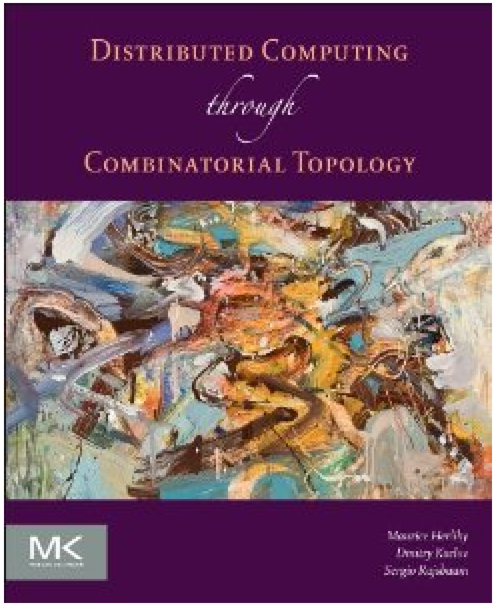
Renaming and Oriented Manifolds

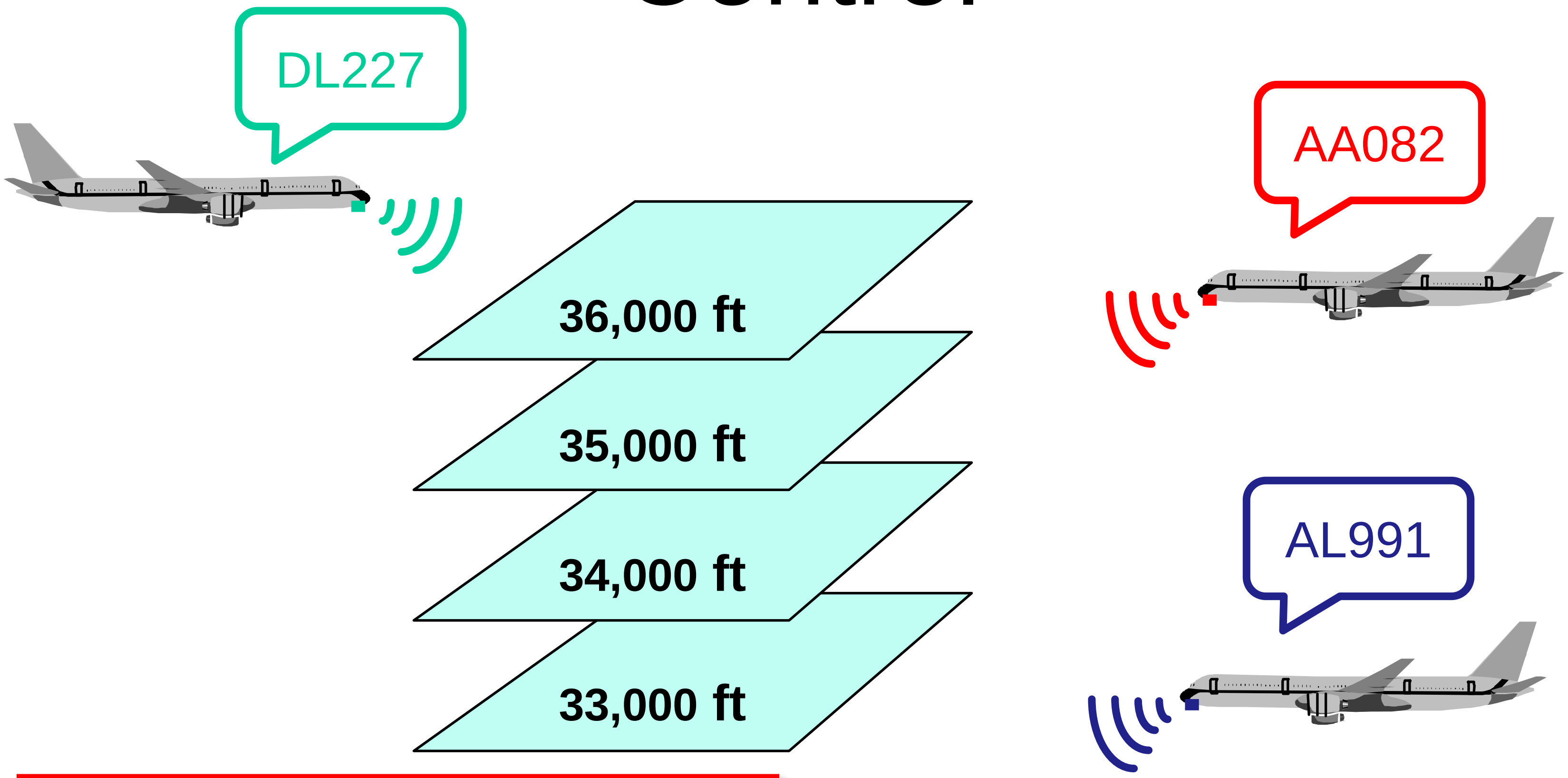


Companion slides for

Distributed Computing Through Combinatorial Topology

Maurice Herlihy & Dmitry Kozlov & Sergio Rajsbaum

Autonomous Air Traffic Control



Pick your own altitude!

How many choices do you need?

Road Map

An Upper Bound: $2n+1$ Names

Weak Symmetry-Breaking

The Index Lemma

Binary Colorings

A Lower Bound for $2n$-Renaming

Road Map

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Index Independence

Avoid trivial solutions …

P_i chooses output name i?

P_i knows its name, but not i

Can test names for order & equality only

Output depends on input and interleavings only

Protocol for 2*n*+1 Names

Let …

$\sigma^n = \{P_0, \ldots, P_n\}$

$\Delta^{2n} = \{0, \ldots, 2n\}$

We will construct

chromatic subdivision

rigid simplicial map

ρ: Rename$(\sigma^n) \to \Delta^{2n}$

Protocol for 2*n* Names

ρ: Rename(σ^n) $\rightarrow \Delta^{2n}$

means that a wait-free immediate snapshot protocol exists

we will also display the protocol …

2 processes,
3 names

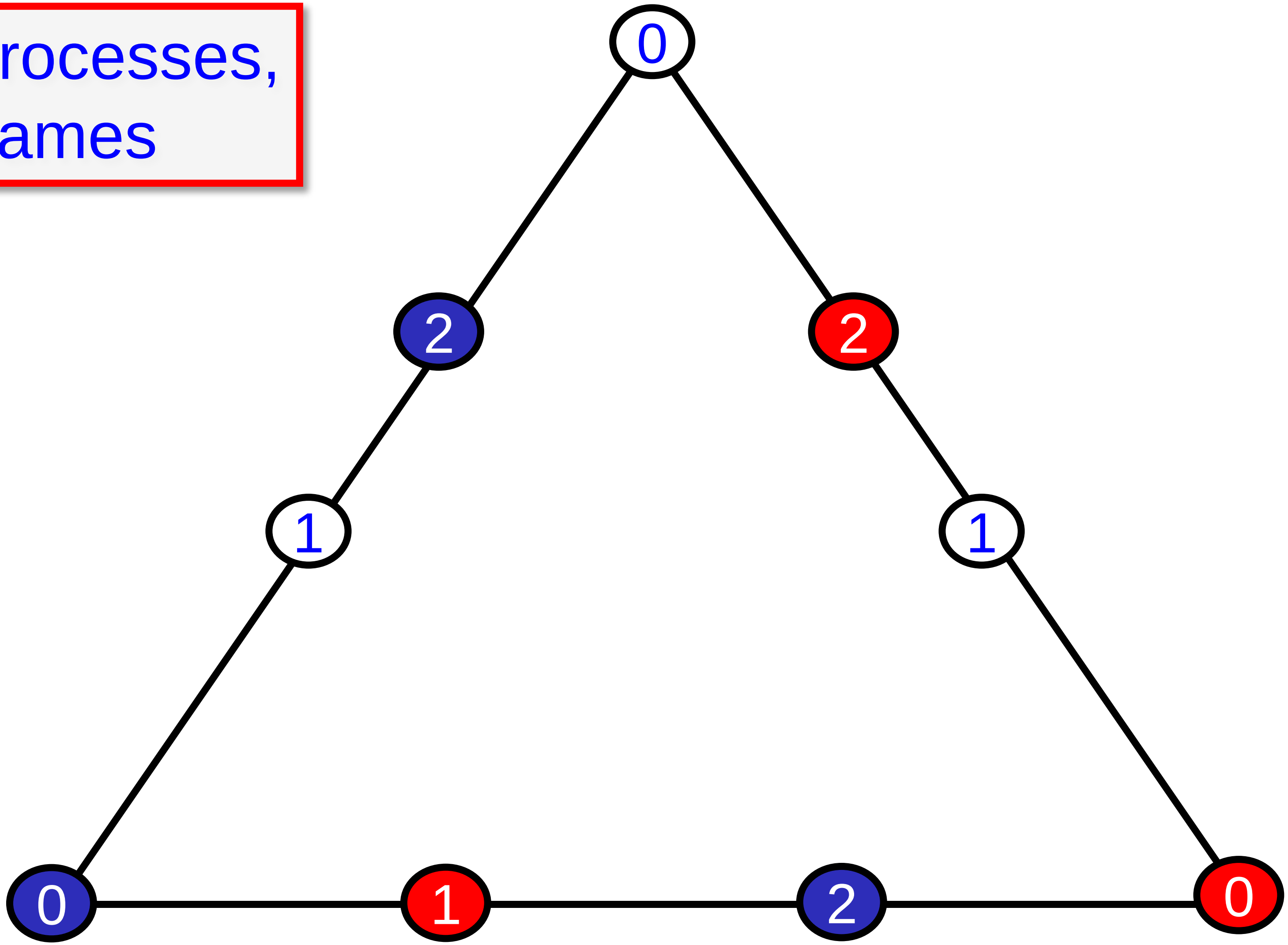
easy to check that map is rigid,
and depends only on order of process names

```
shared Boolean flag[2] = {false, false}
```

```
// code for P_1
flag[1] := true
if (flag[0])
  decide 1
else
  decide 0
```

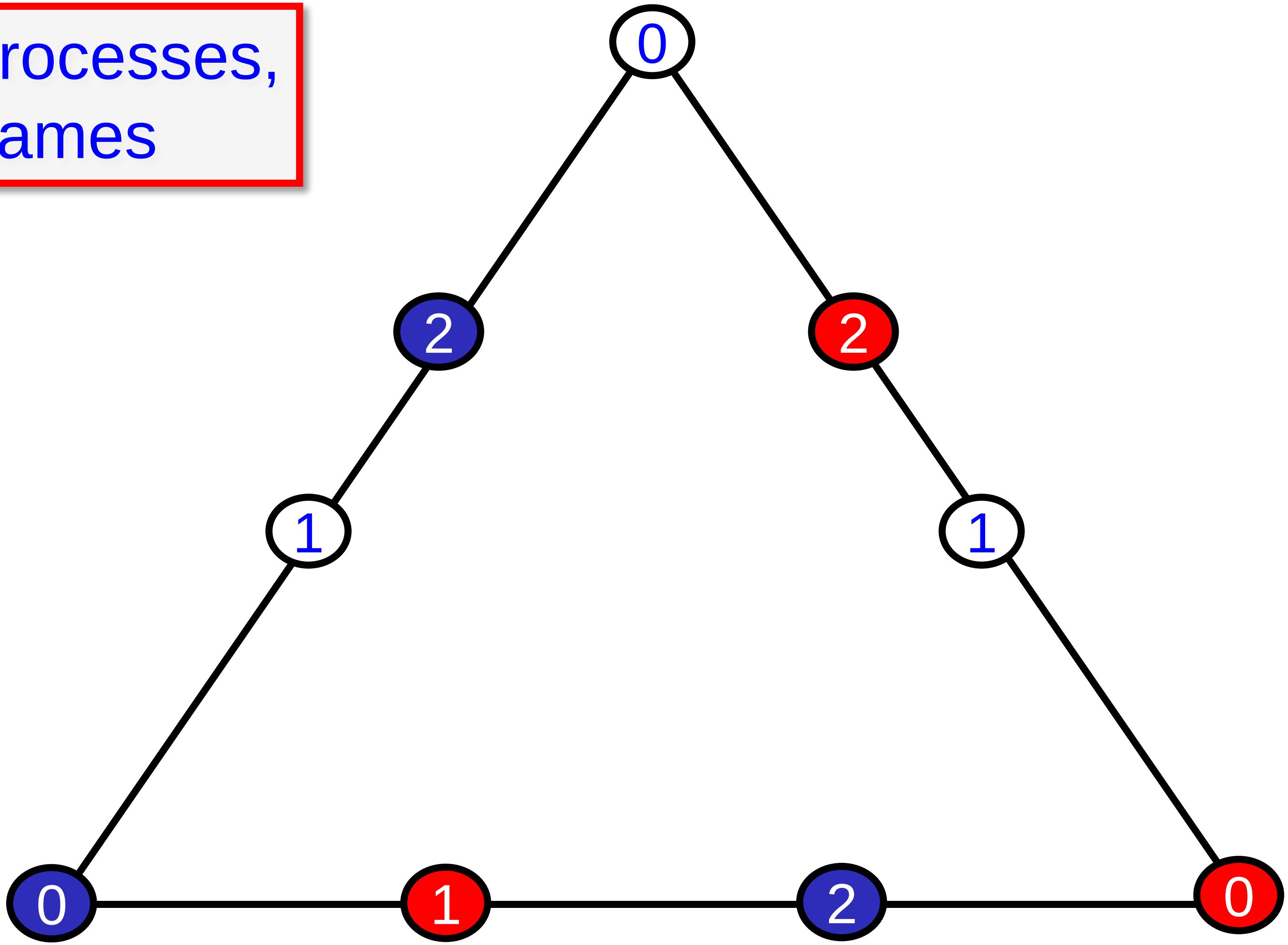
```
// code for P_0
flag[0] := true
if (flag[1])
  decide 2
else
  decide 0
```

