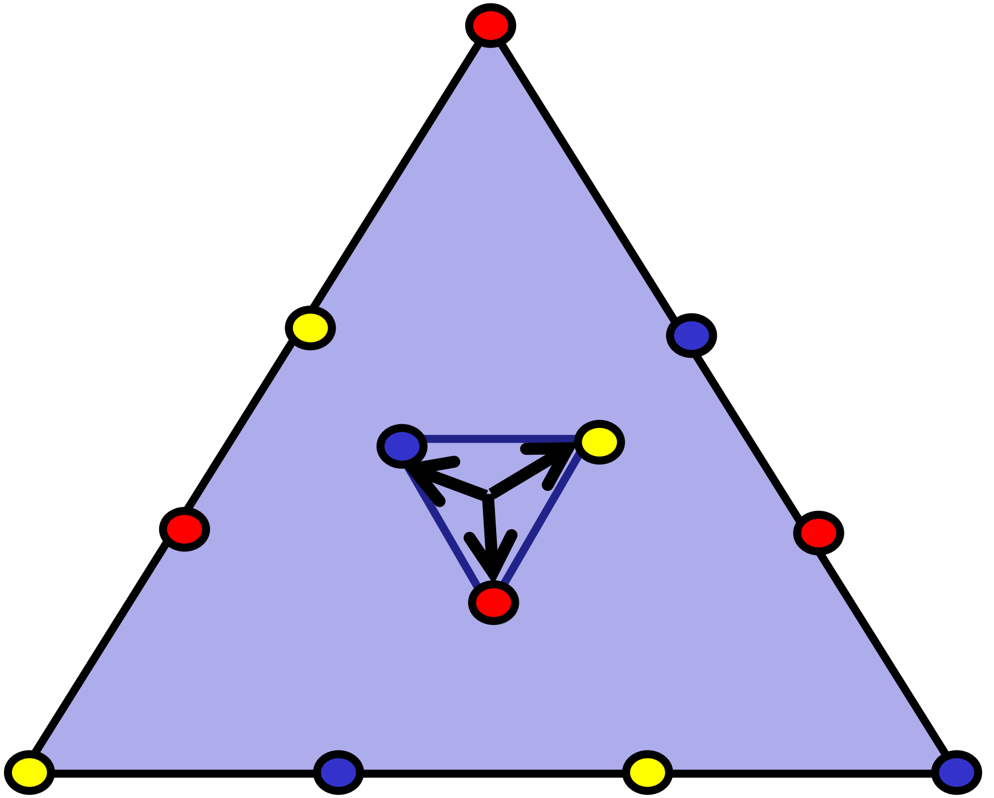


Geometric construction



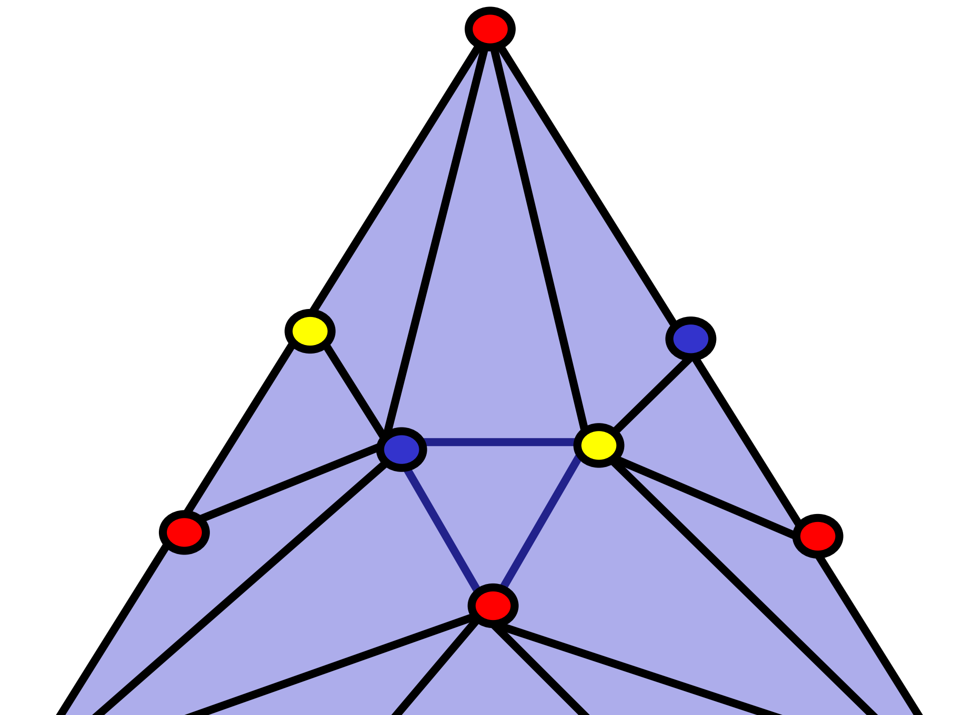
Inductively divide boundary

Geometric construction

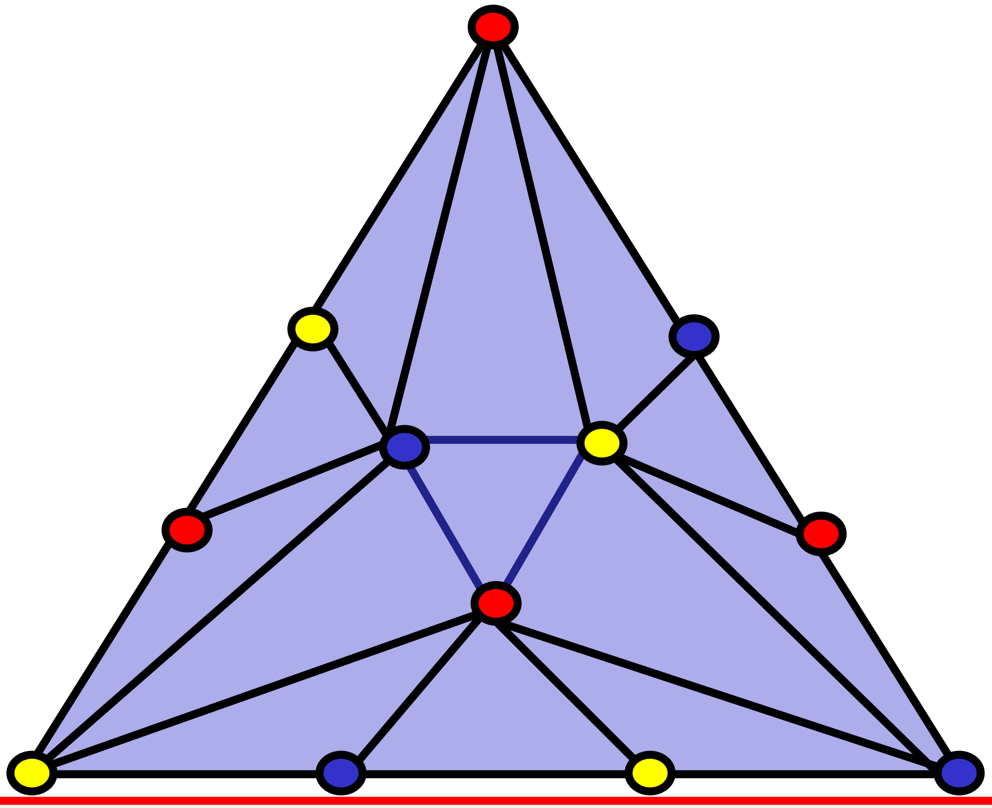


Displace vetexes from barycenter

Geometric construction

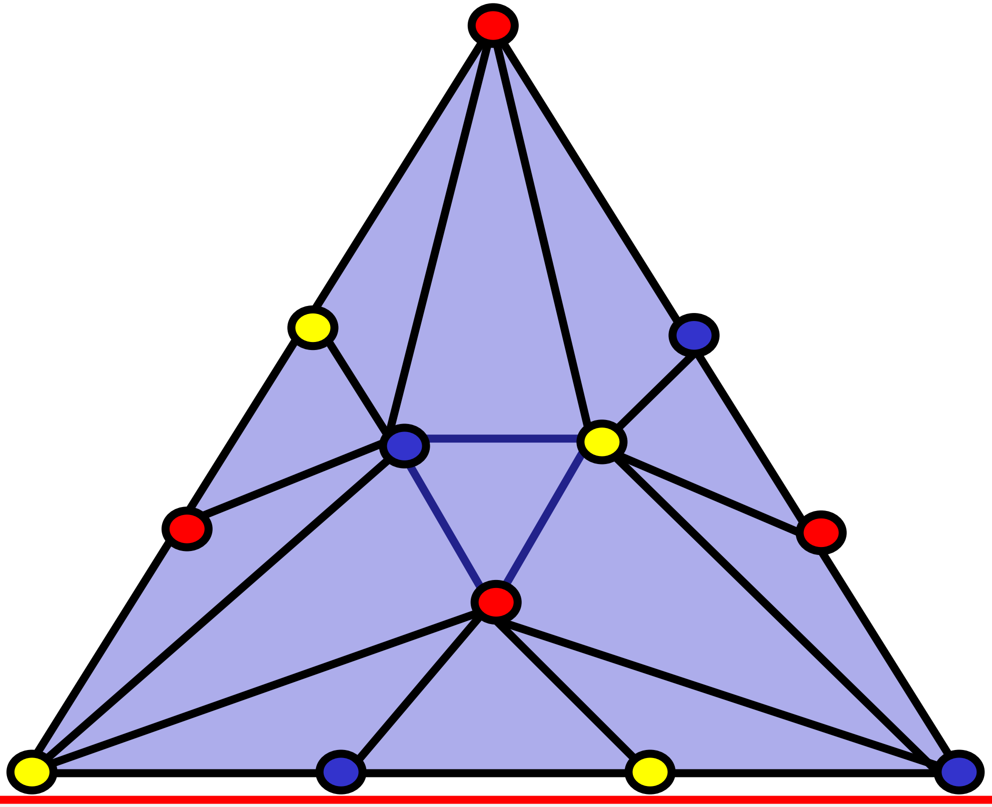


Geometric construction



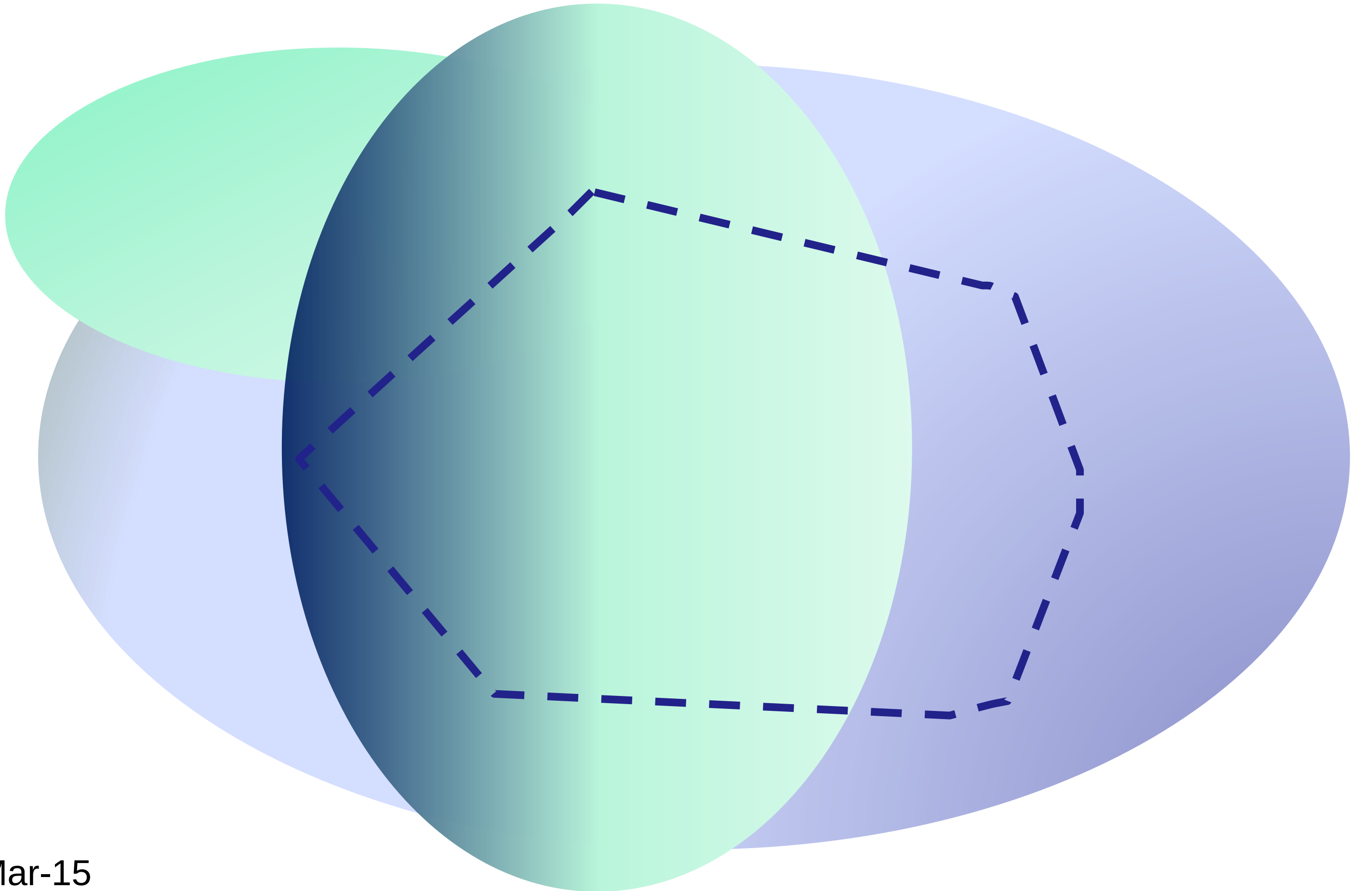
Mesh(Ch σ) is max diameter of a simplex

Subdivision shrinks mesh