3 processes,
5 names

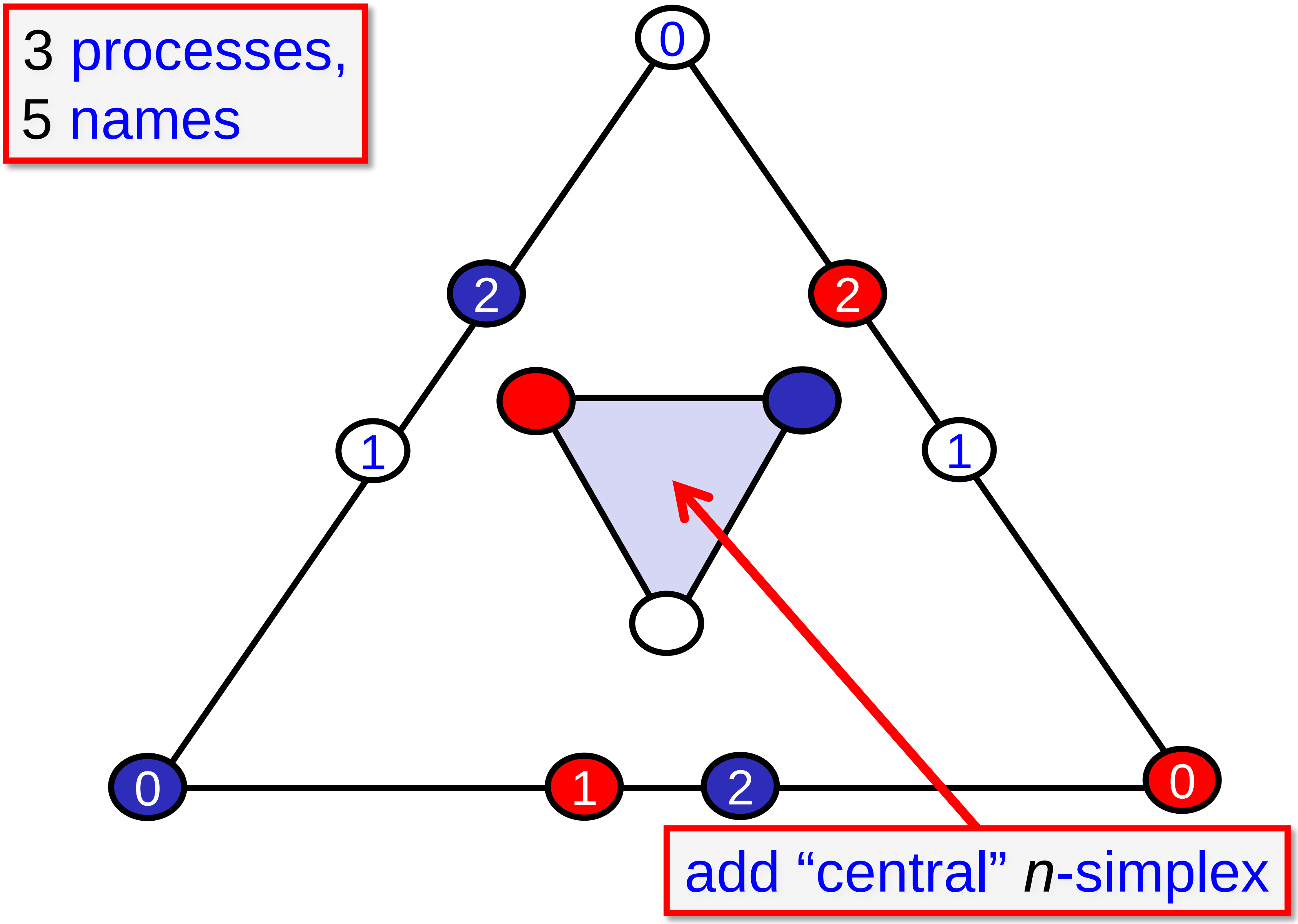


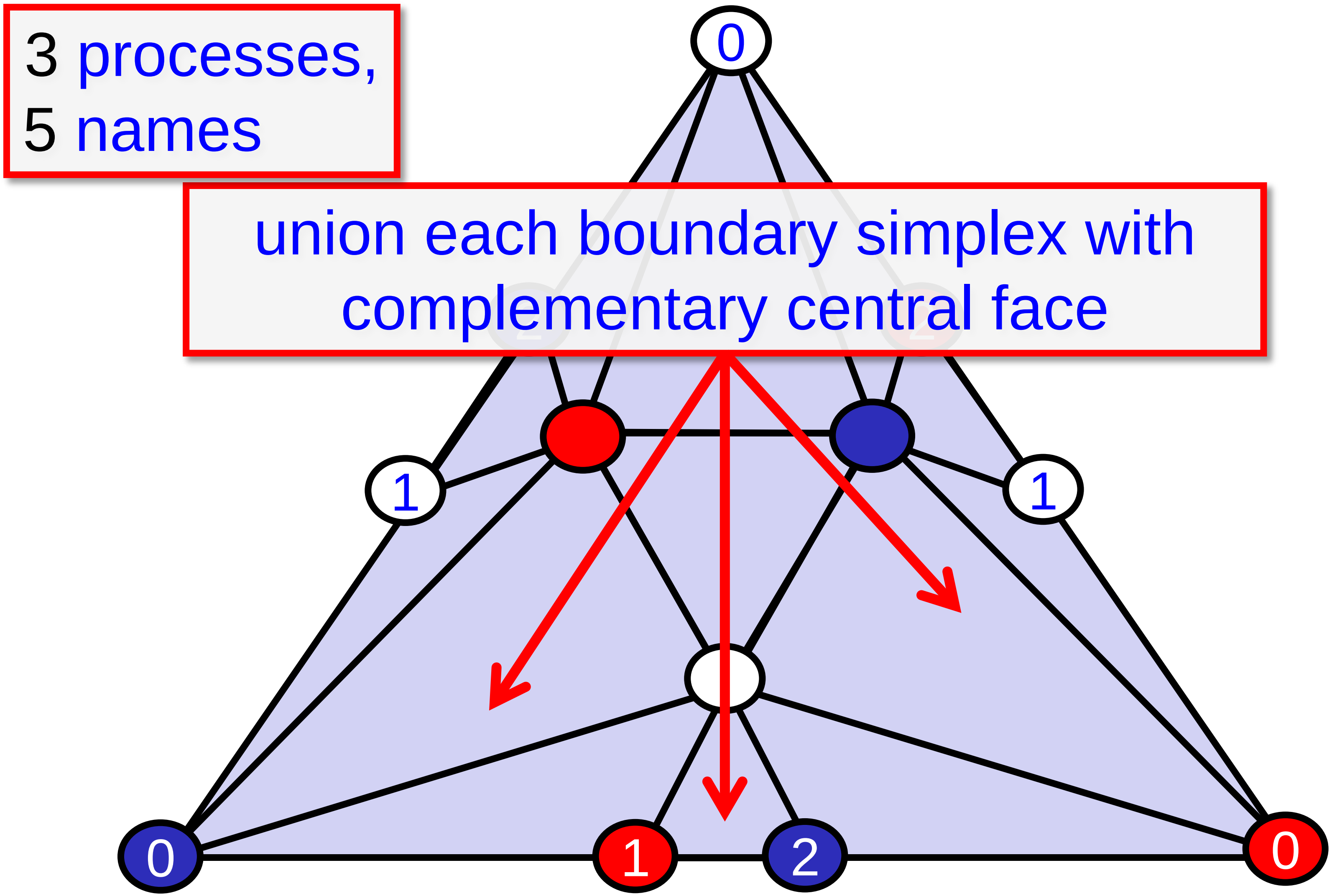
Index-independent ⇒ symmetric on boundary

3 processes,
5 names



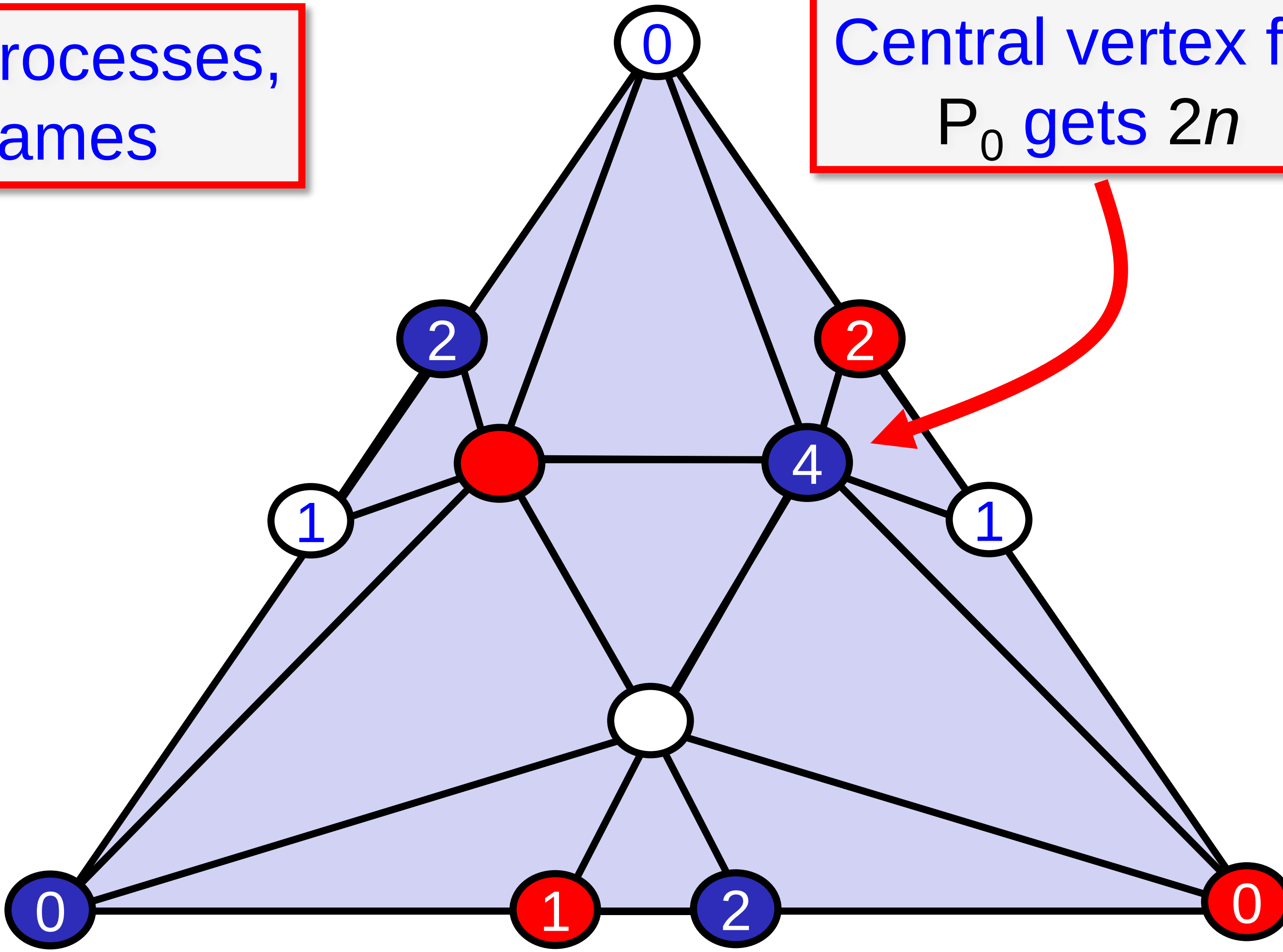
Index-independent ⇒ symmetric on boundary

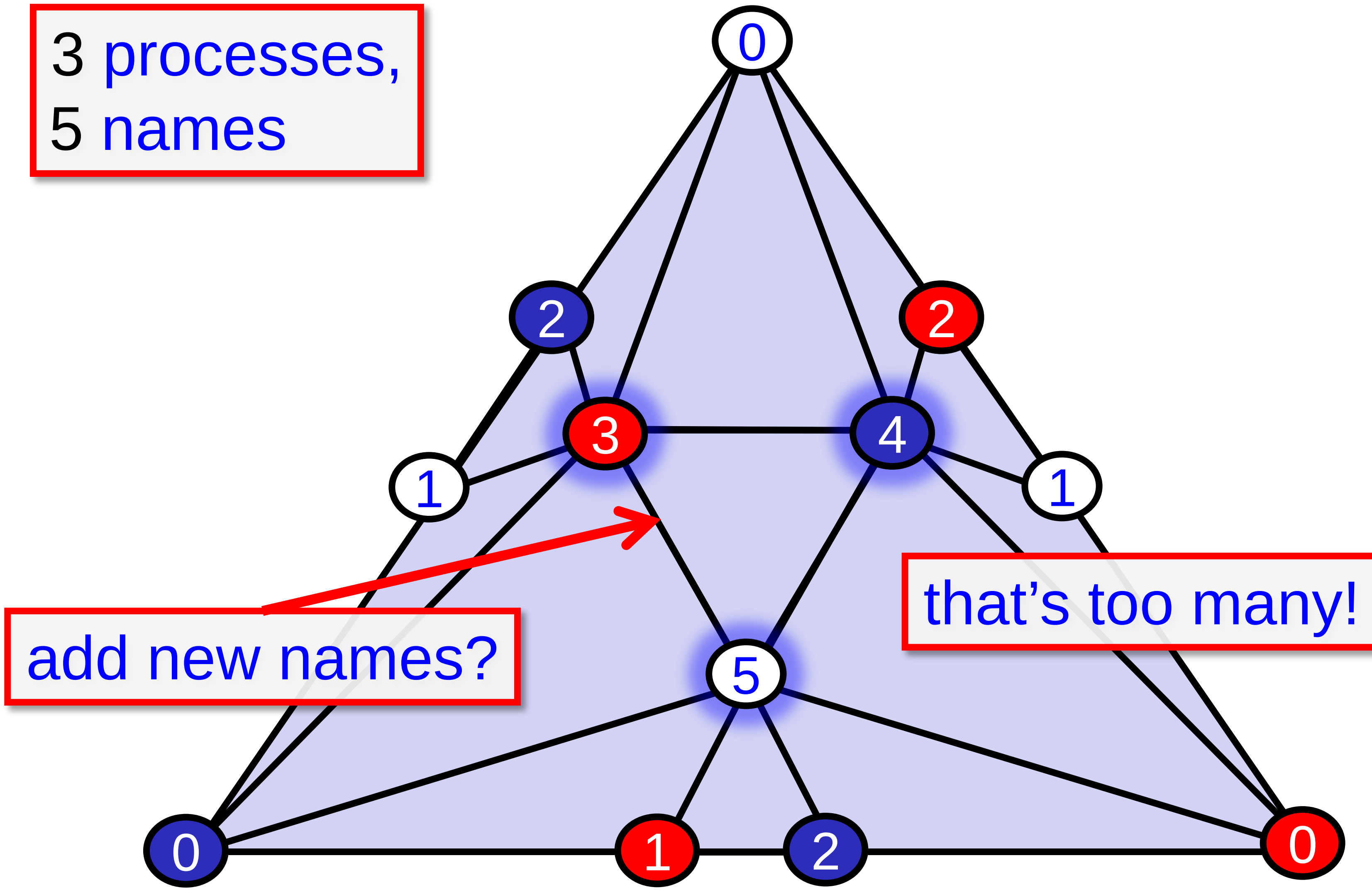

3 processes,
5 names
0
2
2
1
1
0
1
2
0
add "central" *n*-simplex


3 processes,
5 names
union each boundary simplex with
complementary central face
0
1
1
0
1
2
0