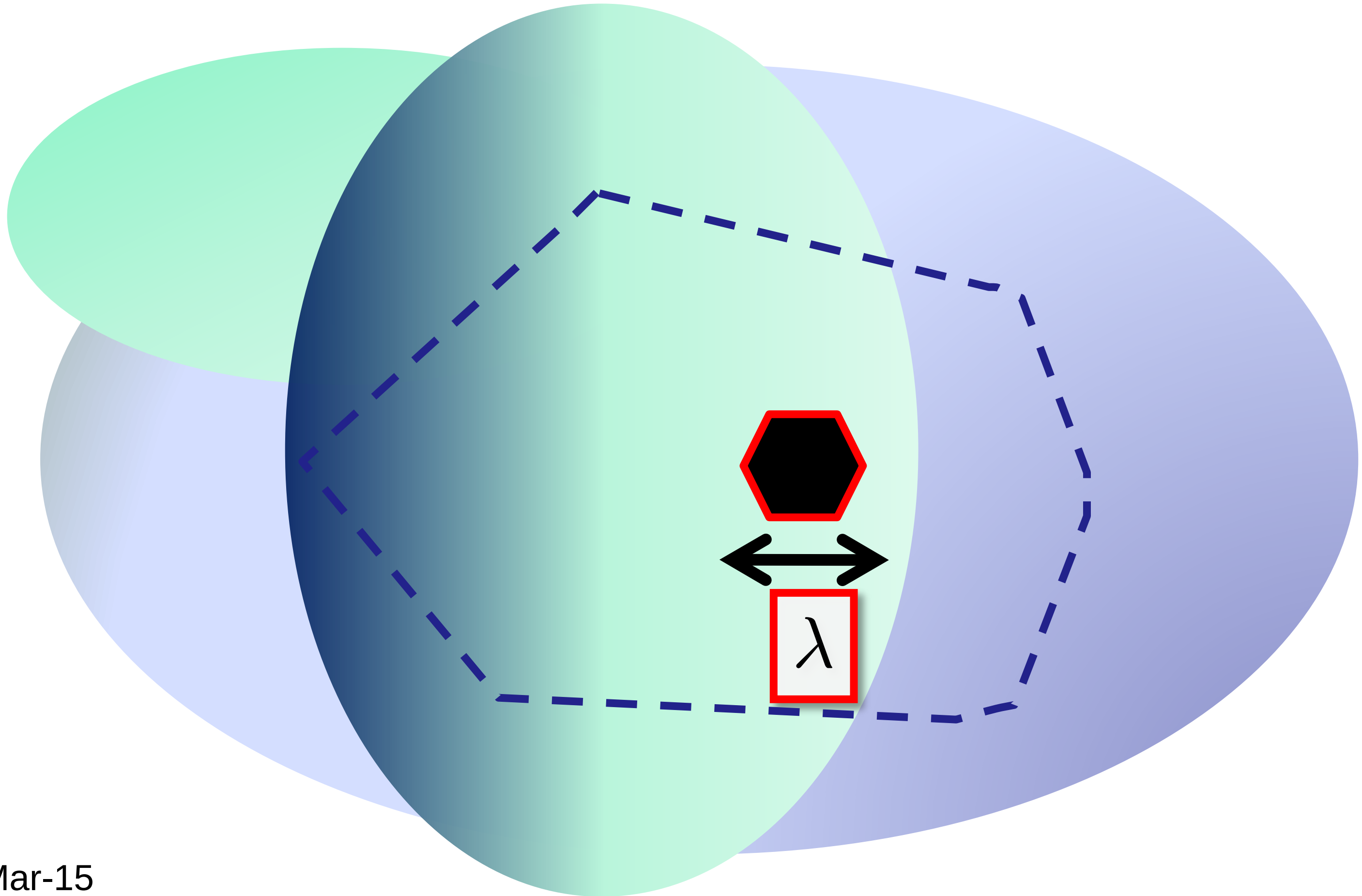


mesh(Ch σ) $\leq$ c mesh(σ) for some $0 < c < 1$

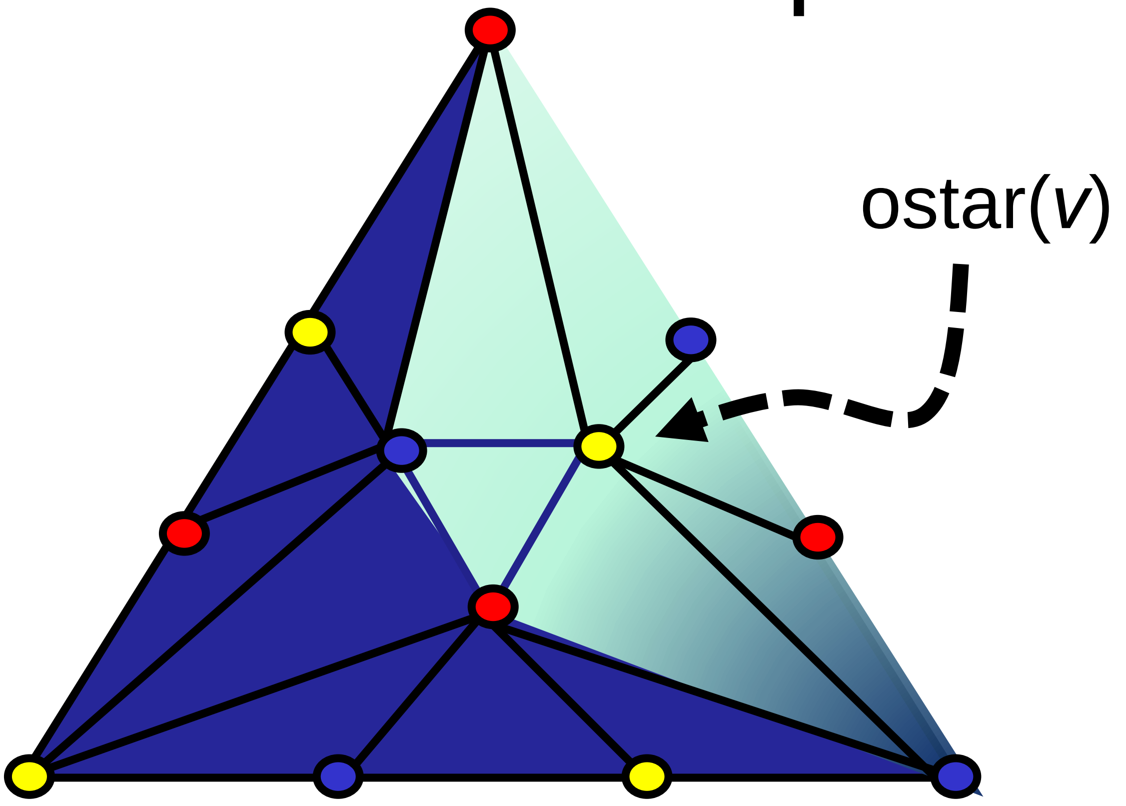
Open cover



Lesbesgue Number

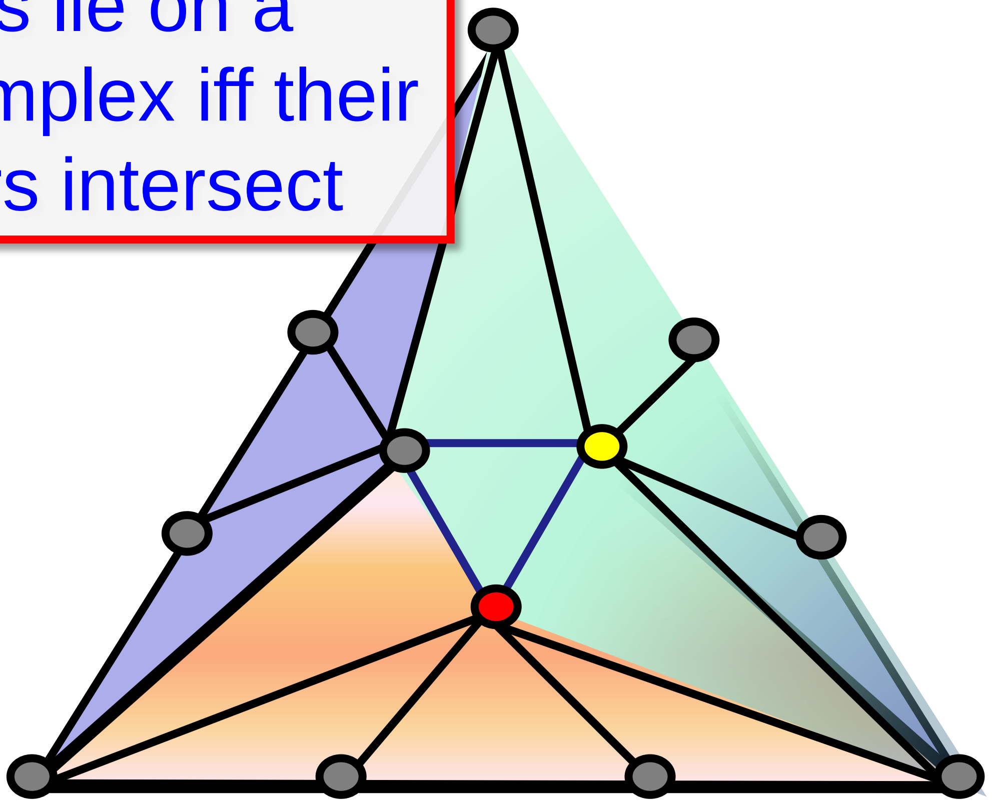


Open stars form an open cover for a complex

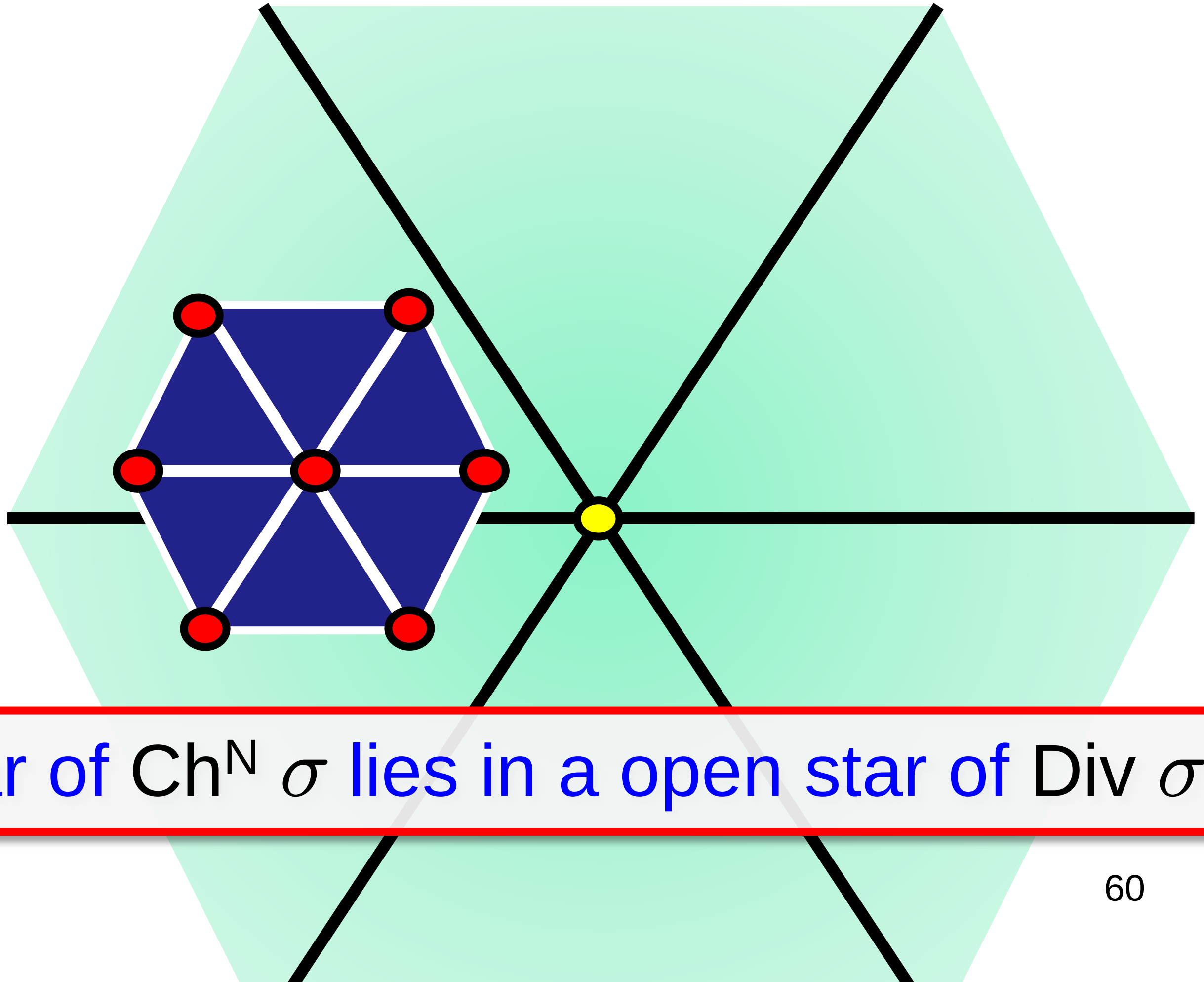


Intersection Lemma

Vertexes lie on a common simplex iff their open stars intersect



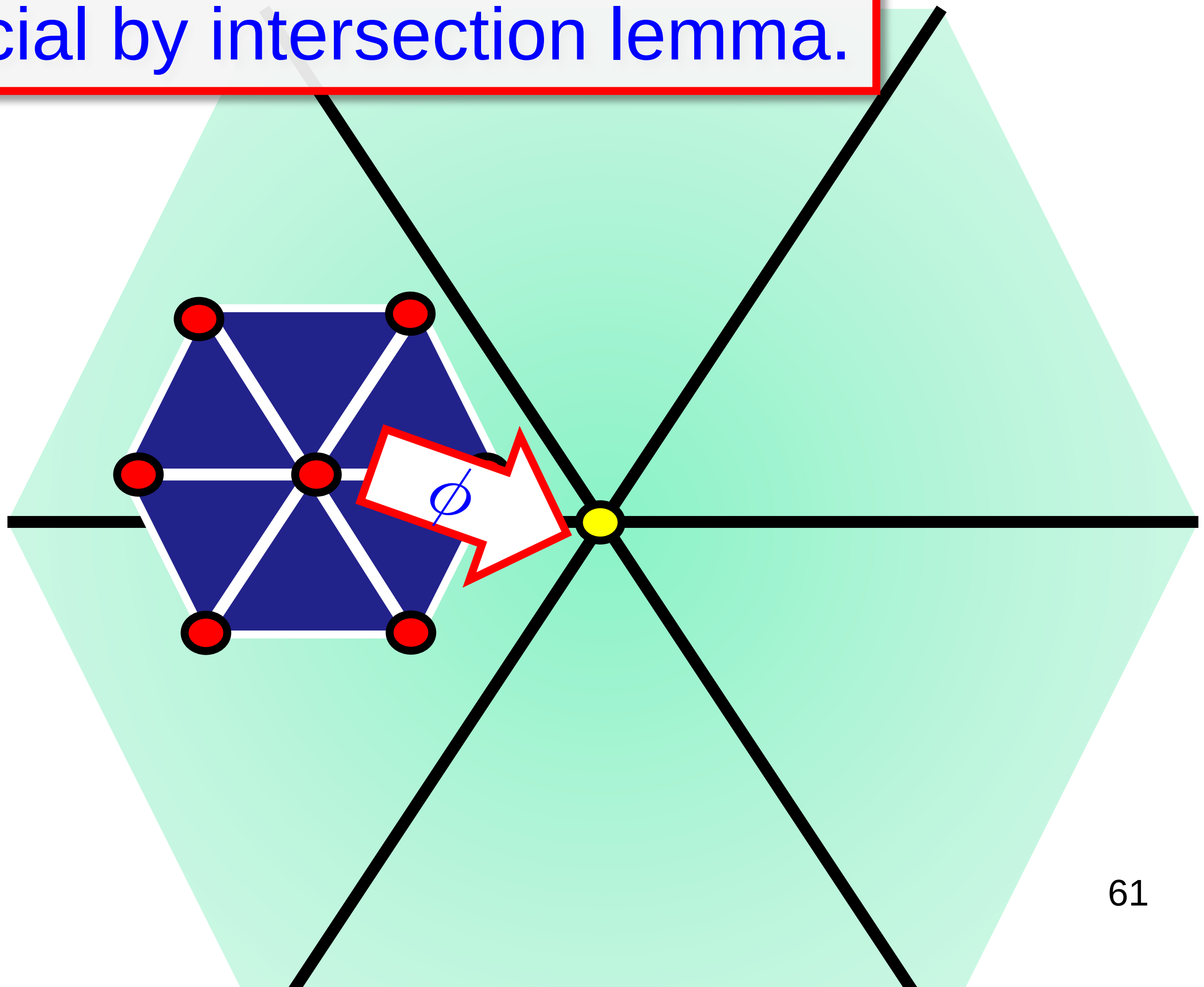