3 processes,
5 names

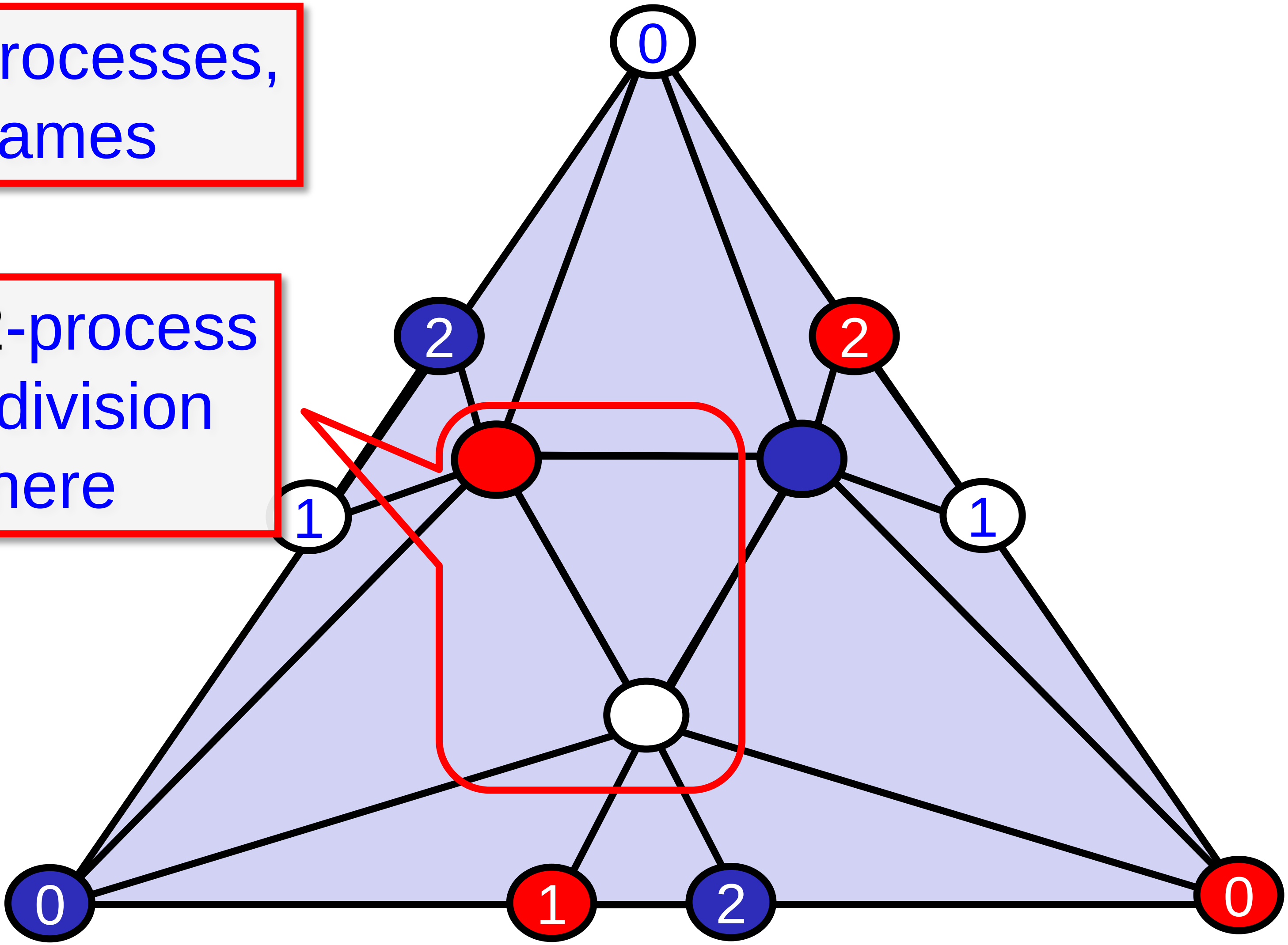
Central vertex for
P_0 gets $2n$




3 processes,
5 names
0
2
2
3
4
1
1
add new names?
that's too many!
5
0
1
2
0

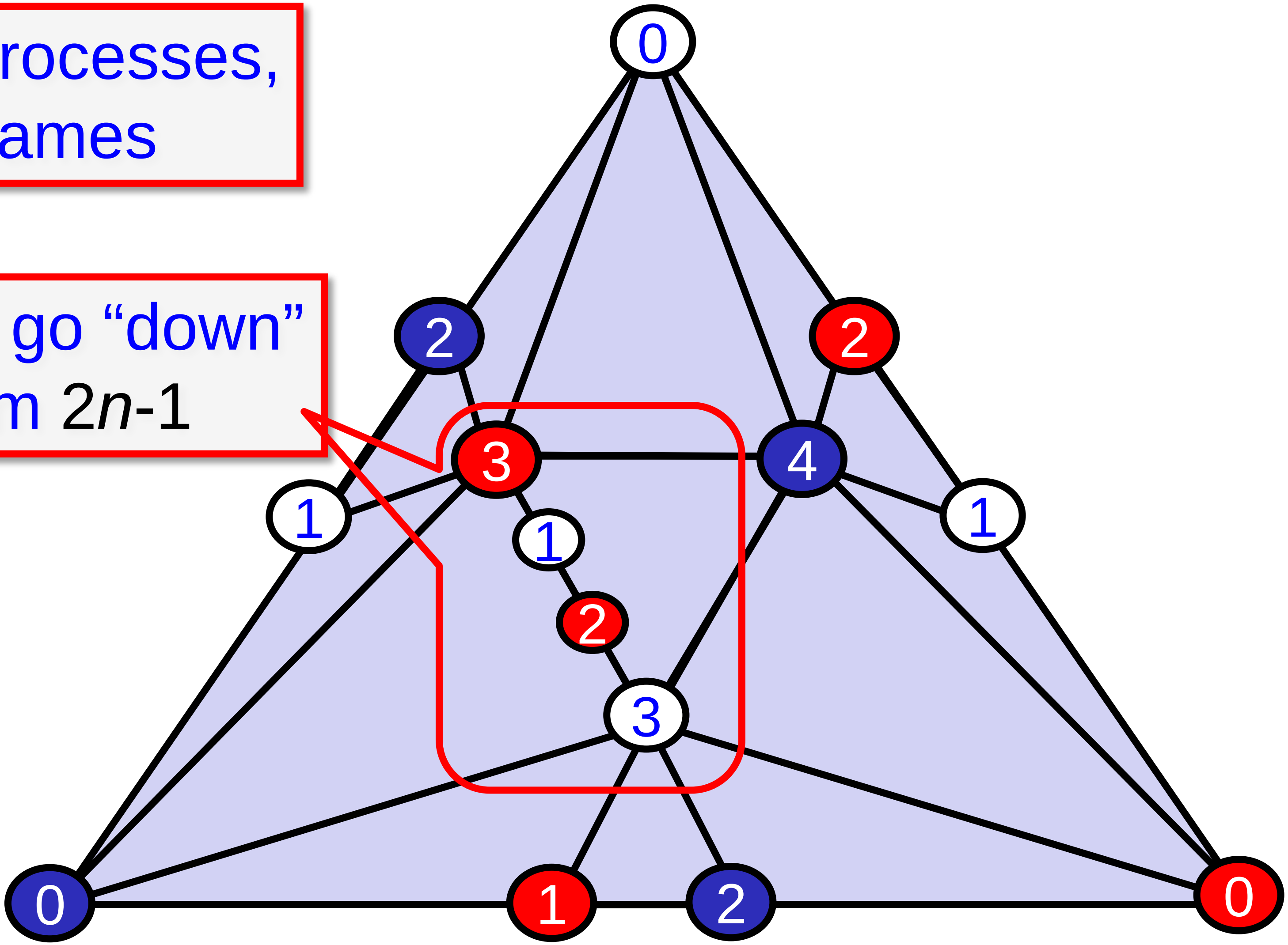
3 processes,
5 names

use 2-process subdivision here

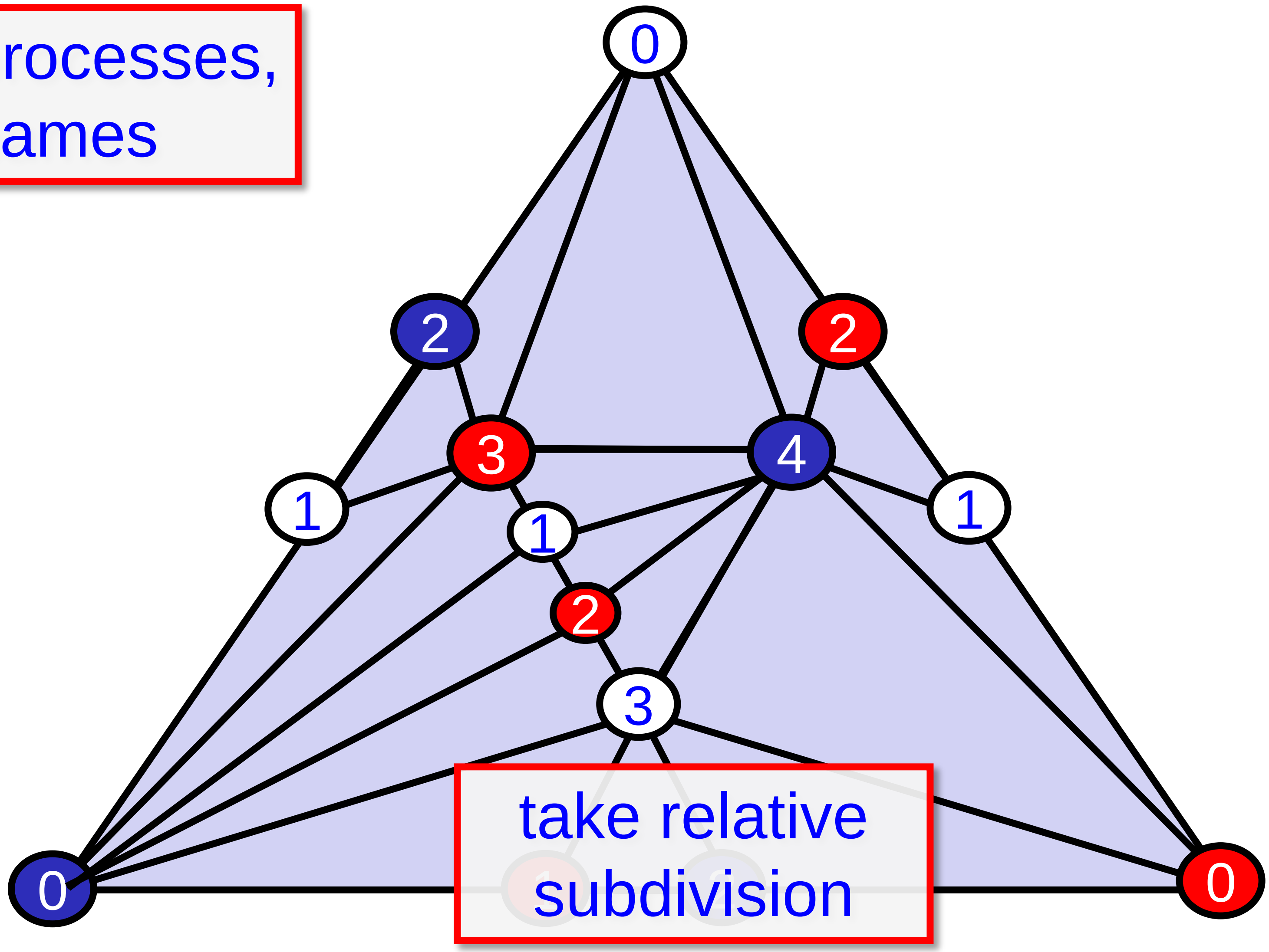


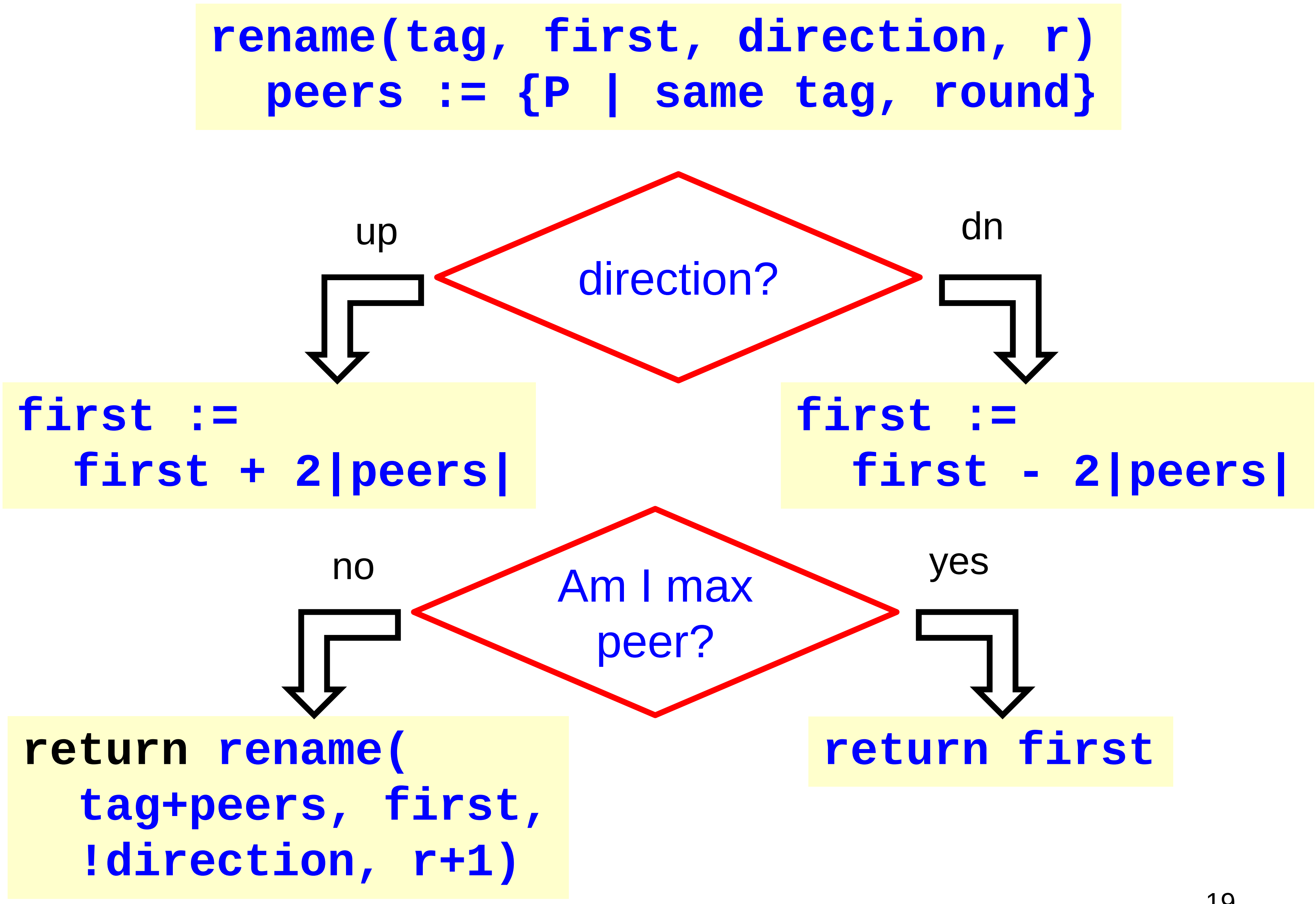
3 processes,
5 names

except go “down”
from $2n$-1



3 processes,
5 names




rename(tag, first, direction, r)
peers := {P | same tag, round}
direction?
up
dn
first :=
first + 2|peers|
first :=
first - 2|peers|
Am I max
peer?
no
yes
return rename(
tag+peers, first,
!direction, r+1)
return first