Pick N large enough that each (closed) star of $Ch^N \sigma$ has diameter less than λ …



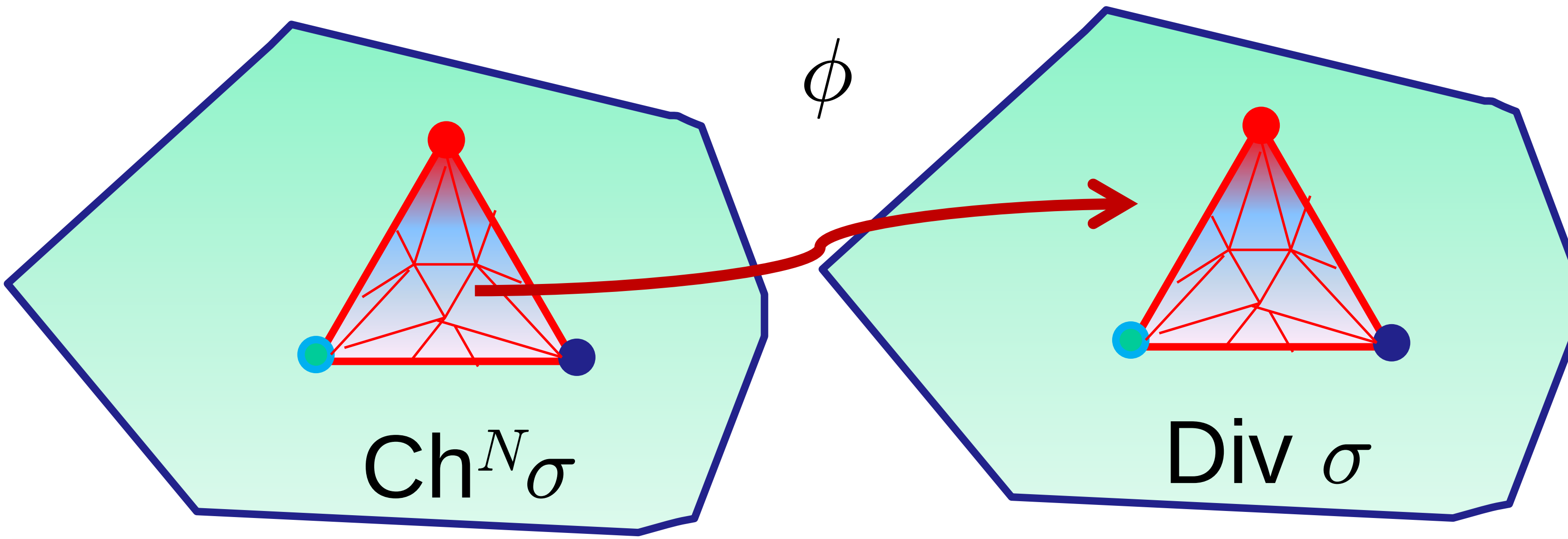
… each star of $Ch^N \sigma$ lies in a open star of Div σ

Defines a vertex map ….

Simplicial by intersection lemma.



We have just proved the Simplicial Approximation Theorem

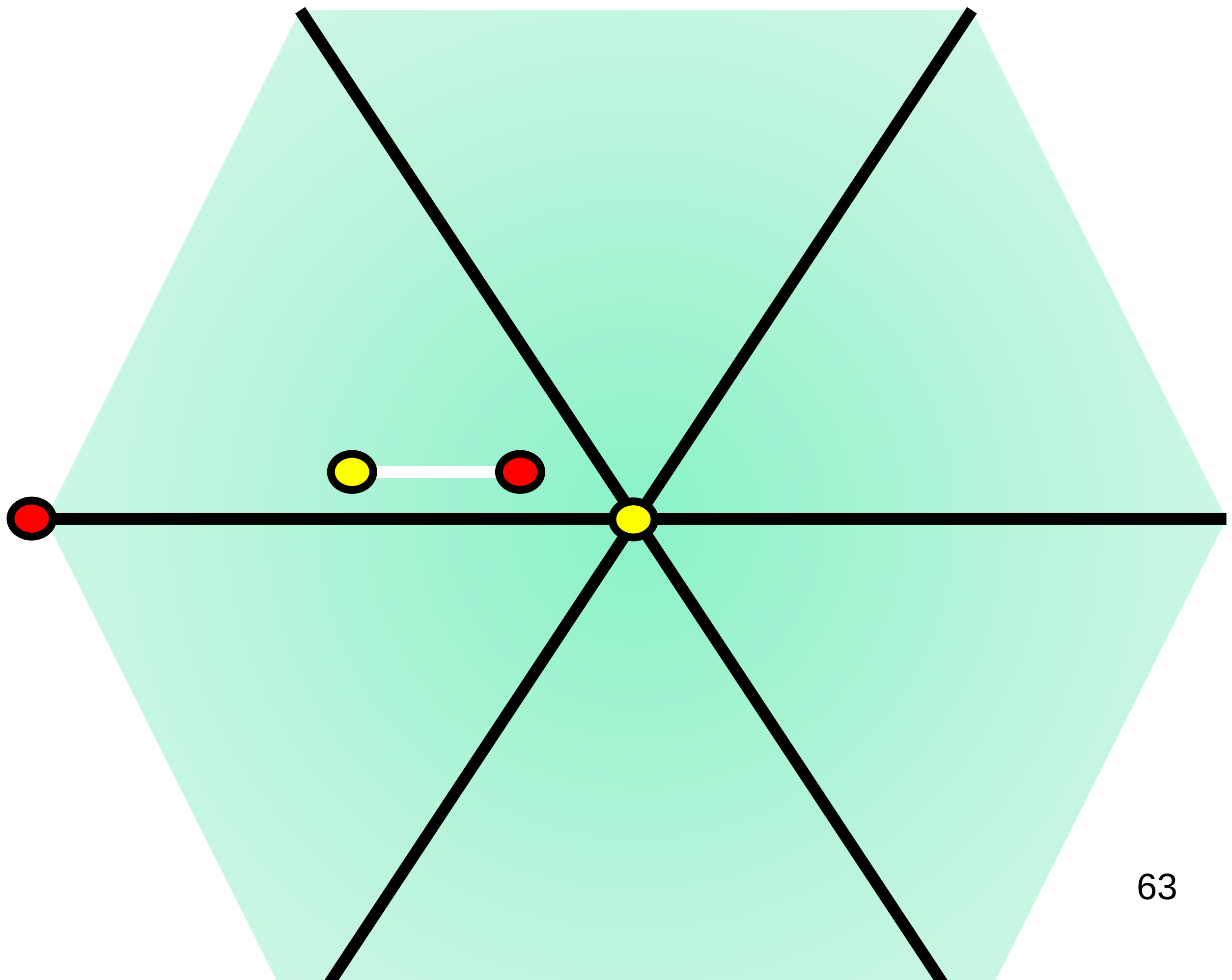


There is a carrier-preserving simplicial map

ϕ: Ch^N $\sigma \to$ Div σ

Not necessarily color-preserving!