Road Map

An Upper Bound: 2n+1 Names

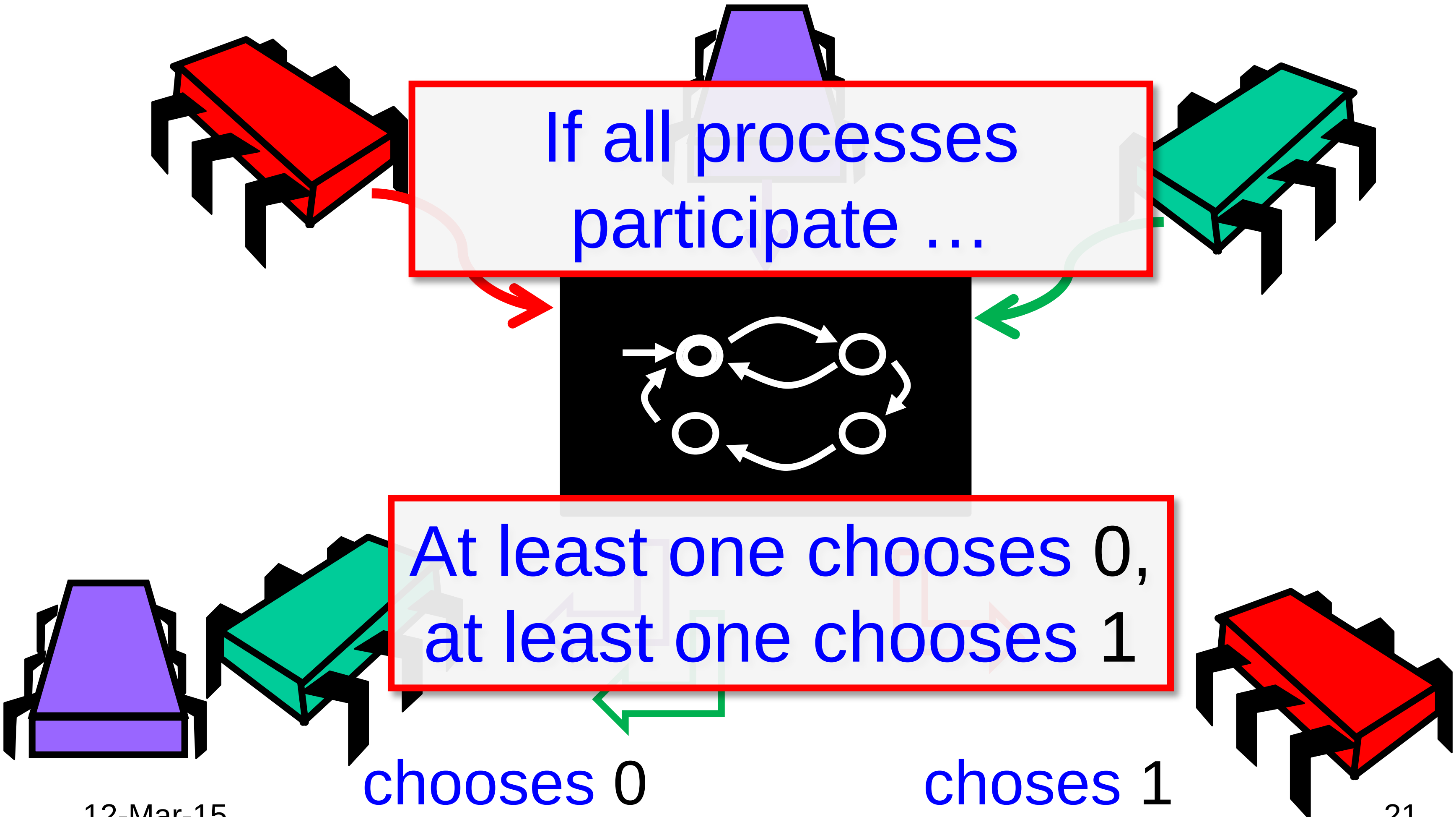
Weak Symmetry-Breaking

The Index Lemma

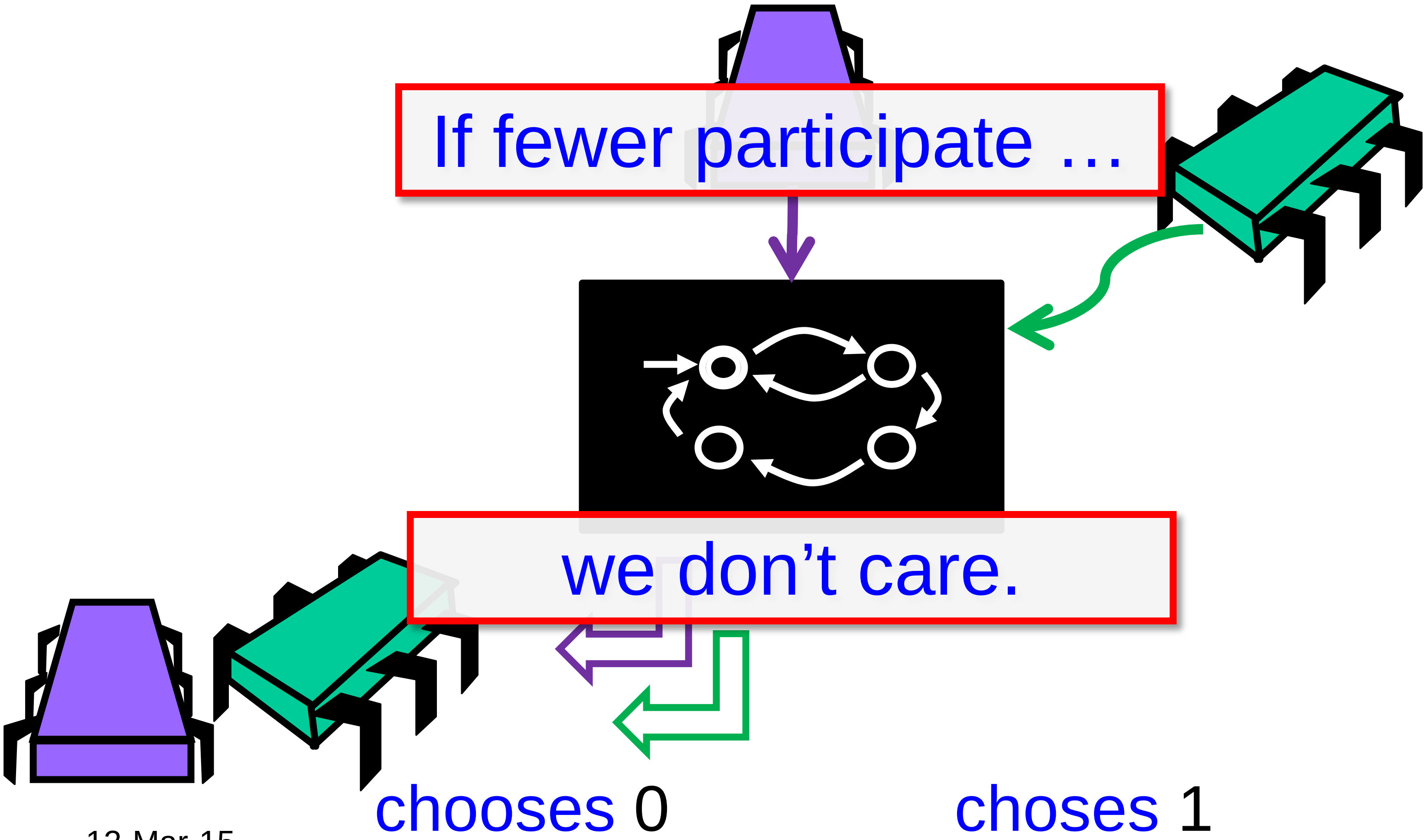
Binary Colorings

A Lower Bound for 2n-Renaming

Weak Symmetry-Breaking



Weak Symmetry-Breaking



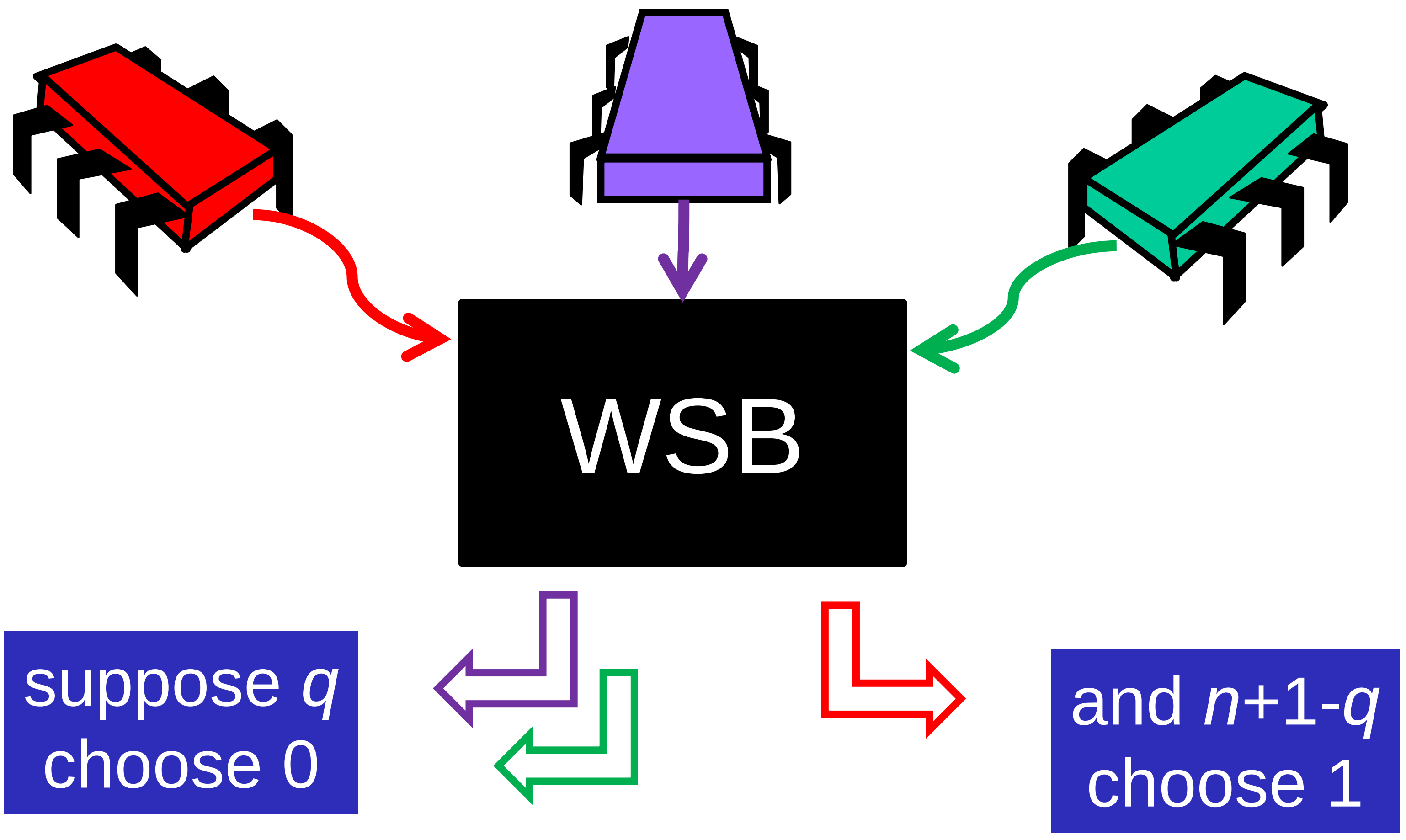
Claim:

Weak symmetry-breaking is *equivalent* to 2*n*-renaming

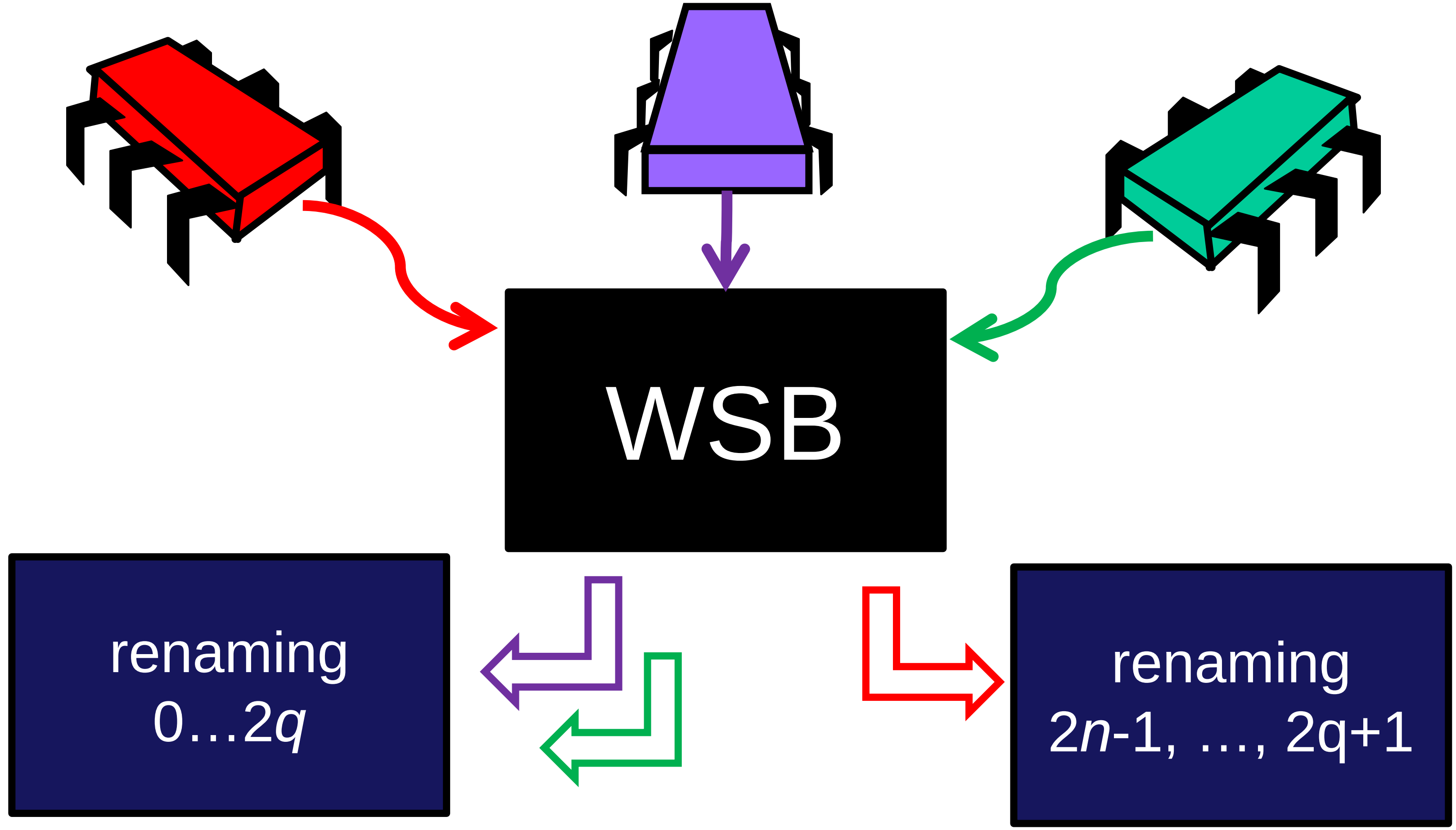
Lower bound for WSB is lower bound for 2*n*-renaming …

Upper bound too …

WSB $\Rightarrow$ 2*n*-Renaming



WSB $\Rightarrow$ 2*n*-Renaming



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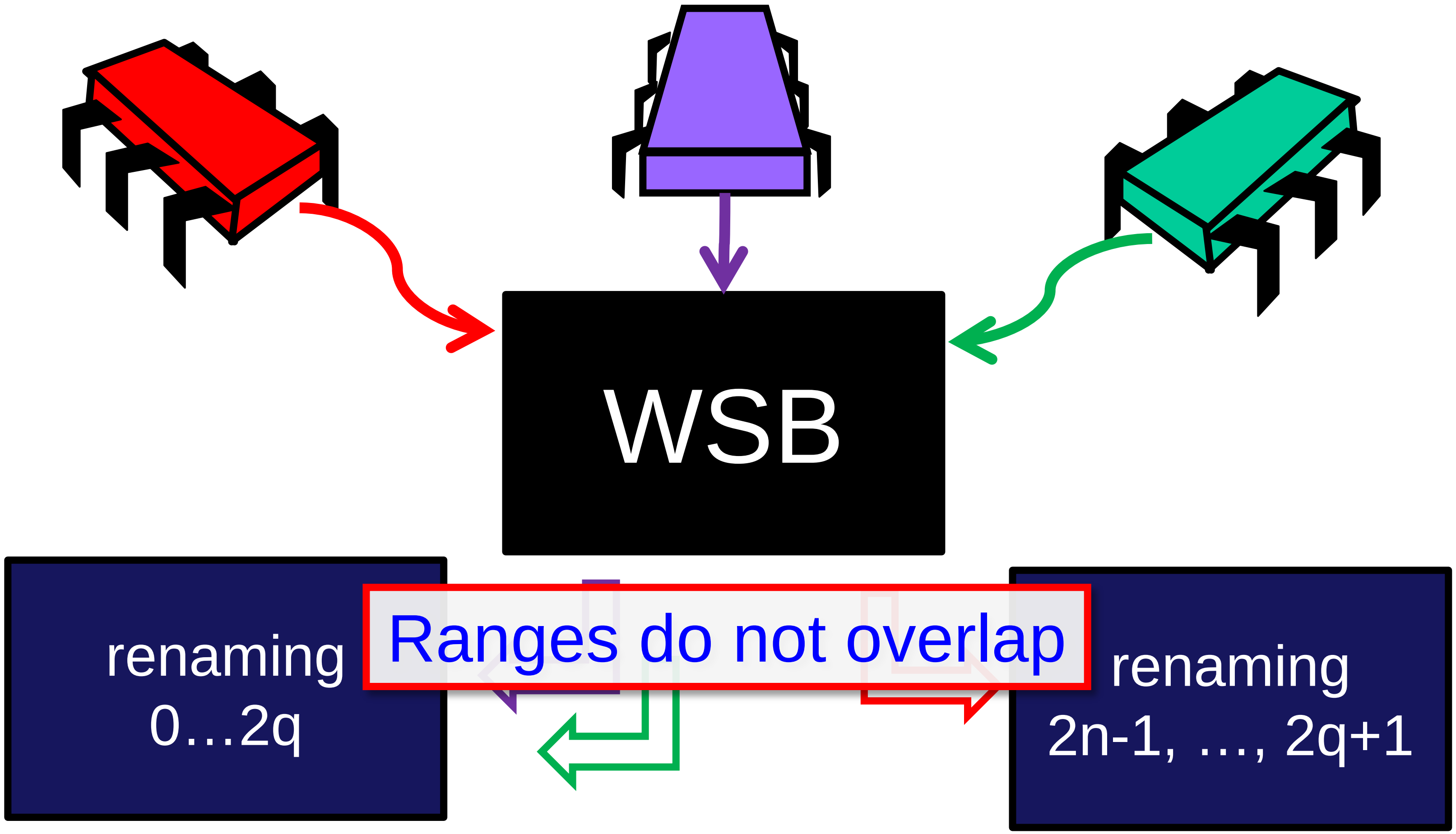
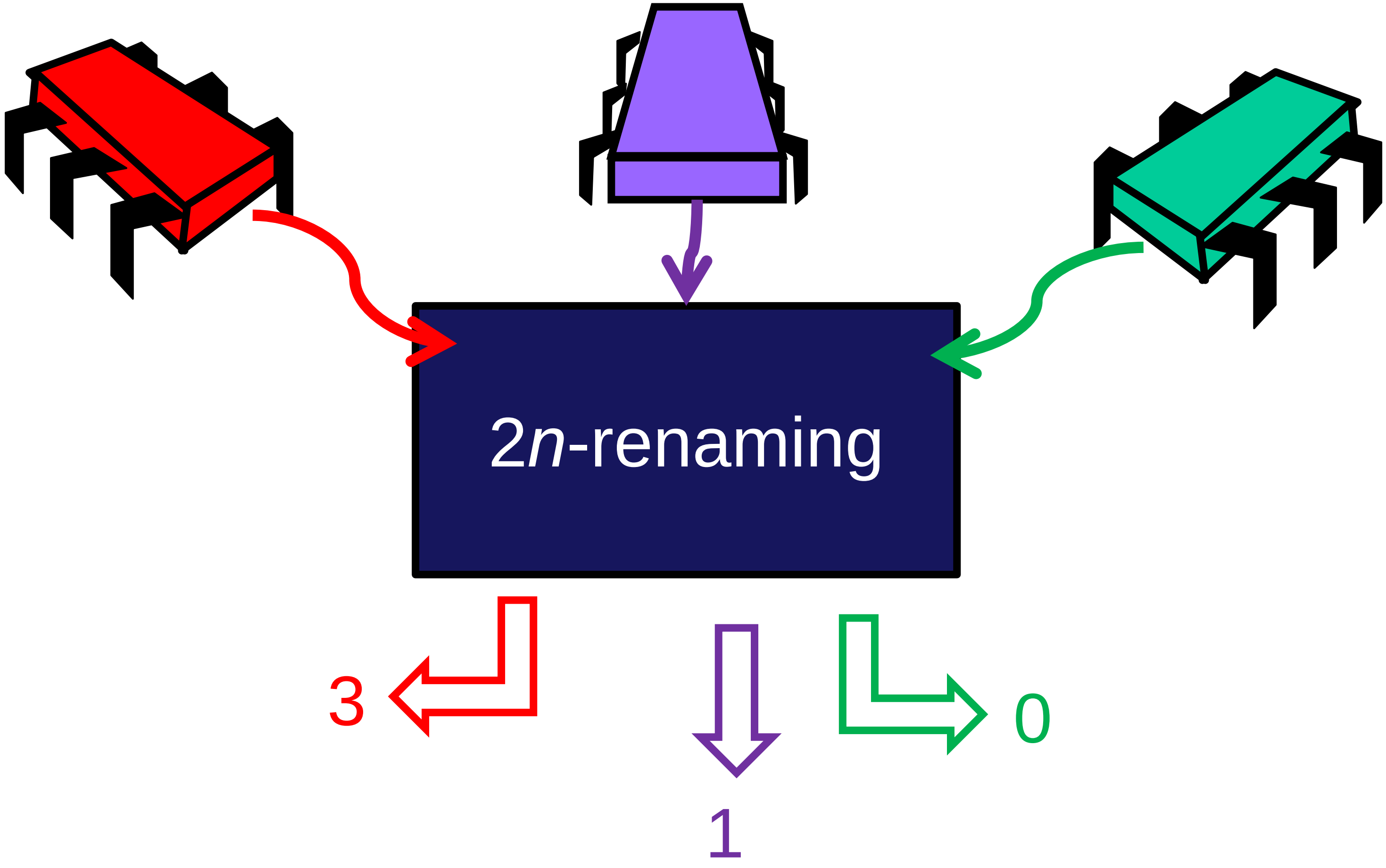
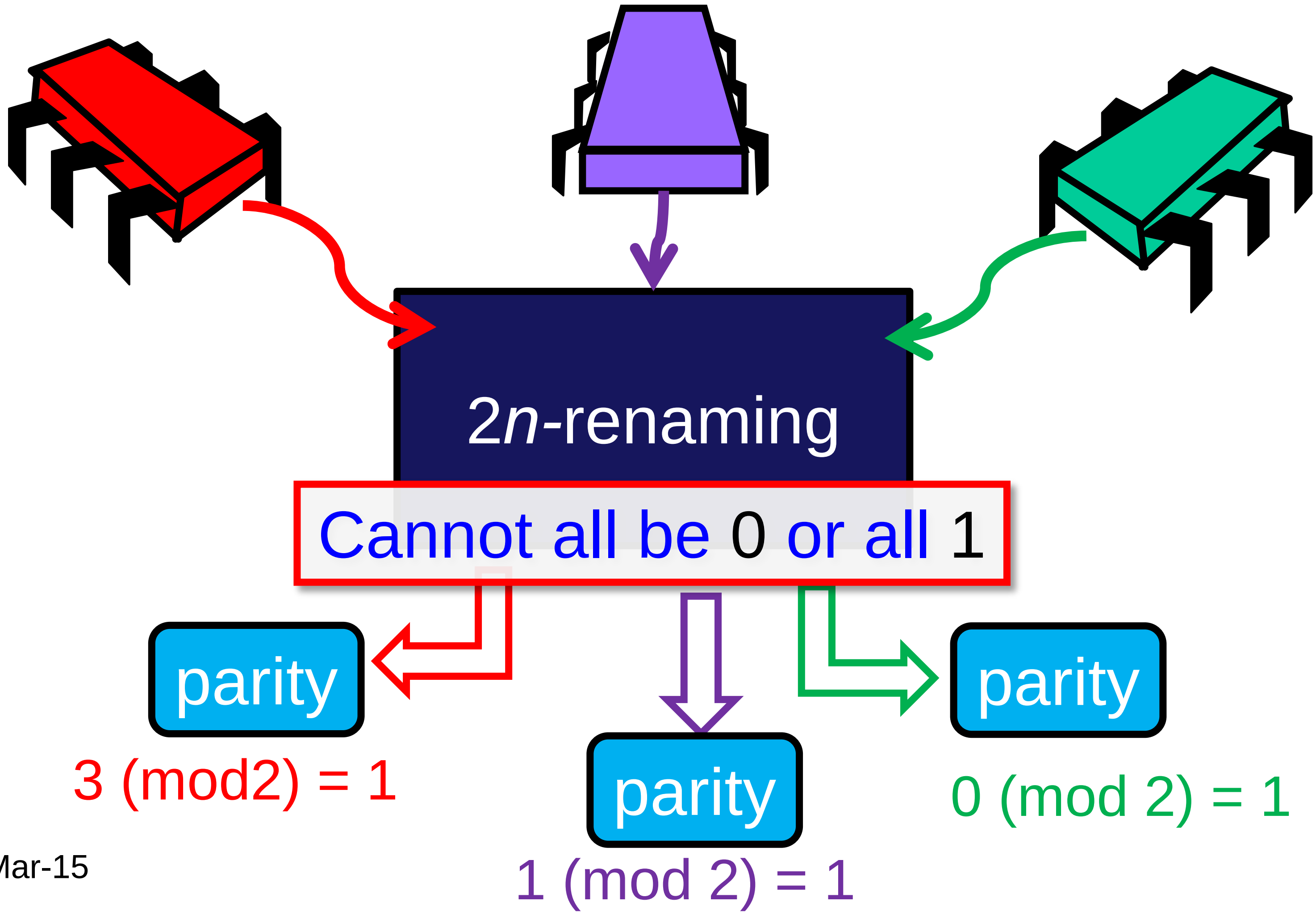


$2n$-Renaming $\Rightarrow$ WSB



2n-Renaming ⇒ WSB

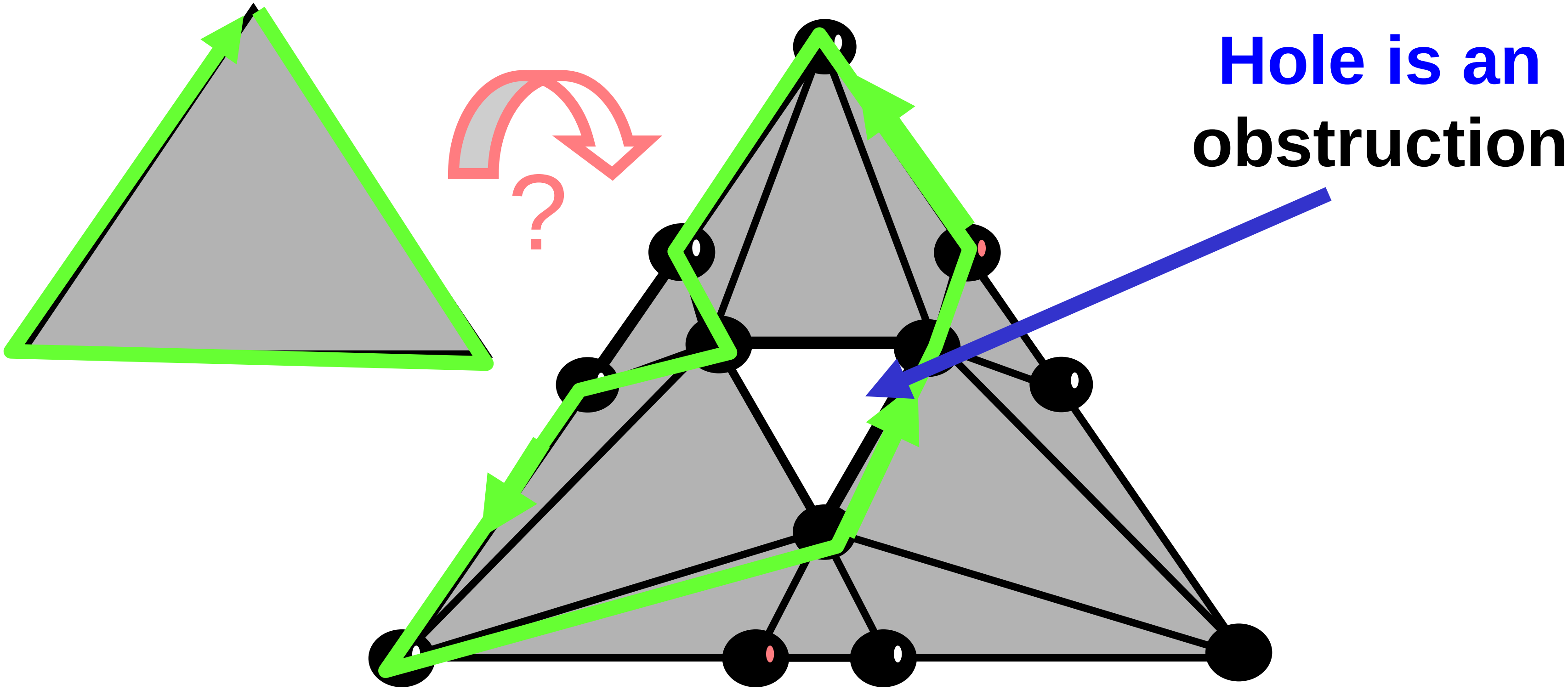


Theorem

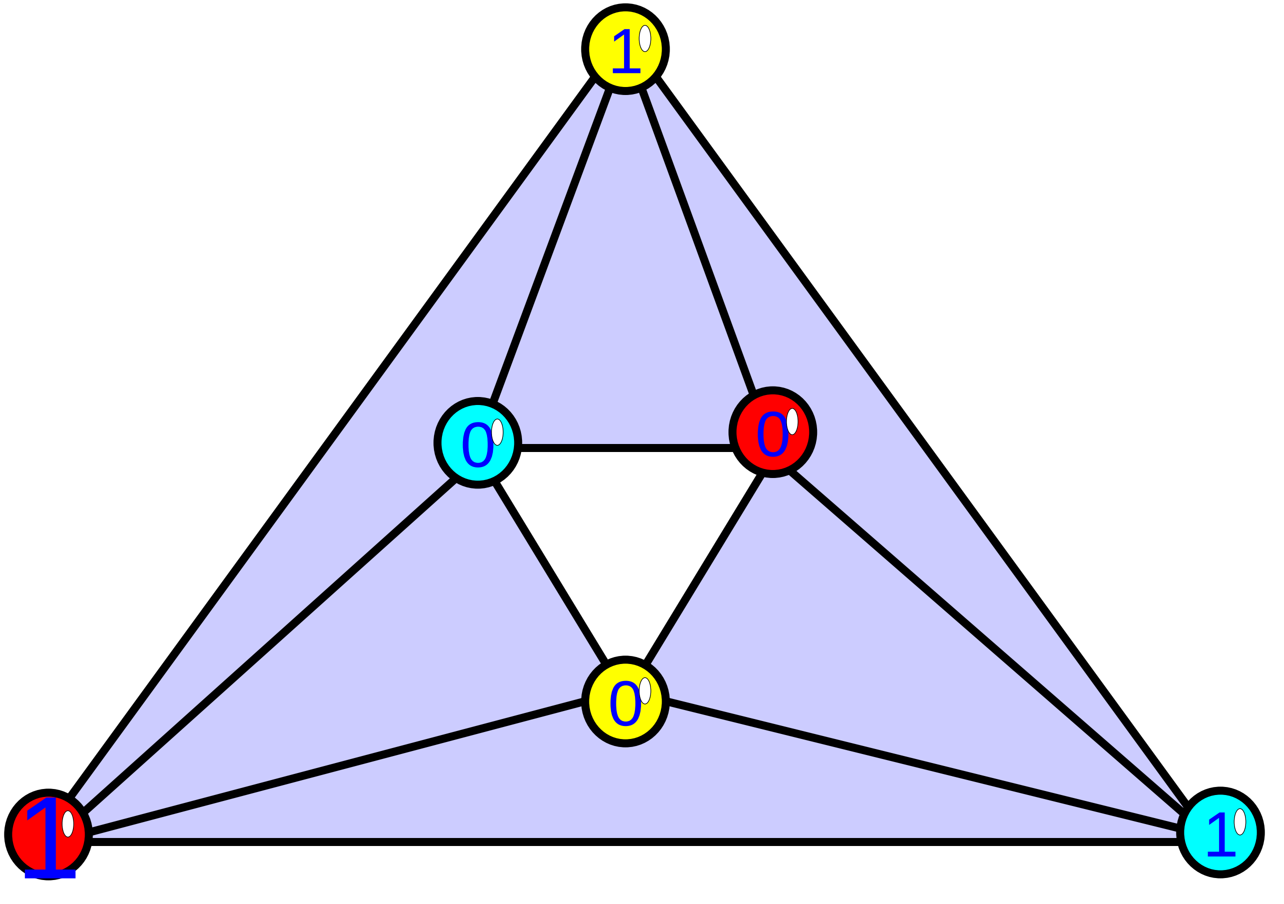
There is no 3-process weak symmetry-breaking protocol

Hence no renaming for 3 processes and 4 names

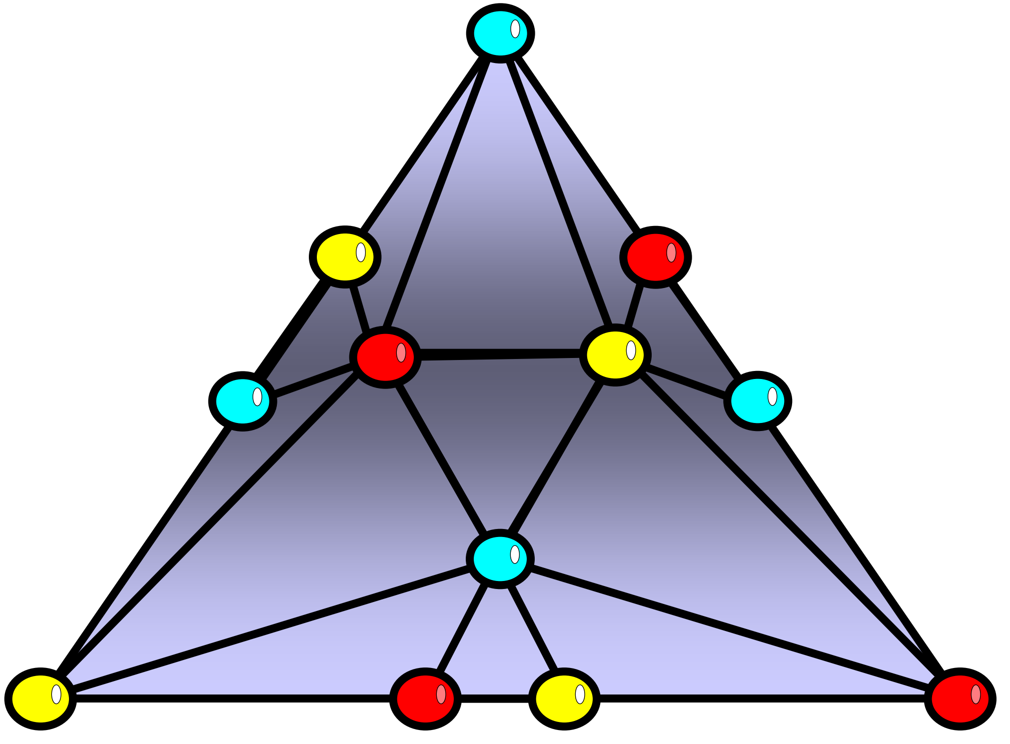
Reminder: Cannot Map Boundary Around a Hole