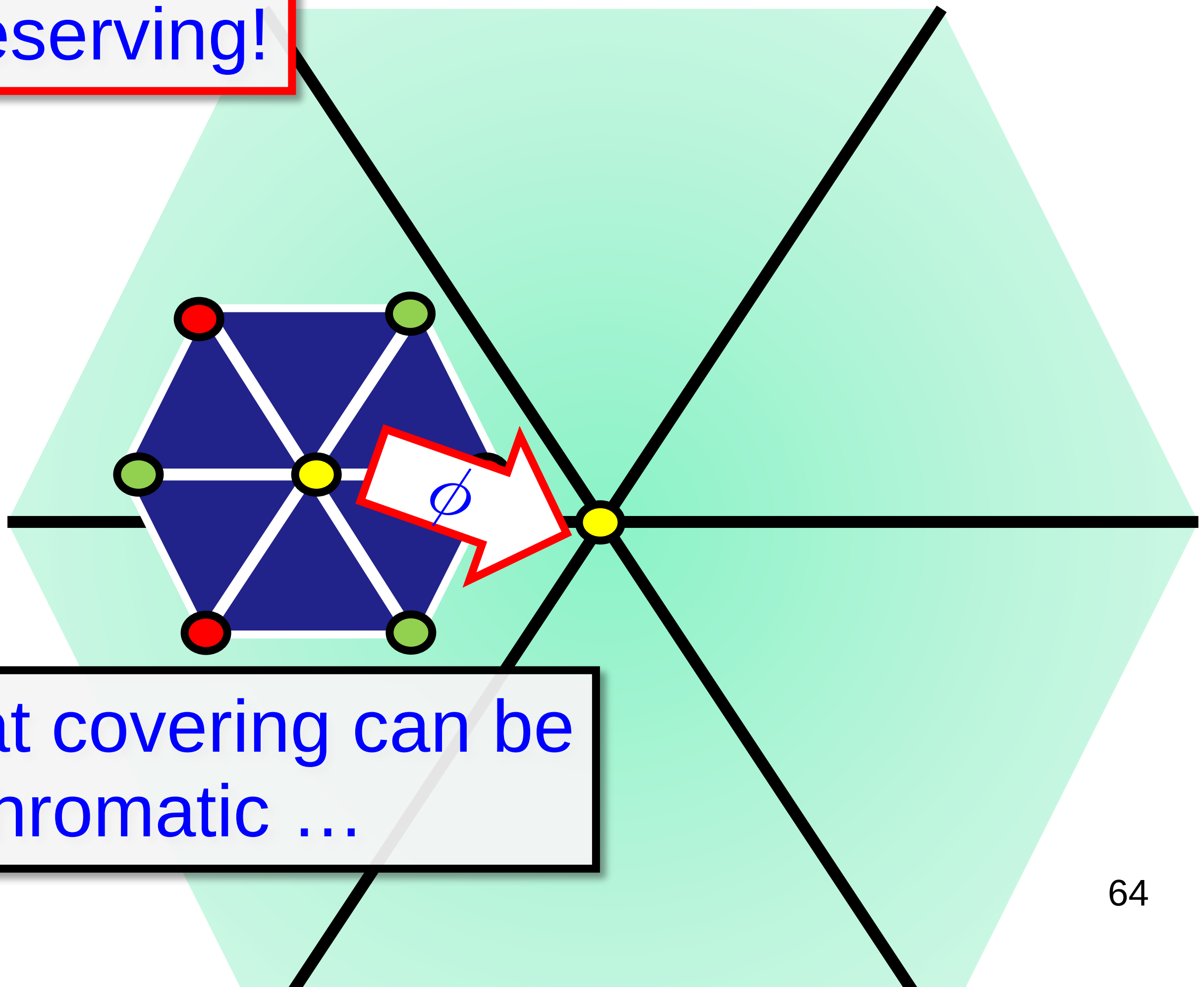
An open-star cover is *chromatic* if every simplex τ of $Ch^N \sigma$ is covered by open stars of of the same color.



If the open-star cover is chromatic ….

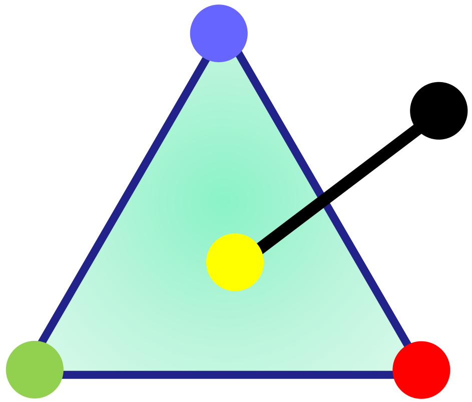
Then the simplicial map ….

Is color preserving!



Must show that covering can be made chromatic …

Open Cover Fail



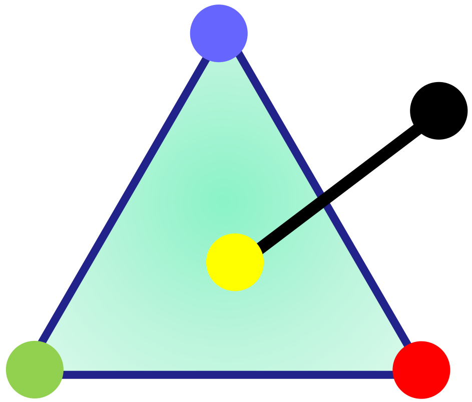
Two simplexes conflict …

If colors disjoint, but …

polyhedrons overlap.

cannot map to same color

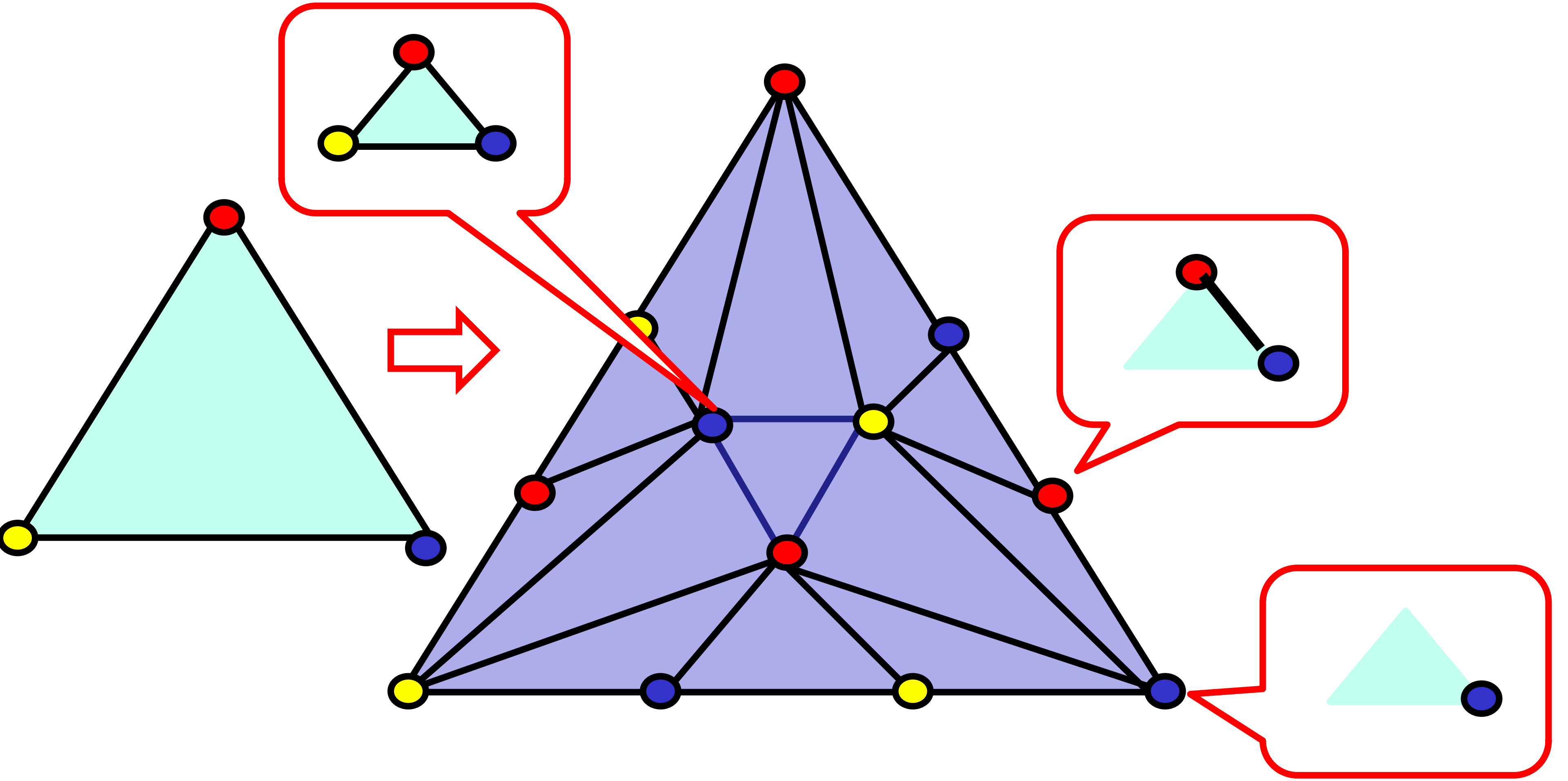
Open Cover Fail



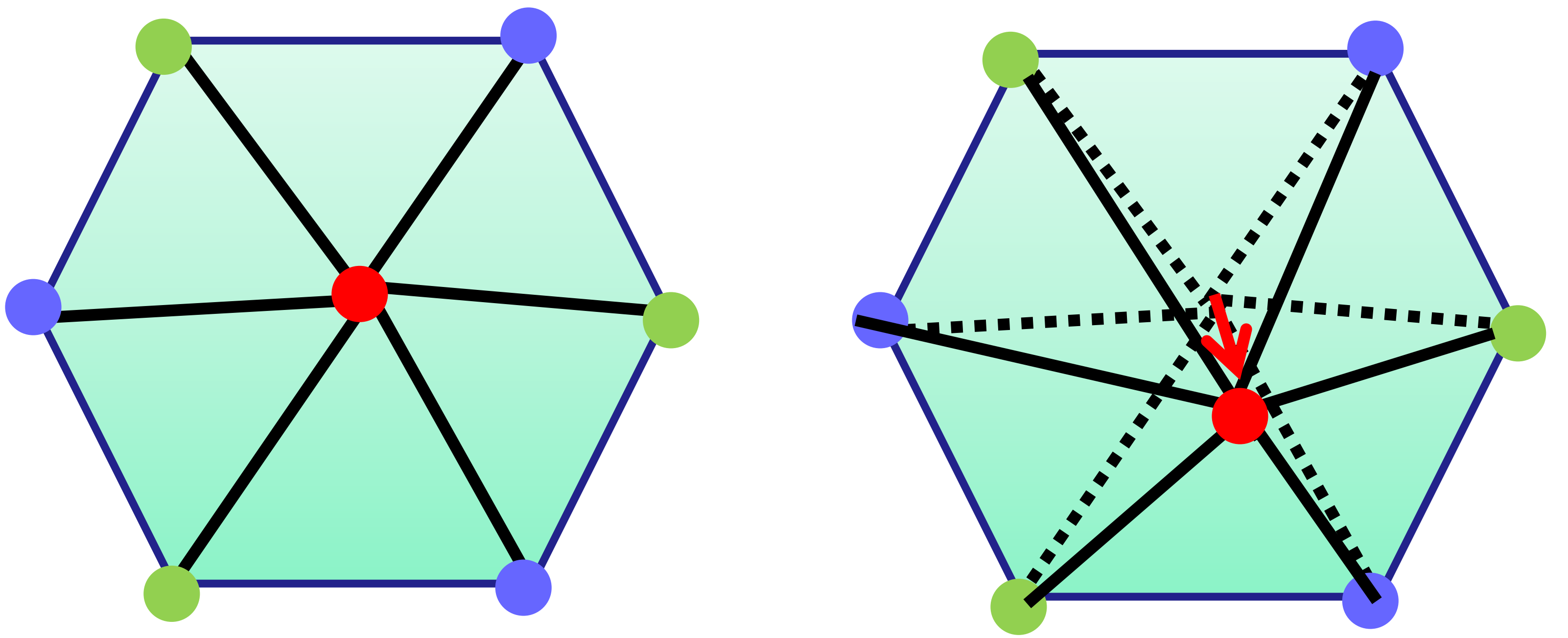
An open-star cover is chromatic iff there are no conflicting simplexes.