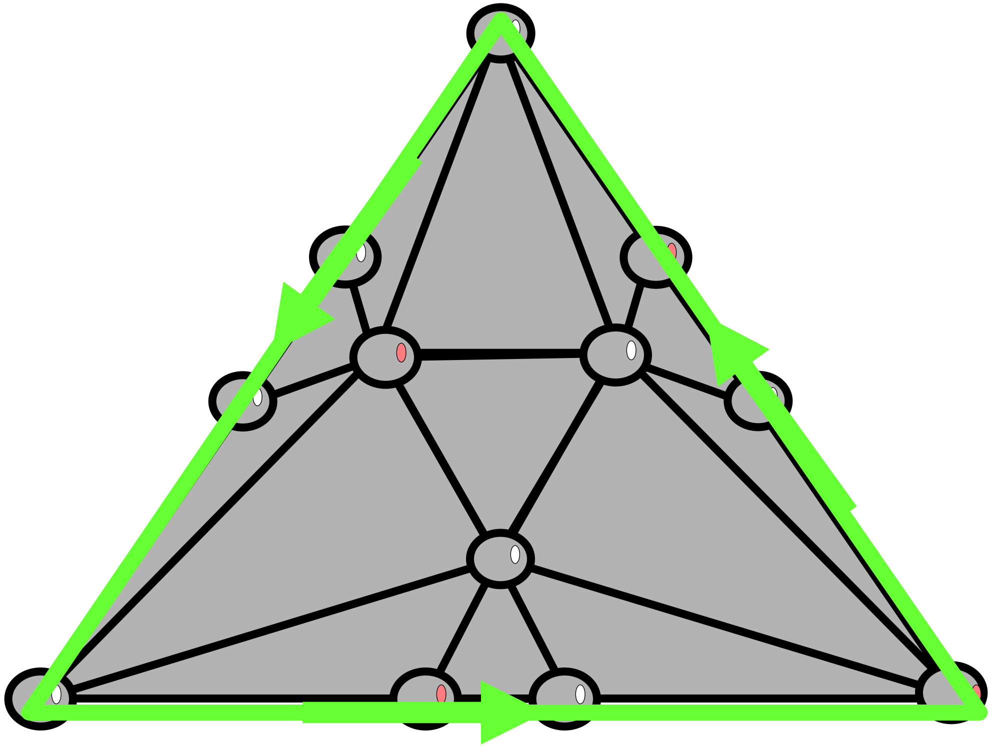
WSB Output Complex



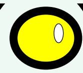
Protocol Complex (schematic)

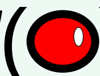


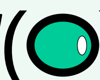
Boundary = 2-Process Executions



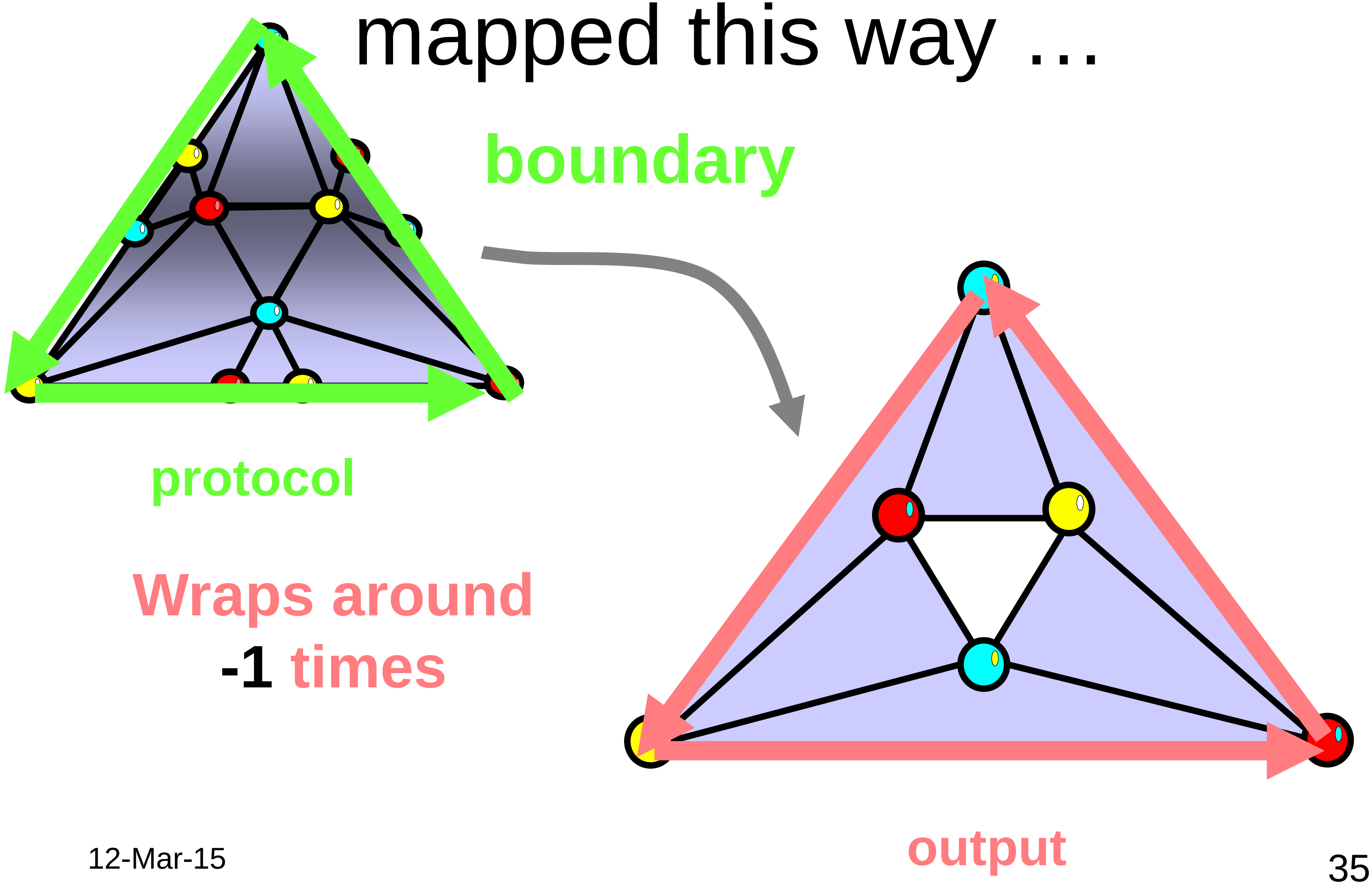
Protocol Complex for One Process Execution

Ξ() decides 1 WLOG

Ξ() decides 1 by symmetry

Ξ() decides 1 by symmetry

2-Process execution might be mapped this way …



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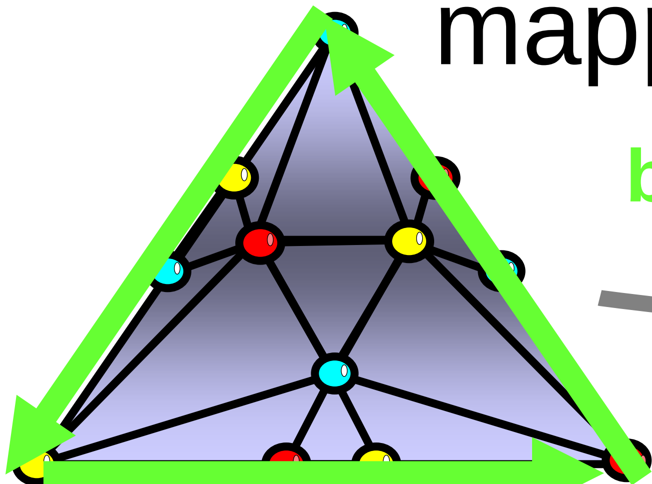
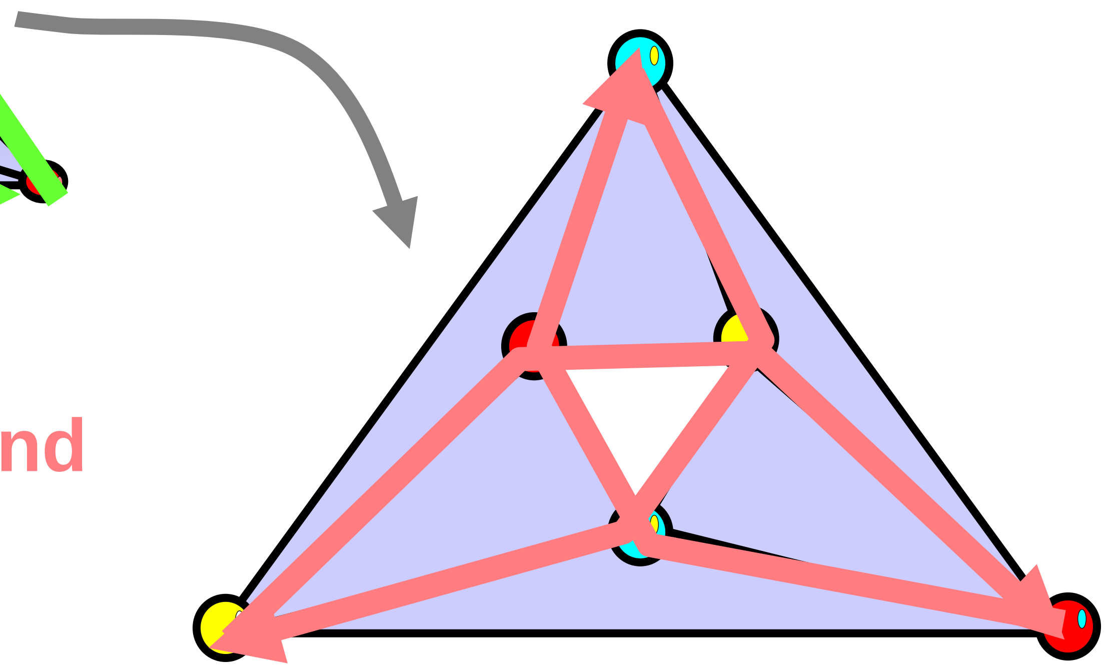
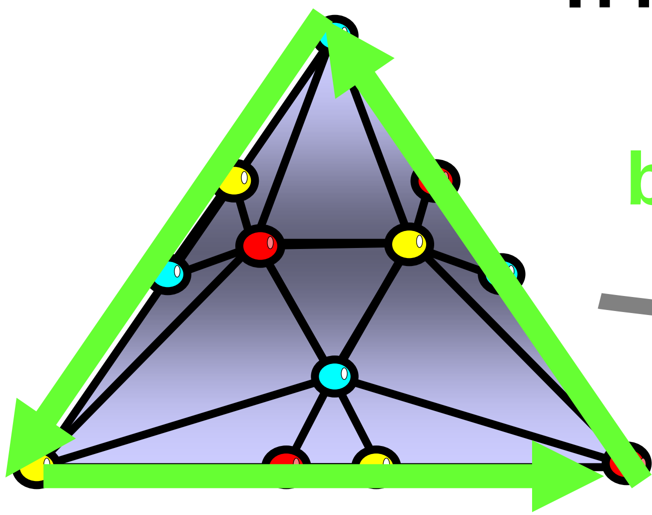


Wraps around
+2 times



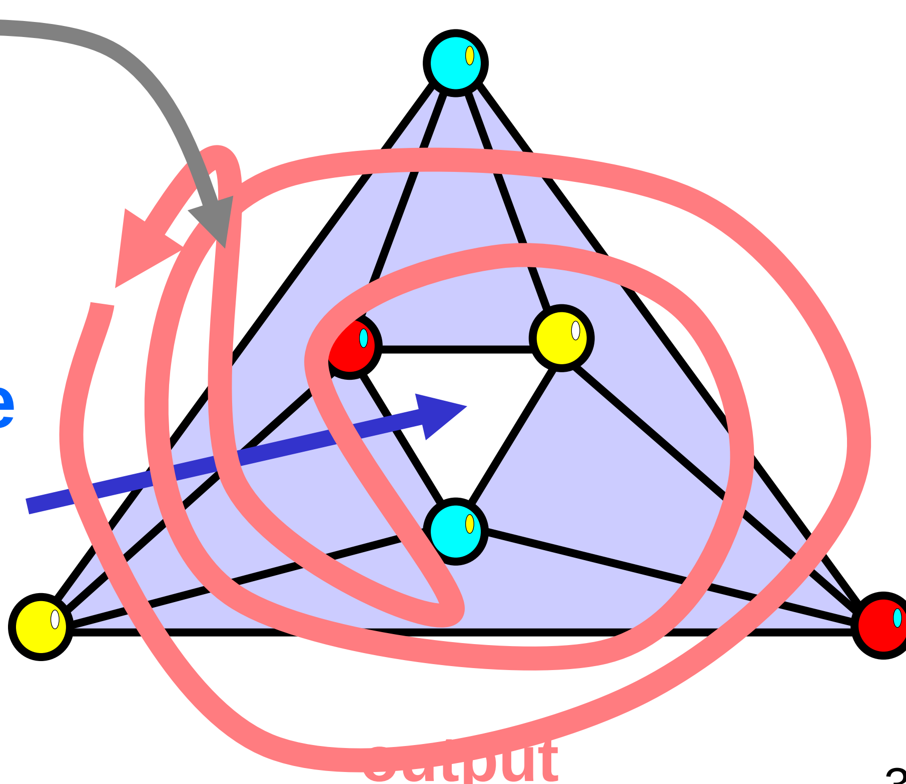
In General …



Wraps around hole

3k-1 ≠ 0 times

QED!



Conjecture

For n+1 processes …

the boundary wraps around the hole …

$(n+1) \cdot k \neq 0$ times …

so 2n-renaming is impossible!

Conjecture

For n+1 processes …

the boundary wraps around the hole …

(n+1) … times …

so 2n-renaming is impossible!

Wrong!

Only holds if n+1 is a prime power!