We will show how to eliminate conflicting simplexes

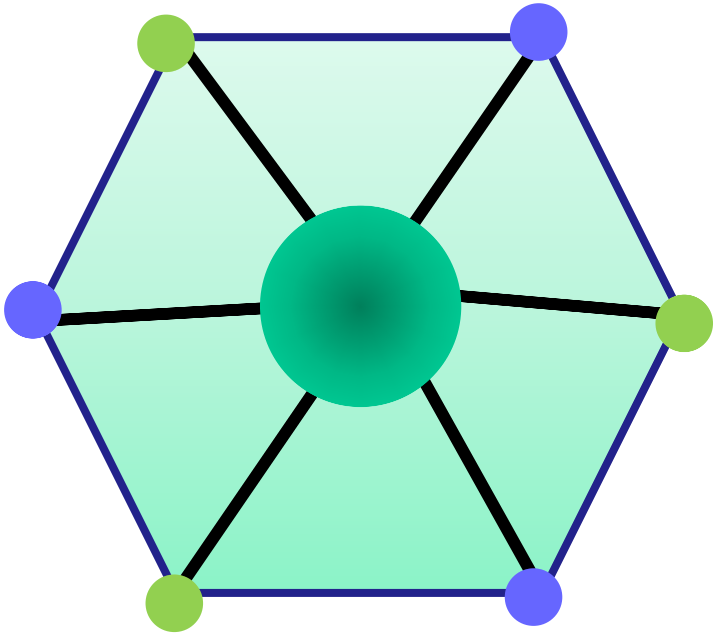
Carriers



Perturbation

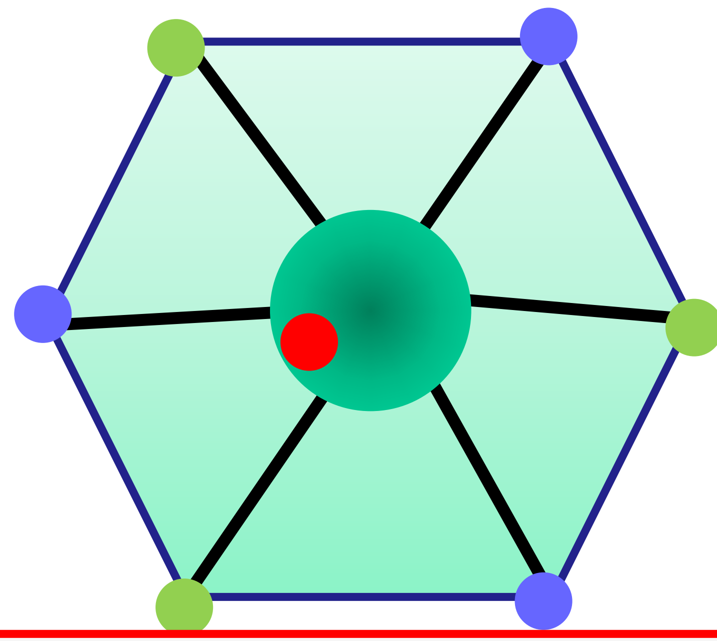


Room for perturbation



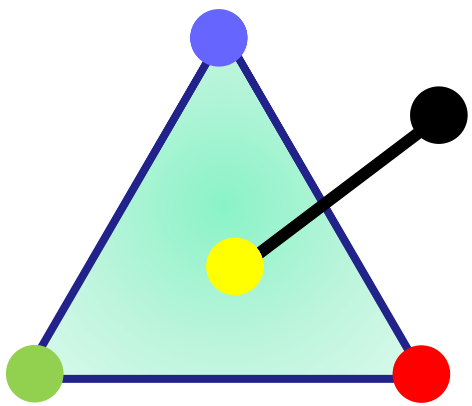
Star contains ϵ ball in carrier around vertex

Room for perturbation



Can perturb to any point within ϵ ball in carrier and still have subdivision

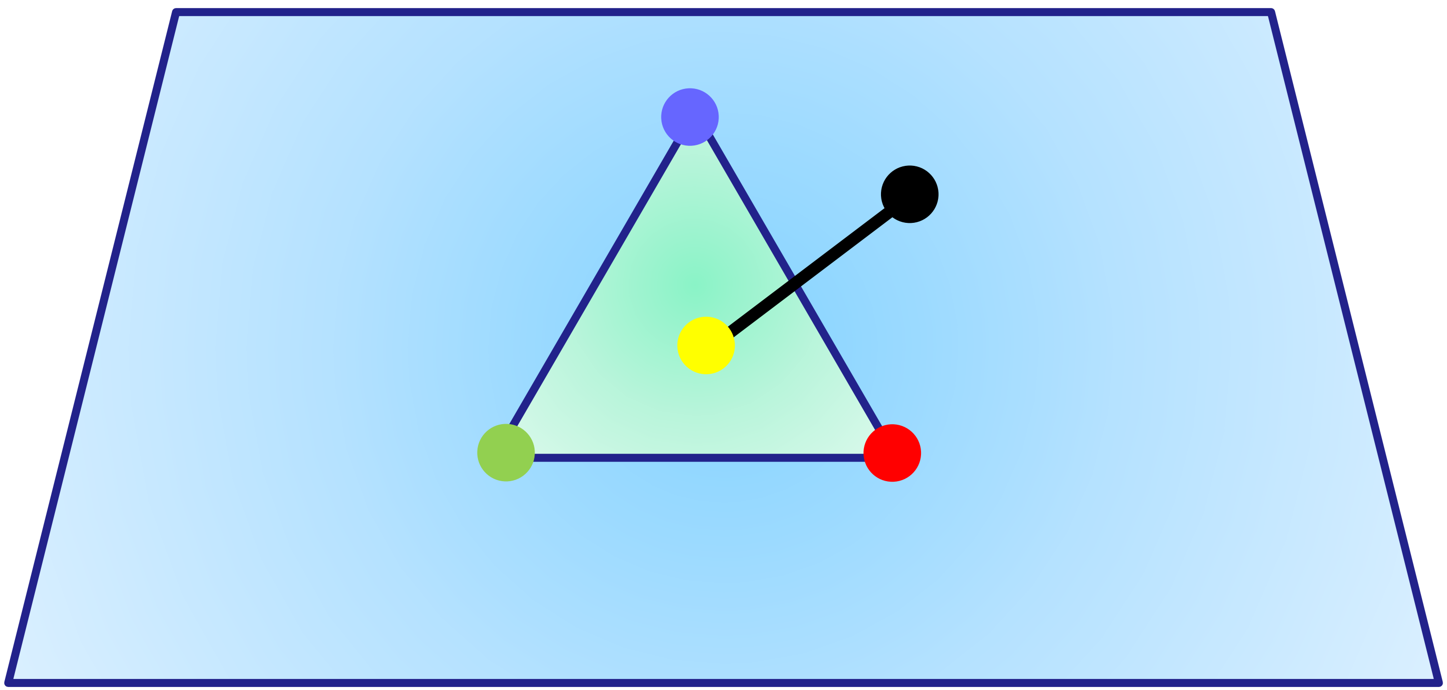
Open Cover Fail

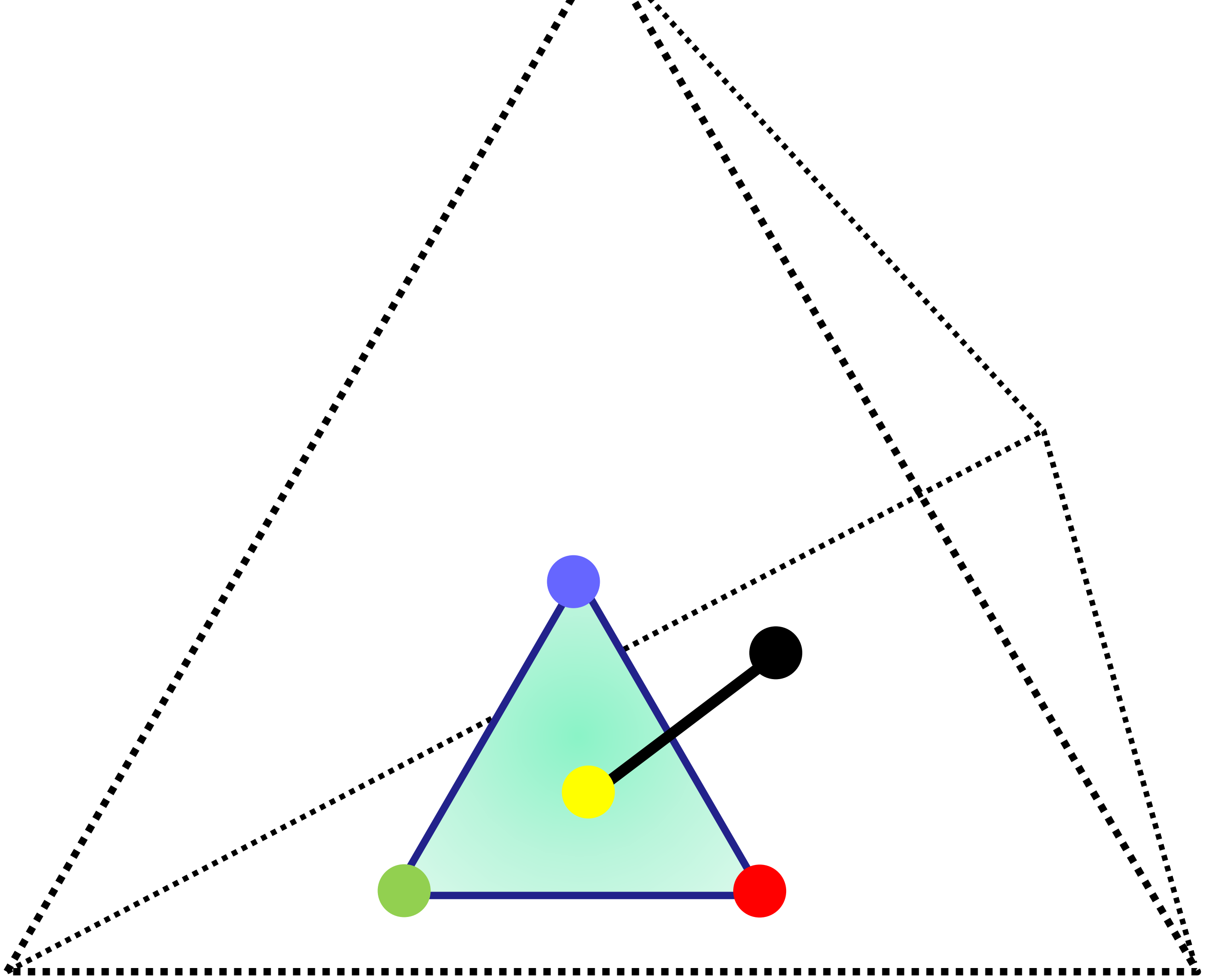


ρ has q+1 colors

τ has p+1 colors

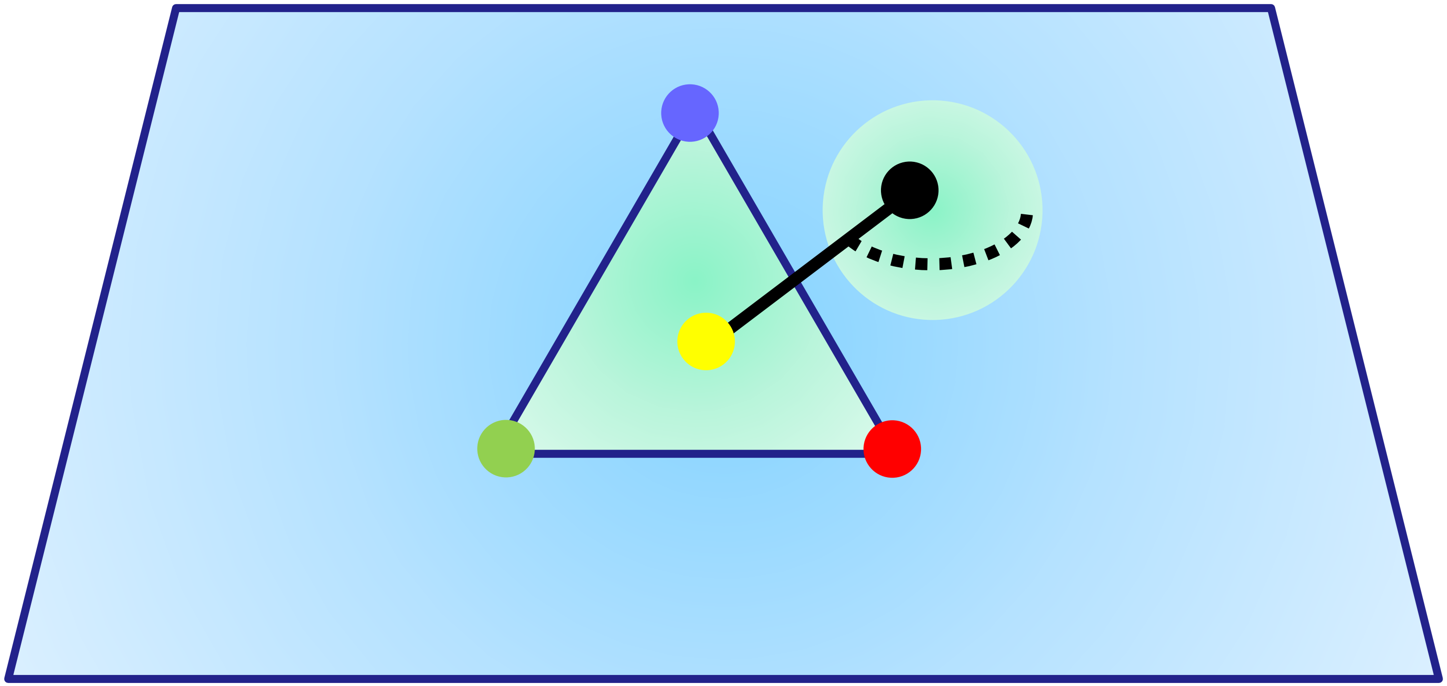
Simplexes lie in hyperplane of dimension $p+q$ (because they overlap)