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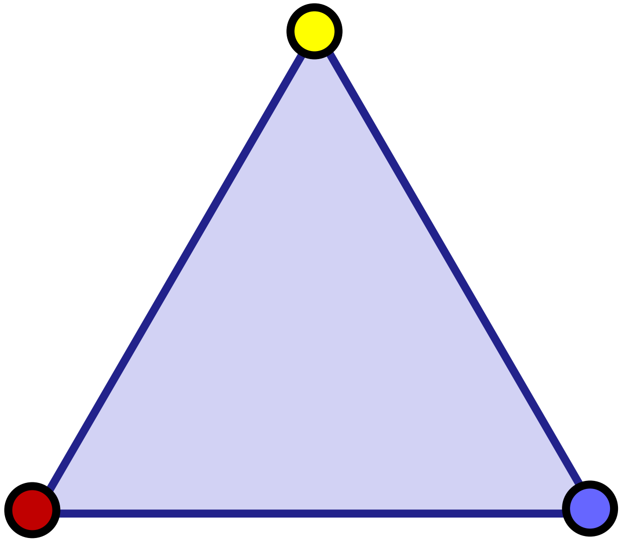
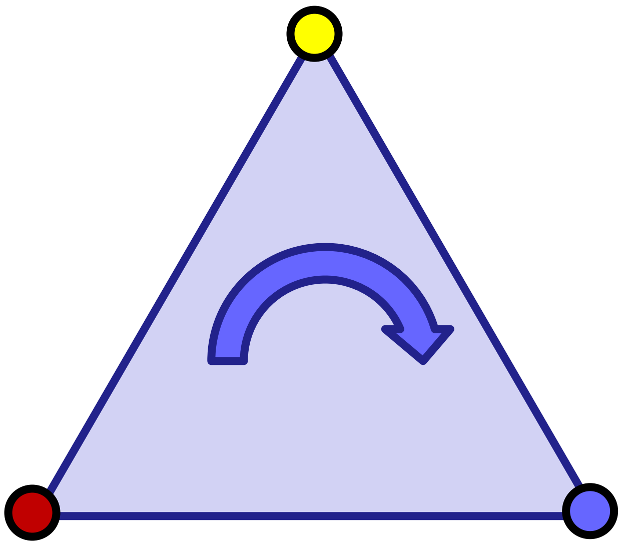
Simplex



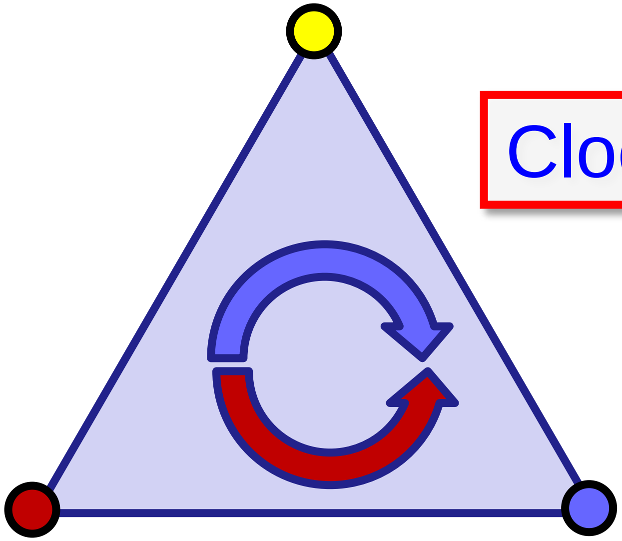
Oriented Simplex



Sequence:

and even permutations:

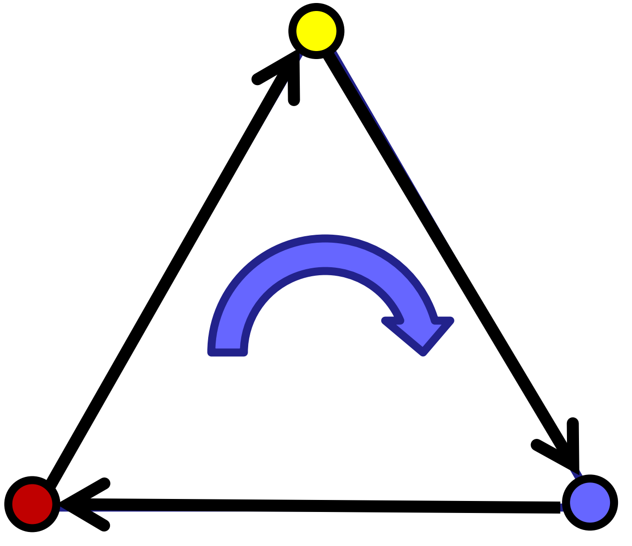
Oriented Simplex



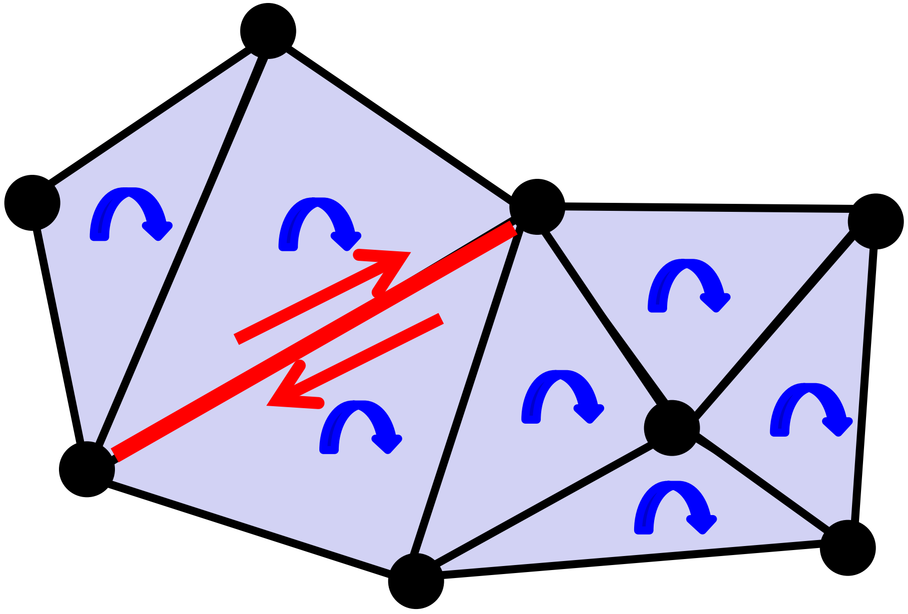
Clockwise ...

Counter-clockwise ...

Induced orientation on faces

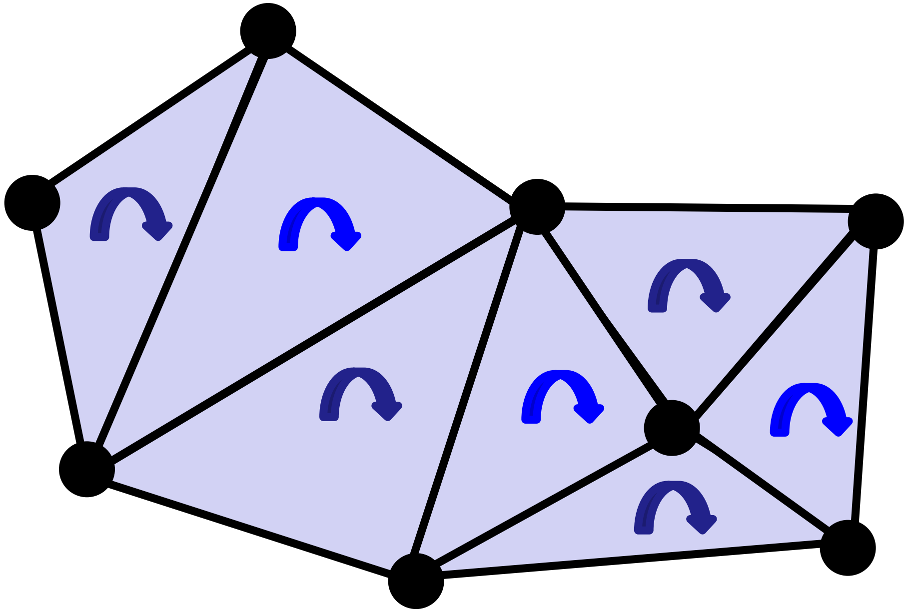


Oriented *n*-manifold with boundary

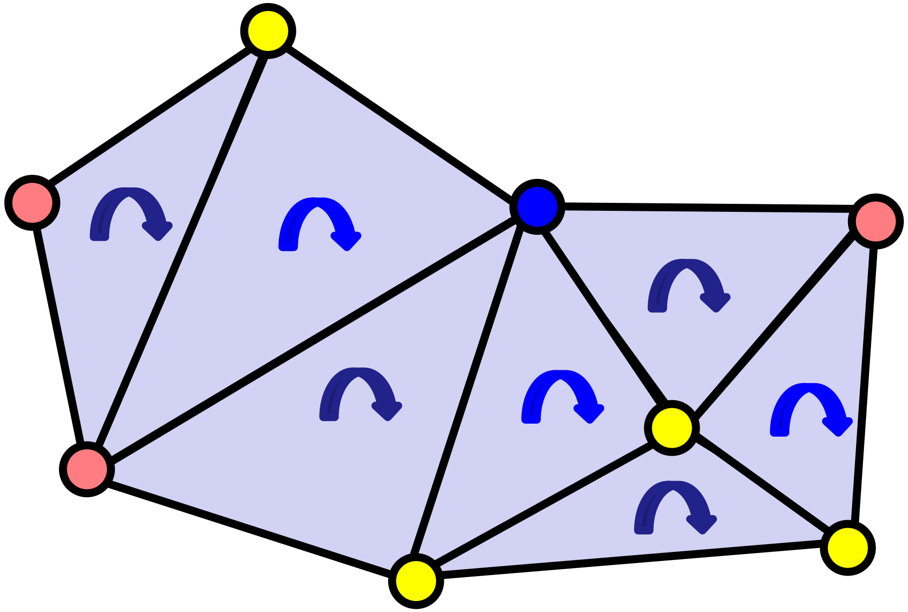


Adjacent *n*-simplexes induce opposite orientations on common face

Oriented *n*-manifold with boundary

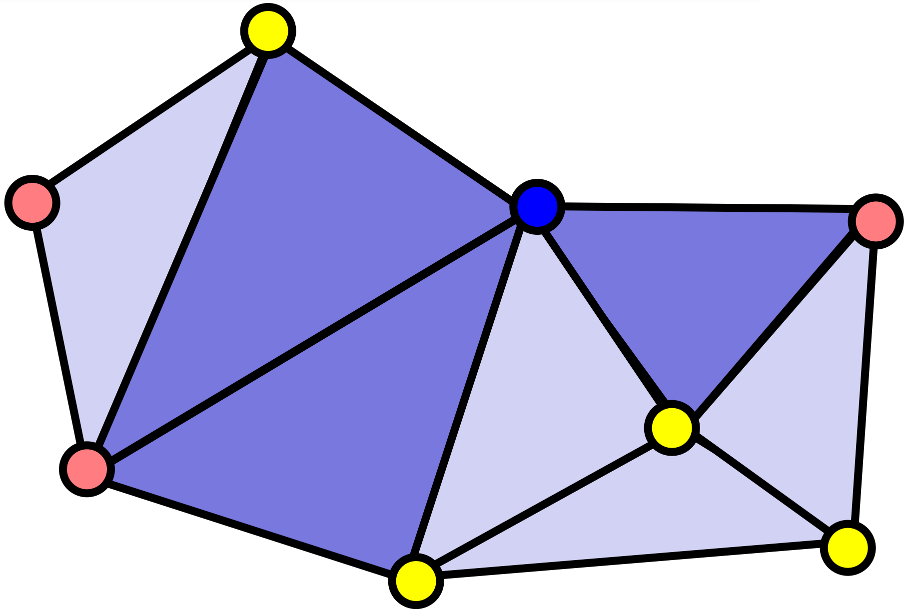


Oriented *n*-manifold with boundary

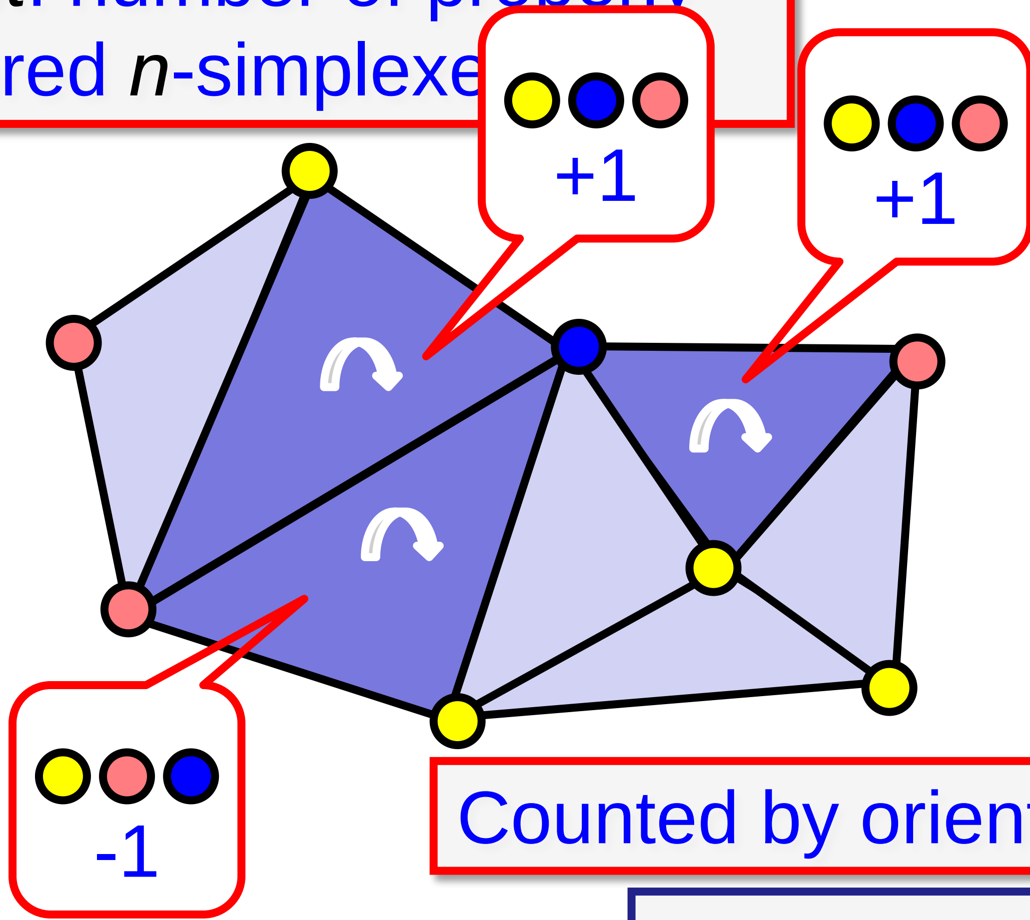


Arbitrary (*n*+1)-coloring

Content: number of properly-colored *n*-simplexes ...



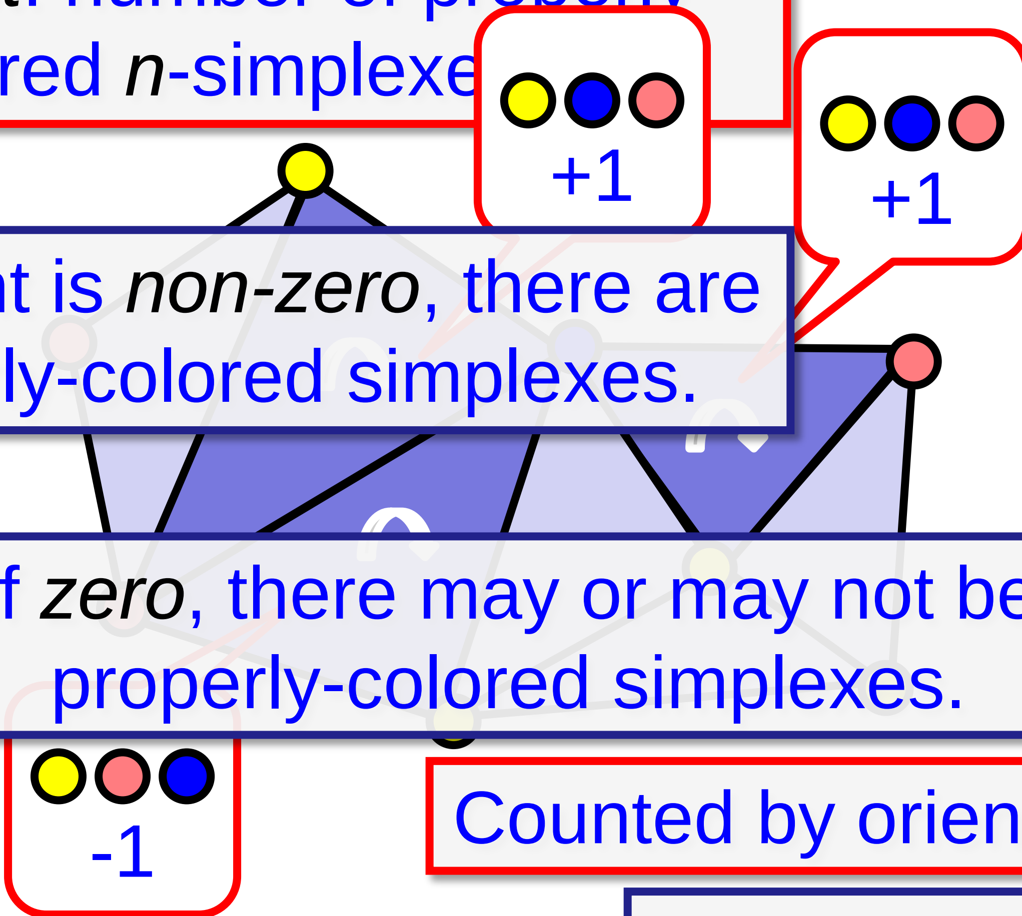
Content: number of properly-colored *n*-simplexe



Counted by orientation.