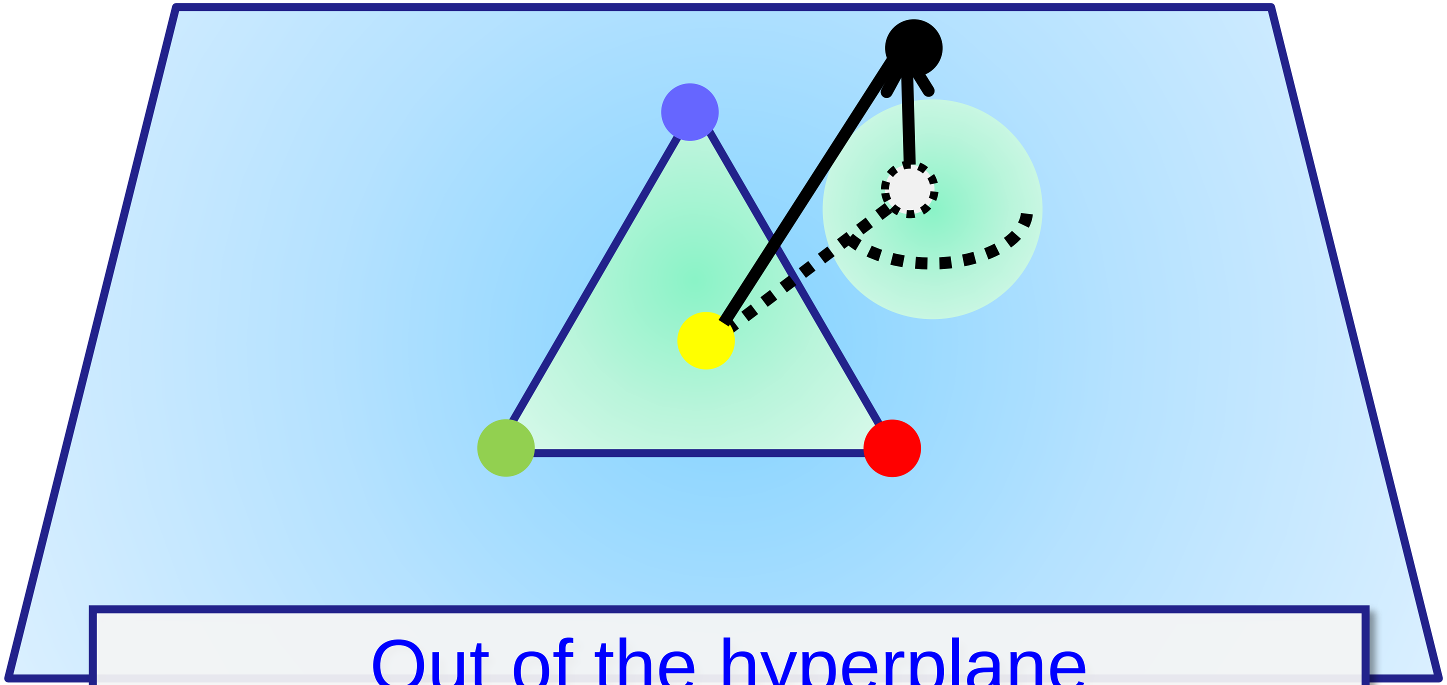


Some vertex has carrier of dimension p+q+1
(because there are p+q+2 colors)

Can perturb vertex within (p+q+1)-dimension ϵ ball …



Can perturb vertex within (p+q+1)-dimension ϵ ball …



Out of the hyperplane

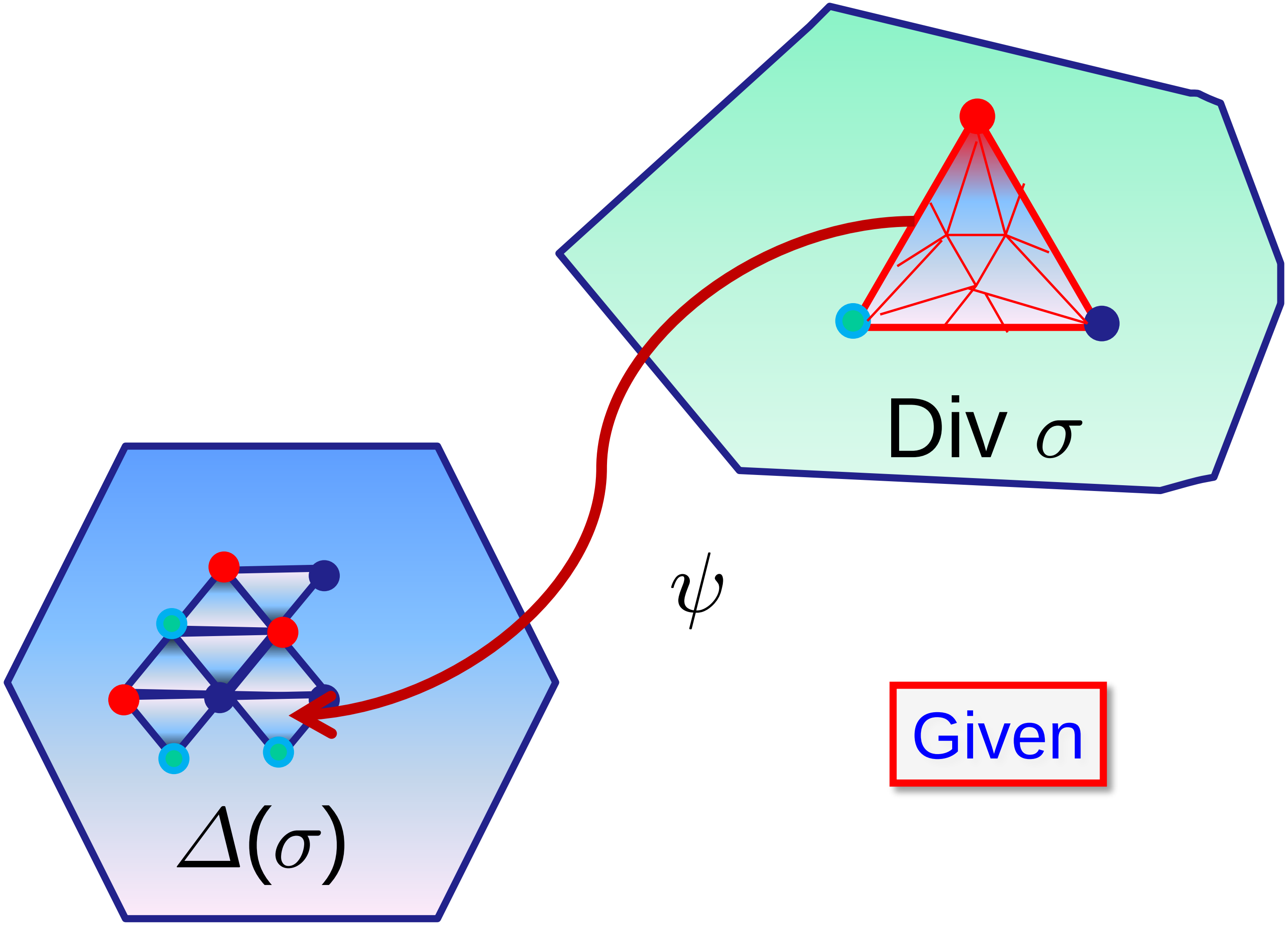
Repeat until star diameter < Lebesgue number:

Construct Ch $Ch^{N-1}_{*}\ \sigma$

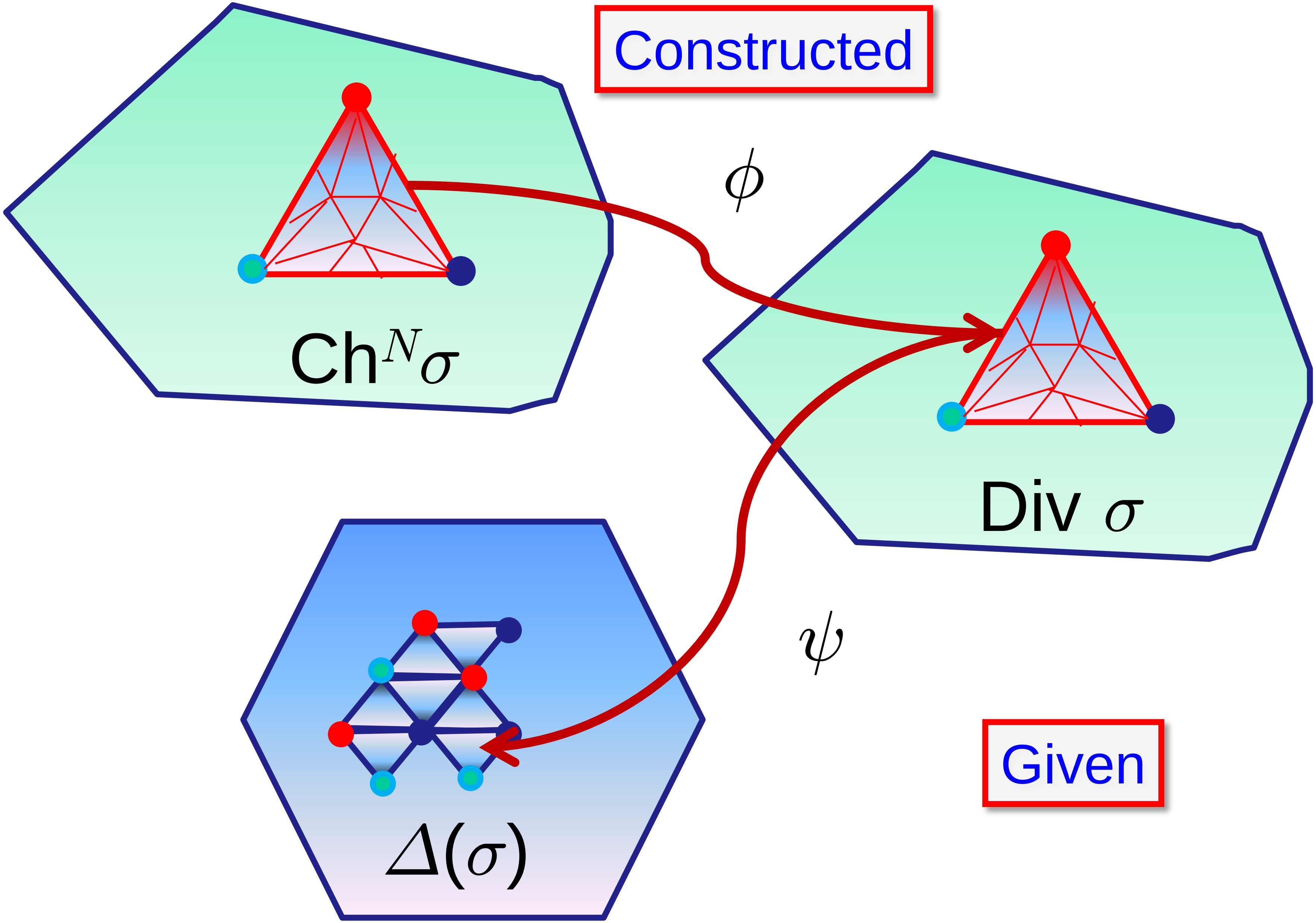
Perturb to $Ch^{N}_{*}\ \sigma$

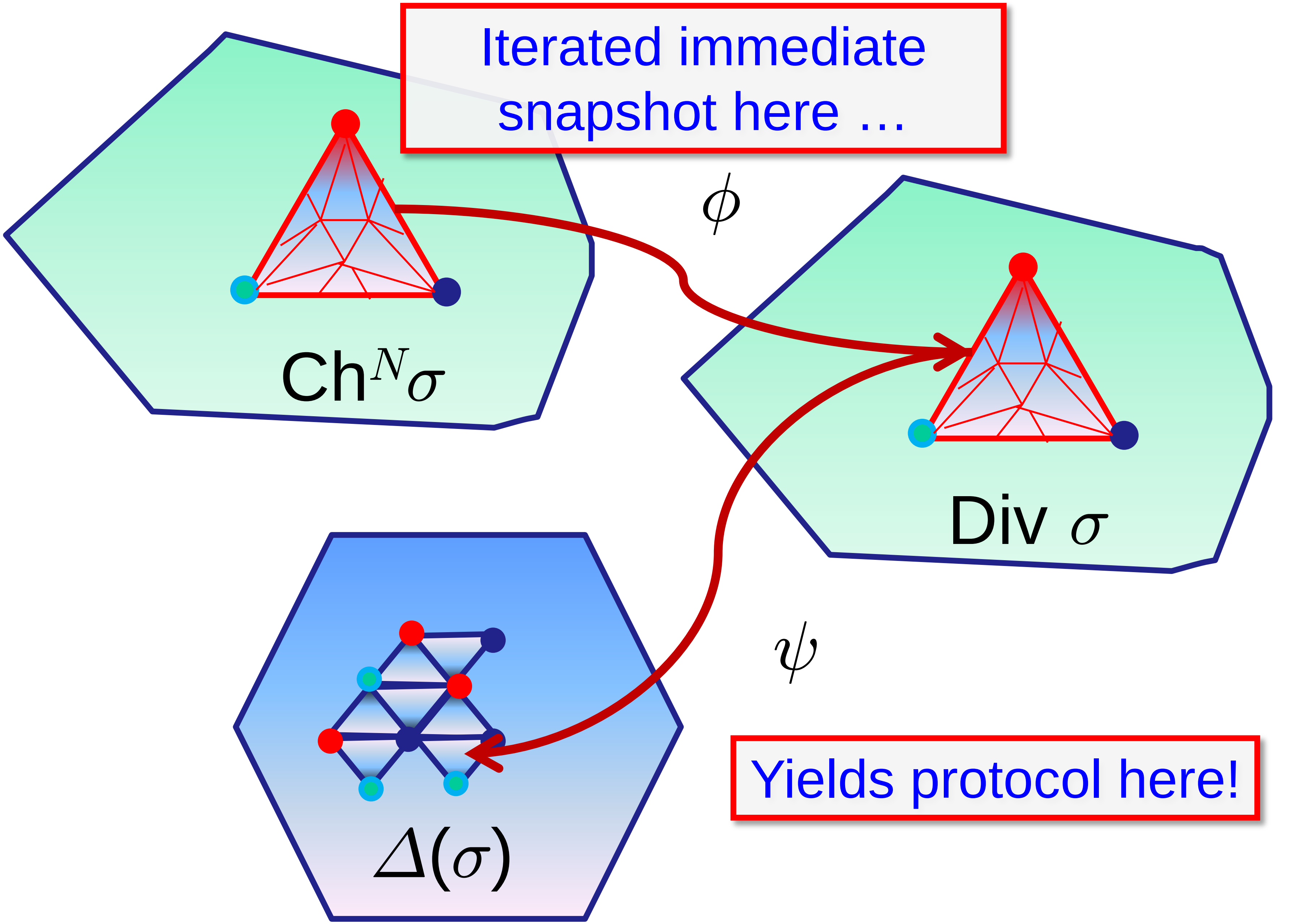
So open-star cover is chromatic

Construct color-preserving simplicial map



Given


Constructed
ϕ
$\mathrm{Ch}^N\sigma$
Div σ
ψ
$\Delta(\sigma)$
Given


Iterated immediate snapshot here …
ϕ
$\mathrm{Ch}^N\sigma$
$\mathrm{Div}\ \sigma$
ψ
$\Delta(\sigma)$
Yields protocol here!

Road Map

Inherently colored tasks

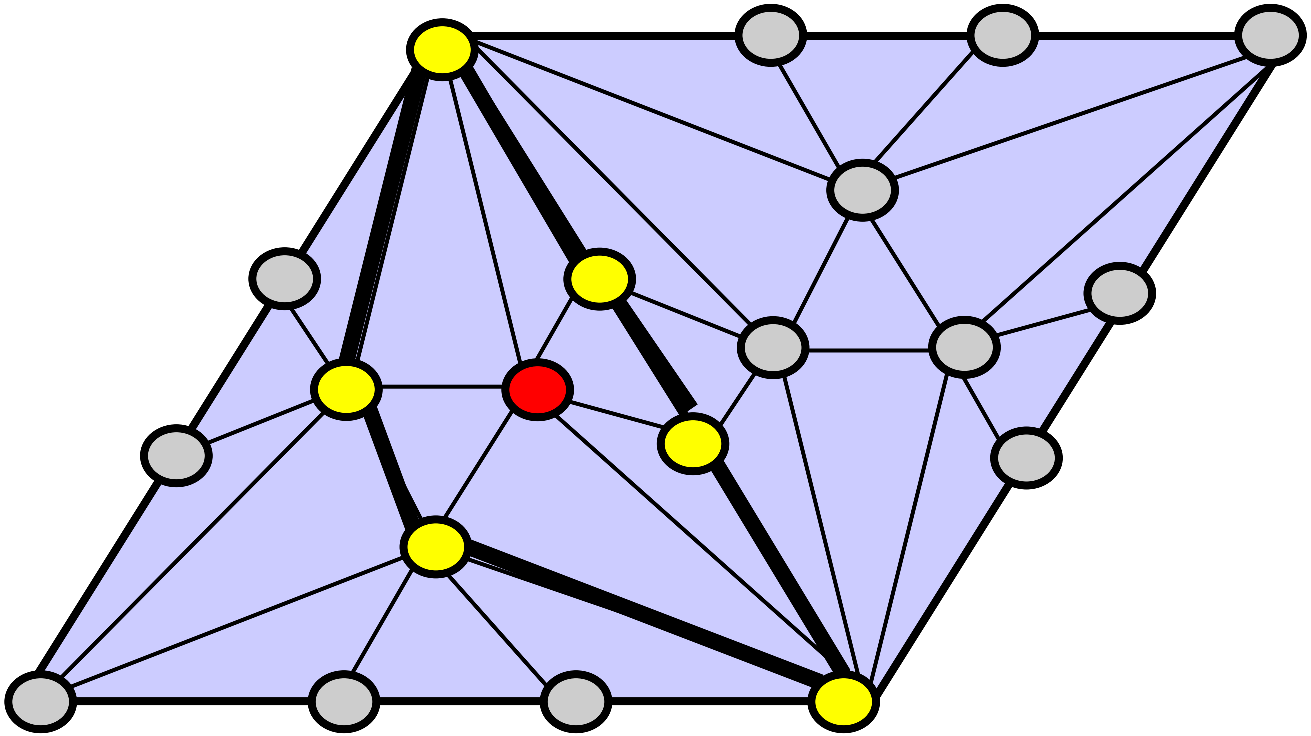
Solvability for colored tasks

Protocol $\Rightarrow$ map

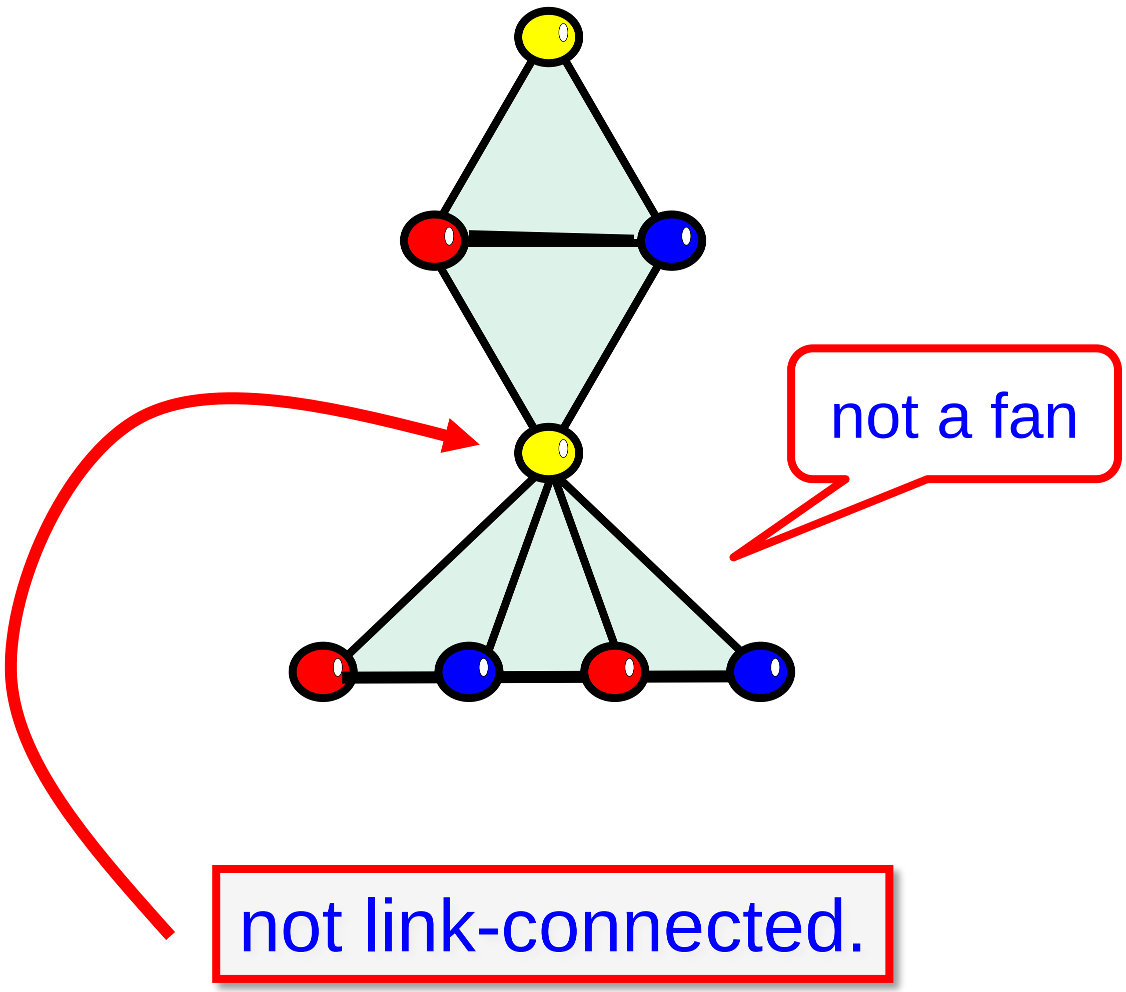
Map $\Rightarrow$ protocol

A Sufficient Topological Condition

Link-Connected



$\mathcal{O}$ is *link-connected* if for each $\tau \in \mathcal{O}$, link(τ,$\mathcal{O}$) is (n - 2 - dim τ)-connected.


not a fan
not link-connected.

Theorem

If, for all $\sigma \in \mathcal{I}$, $\Delta(\sigma)$ is

((dim σ)-1)-connected, and

$\mathcal{O}$ is link-connected

then $(\mathcal{I},\mathcal{O},\Delta)$ has a wait-free IS protocol

Proof Strategy

If, for all $\sigma \in \mathcal{I}$, $\Delta(\sigma)$ is

((dim σ)-1)-connected, and

$\mathcal{O}$ is link-connected,

there exists subdivision Div
& color-preserving simplicial map
μ: Div $\mathcal{I} \rightarrow \mathcal{O}$ carried by Δ.

Proof Strategy

If, for all $\sigma \in \mathcal{I}$, $\Delta(\sigma)$ is

((dim σ)-1)-connected, and

$\mathcal{O}$ is link-connected

so protocol exists
by colored theorem

there exists subdivision Div
& color-preserving simplicial map
μ: Div $\mathcal{I} \rightarrow \mathcal{O}$ carried by Δ.

Lemma

rigid & color-preserving on boundary means color-preserving everywhere

suppose we have a rigid simplicial map

ϕ: Div $\sigma \to \mathcal{O}$

that is color-preserving on Div $\partial\, \sigma$

then ϕ is color-preserving on Div σ

Lemma

If $\mathcal{O}$ is link-connected …

can extend rigid simplicial map

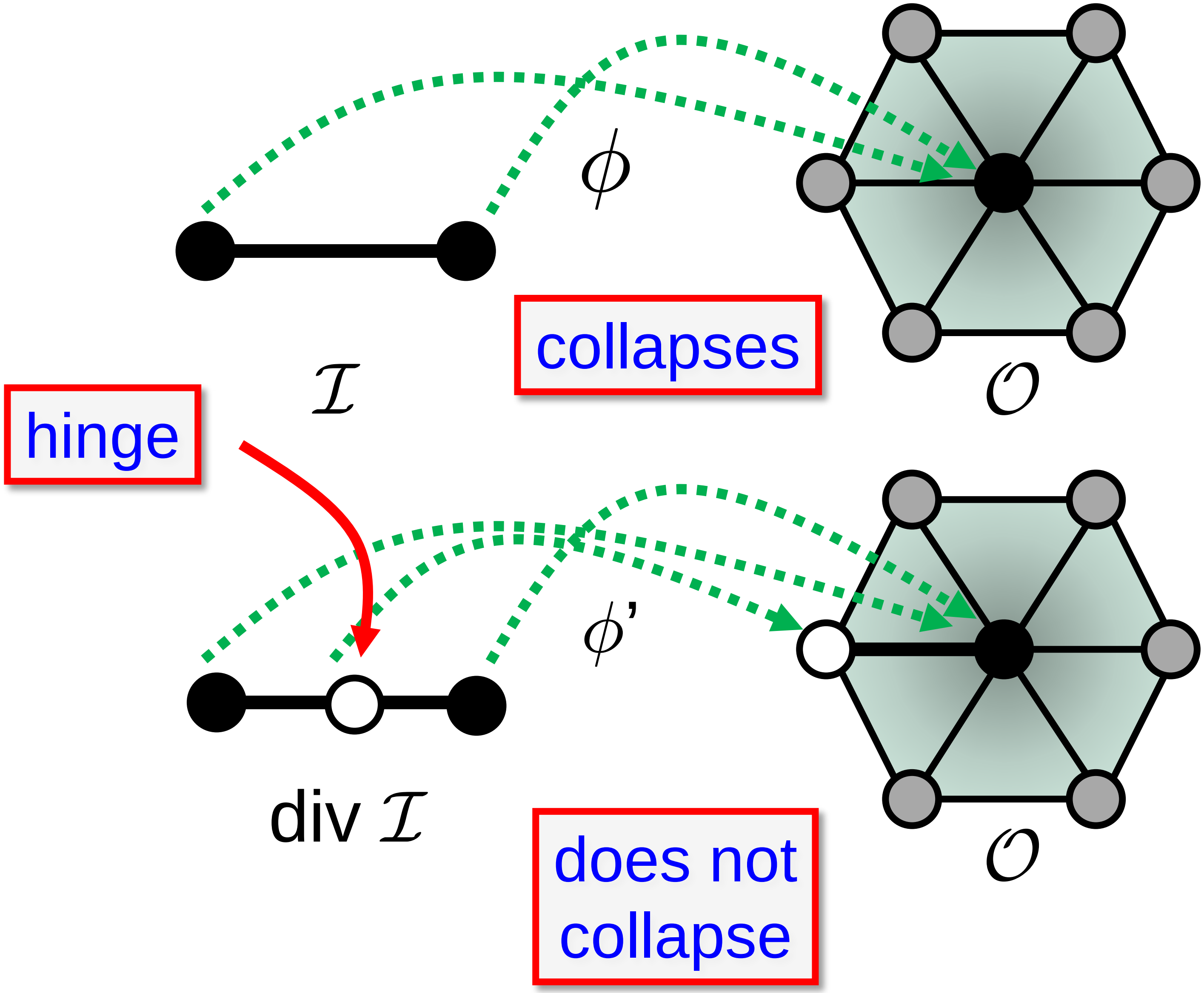
ϕ_{n-1}: skel^{n-1} $\mathcal{I} \to \mathcal{O}$

to a rigid simplicial map

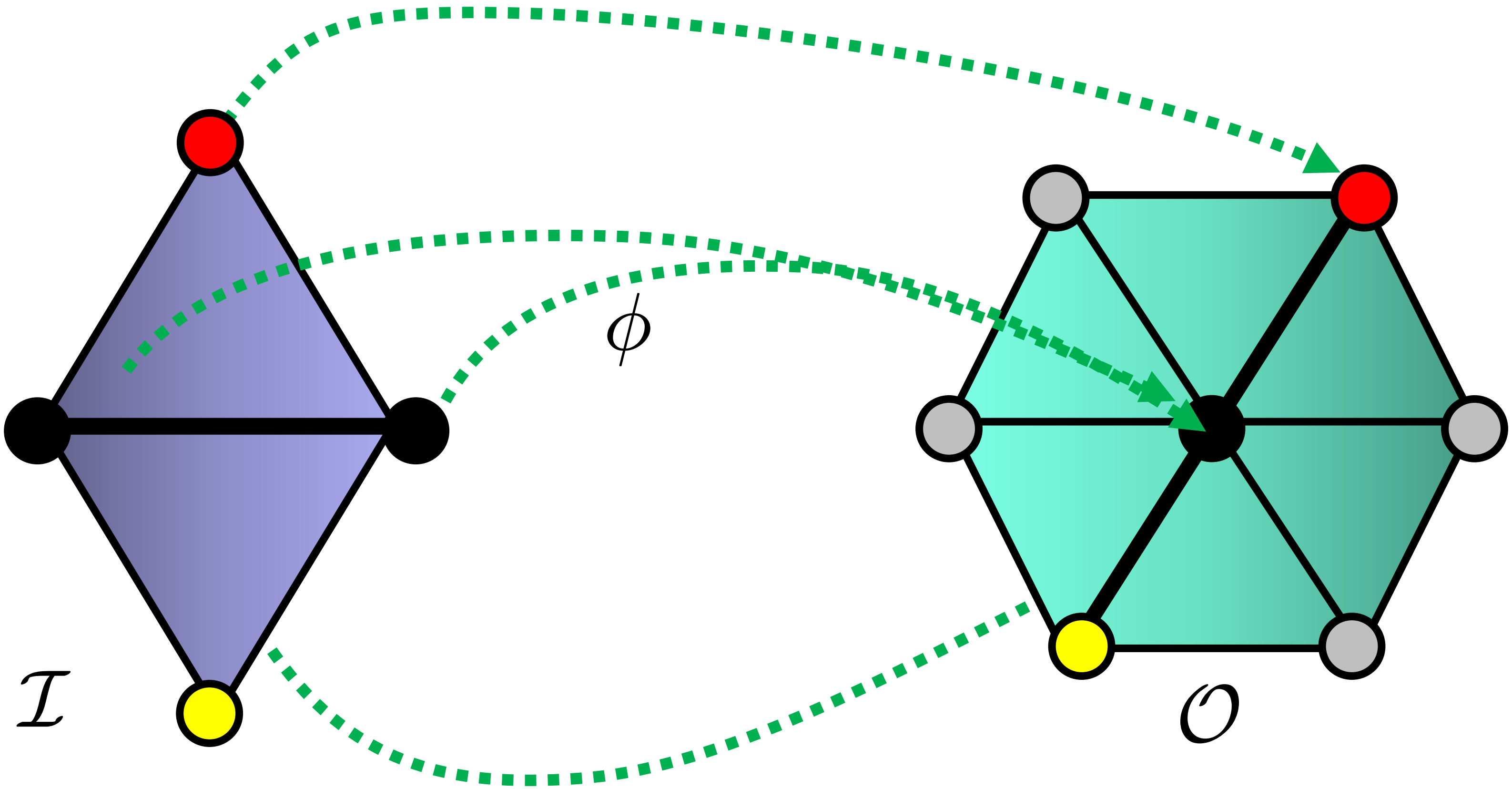
ϕ_n: Div $\mathcal{I} \to \mathcal{O}$

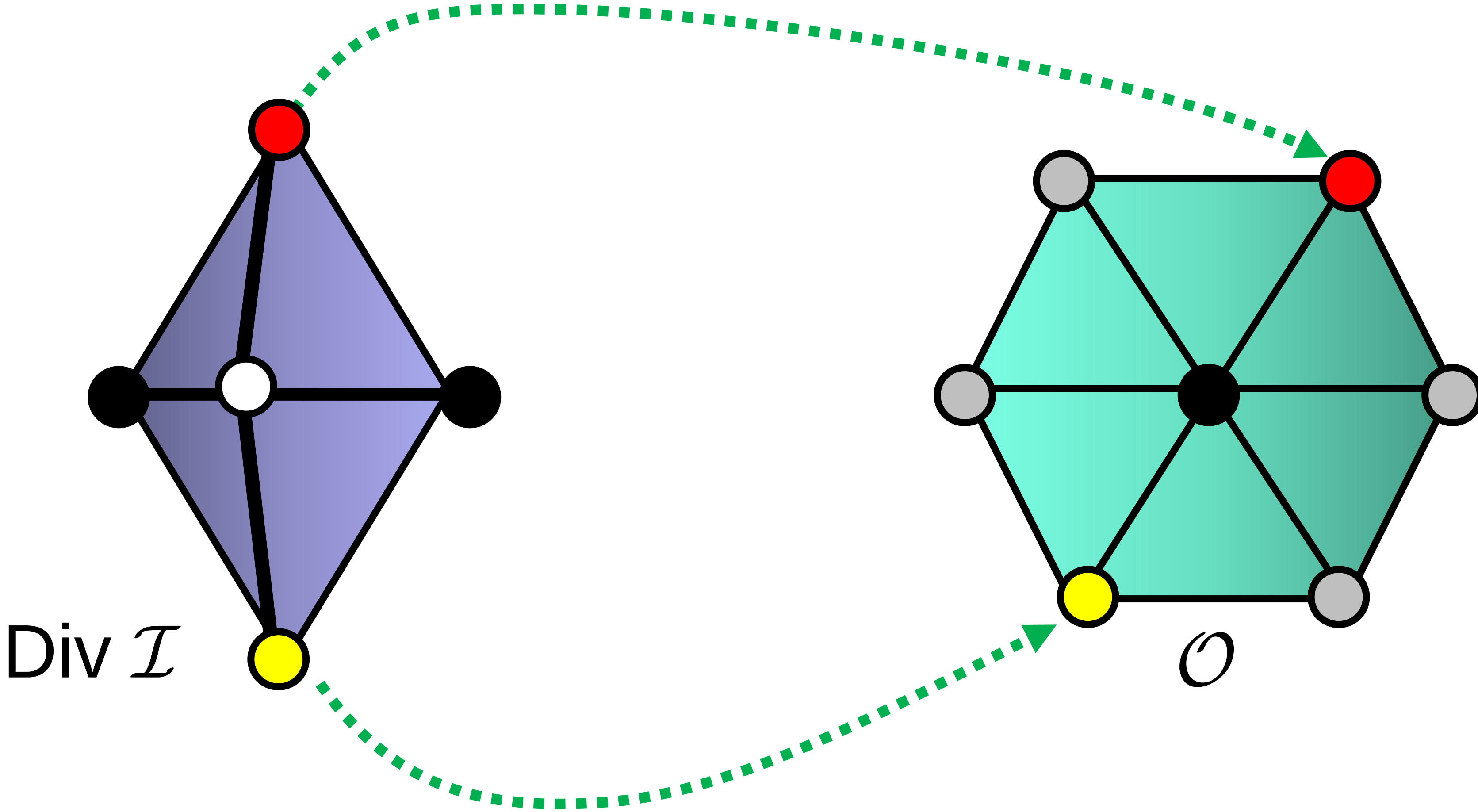
where Div skel^{n-1} $\mathcal{I}$ = skel^{n-1} $\mathcal{I}$

Induction Base: $n = 1$

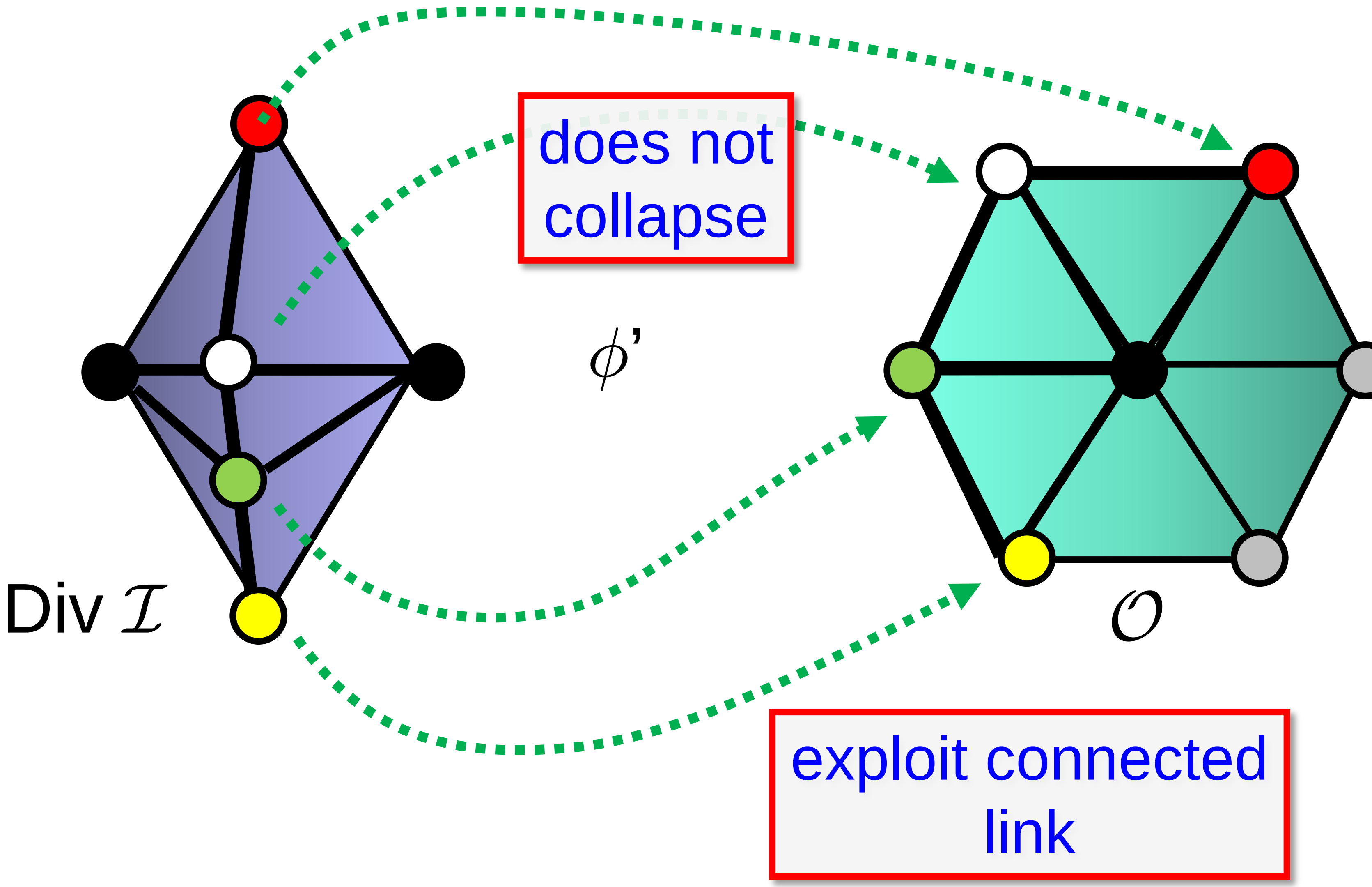


Induction Step: ϕ does not collapse (n-1)-simplexes





Div $\mathcal{I}$
$\mathcal{O}$



does not collapse
ϕ'
Div $\mathcal{I}$
$\mathcal{O}$
exploit connected link

Summary

Inductively use …

$\left\{ \begin{array}{l} \text{connectivity of } \Delta(\sigma) \\ \text{link-connectivity of } \mathcal{O} \end{array} \right.$

to construct a color-preserving simplicial map

ϕ: Div $\mathcal{I} \rightarrow \mathcal{O}$ carried by Δ.

protocol follows from main theorem