C = 1-1+1=1

Content: number of properly-colored *n*-simplexes



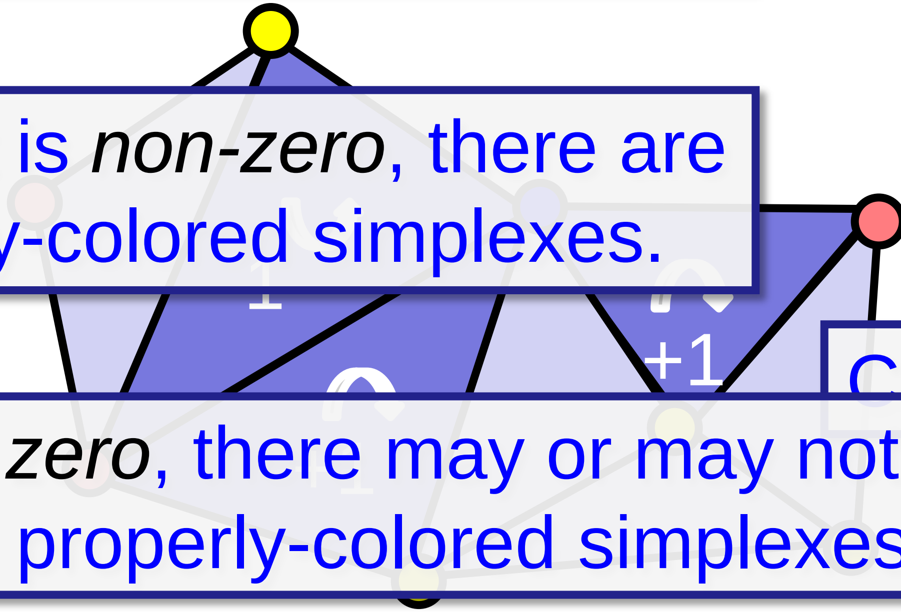
If content is *non-zero*, there are properly-colored simplexes.

If *zero*, there may or may not be properly-colored simplexes.

Counted by orientation.

C = 1-1+1=1

Content: number of properly-colored *n*-simplexes …



If content is *non-zero*, there are properly-colored simplexes.

C = 1-1+1=1

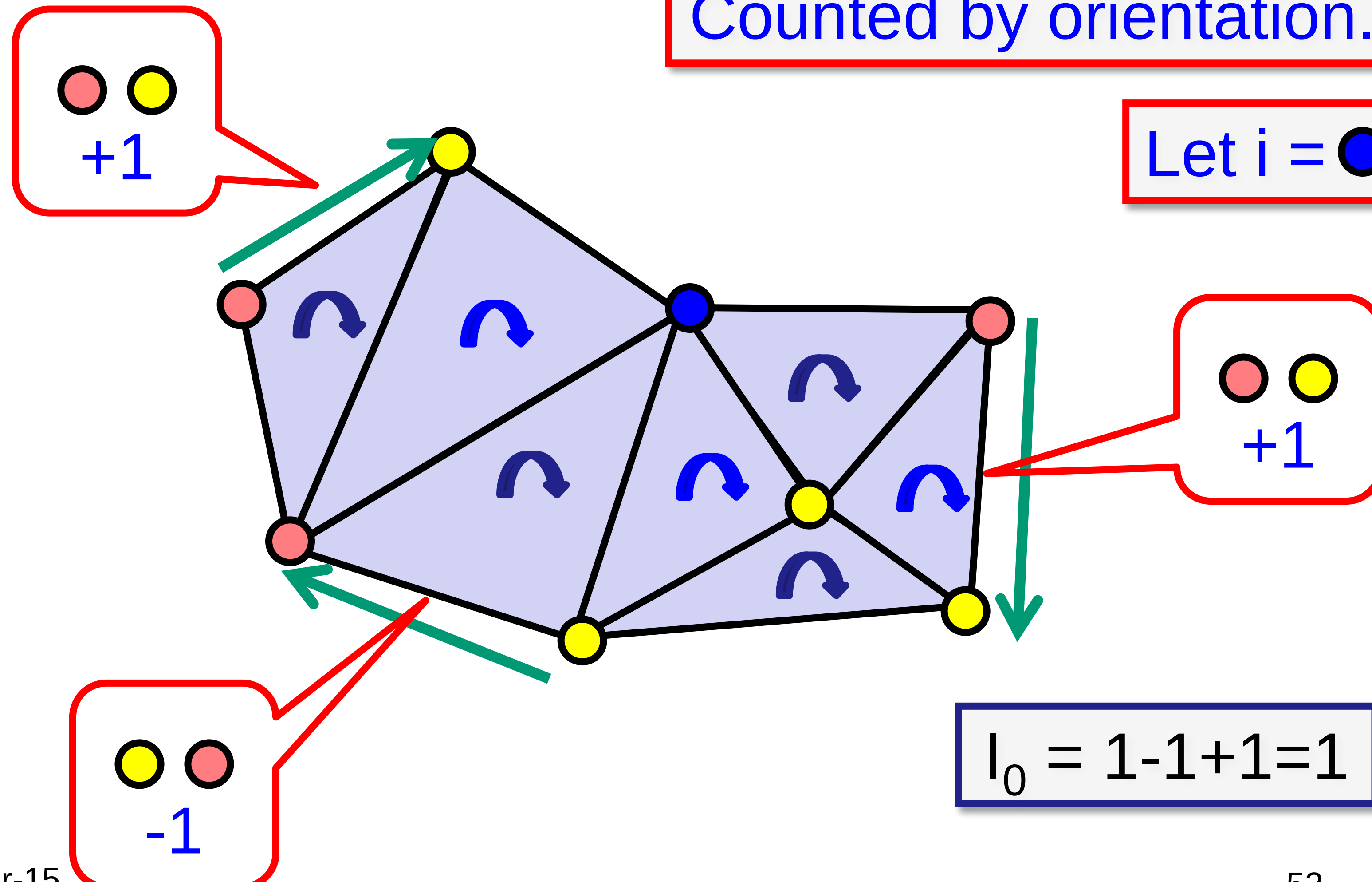
If *zero*, there may or may not be properly-colored simplexes.

Counted by orientation.

Index$_i$: number of boundary (n-1)-simplexes properly colored by colors other than i ...

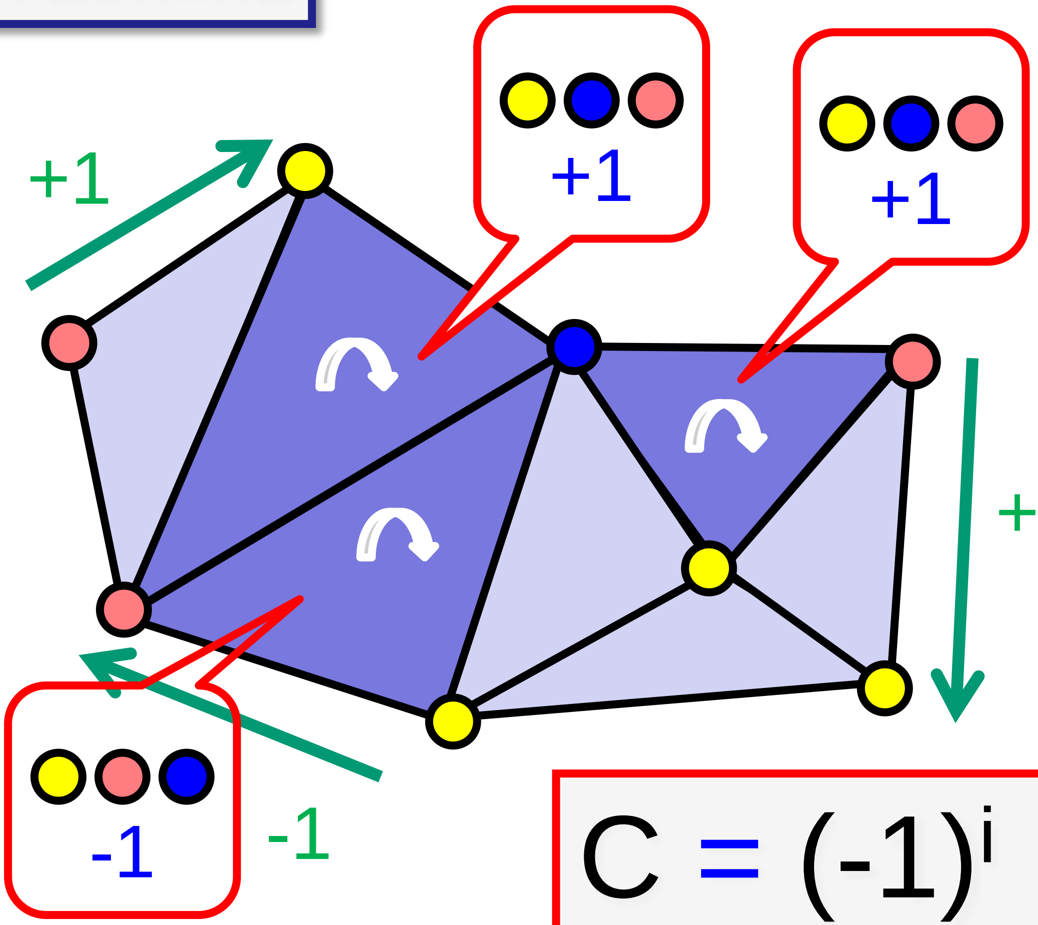
Counted by orientation.

Let i =



I_0 = 1-1+1=1

Index Lemma

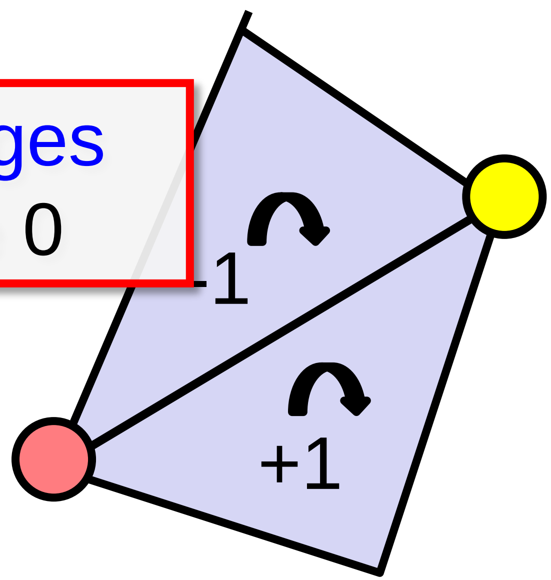


$$C = (-1)^i \, I_i$$

Proof for Dim 2

Let S be the number of 01 edges counted by orientation

Internal edges contribute 0

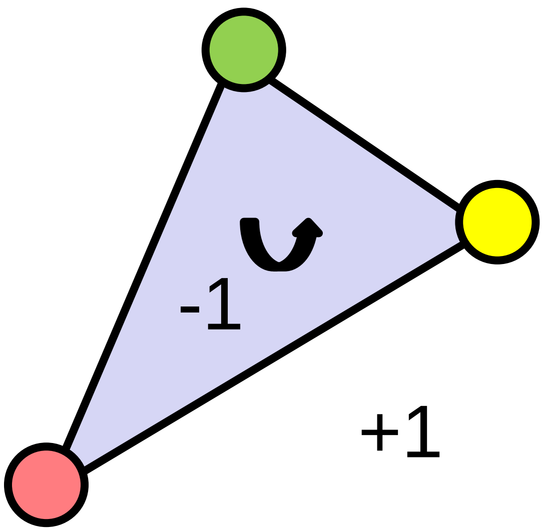


boundary edges contribute to I_i

So $S = I_i$

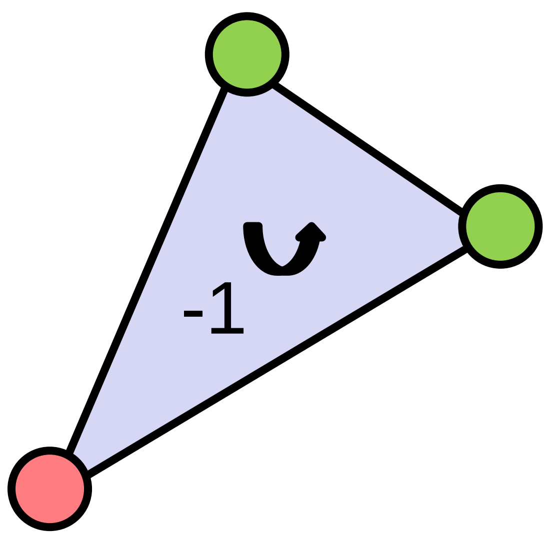
Proof for Dim 2

For properly colored triangle, 01 edge adds same value to both C and I_i



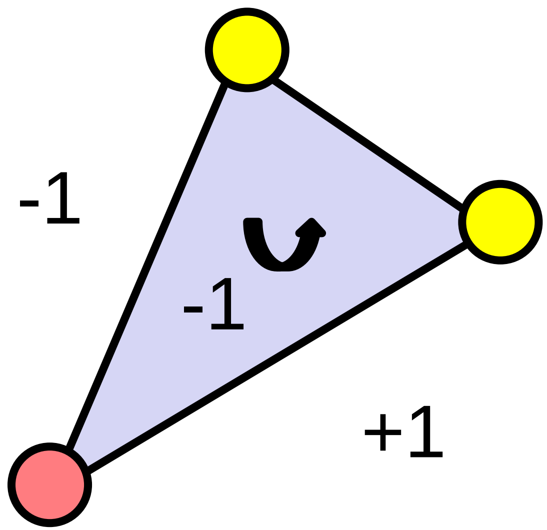
Proof for Dim 2

For non-properly colored triangle,
either no 01-edges …



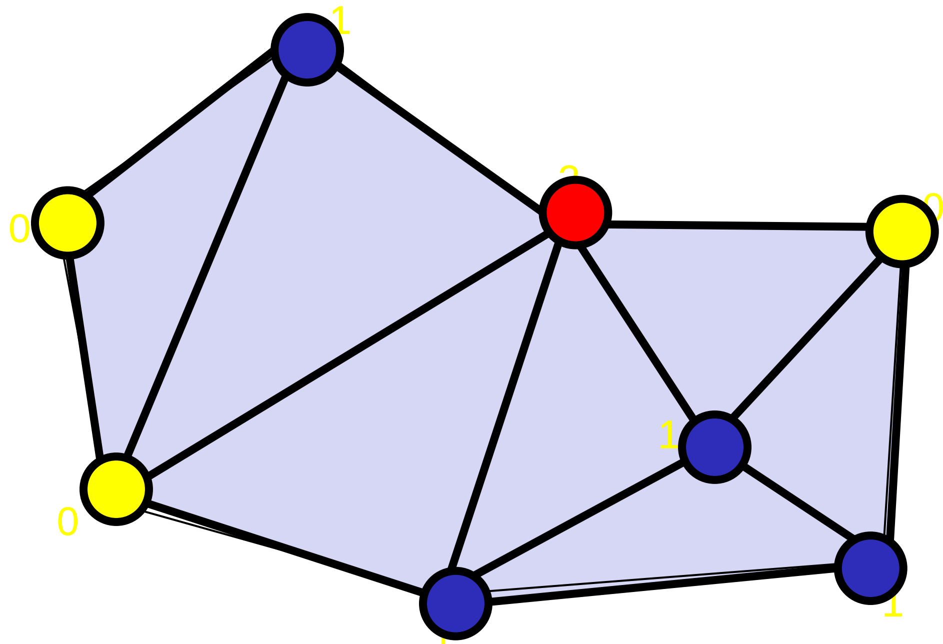
Proof for Dim 2

For non-properly colored triangle, either or two 01-edges that cancel



So S = C

Think of the index as the number of times the boundary of $\mathcal{K}$ is wrapped around the boundary of Δ^{2n}



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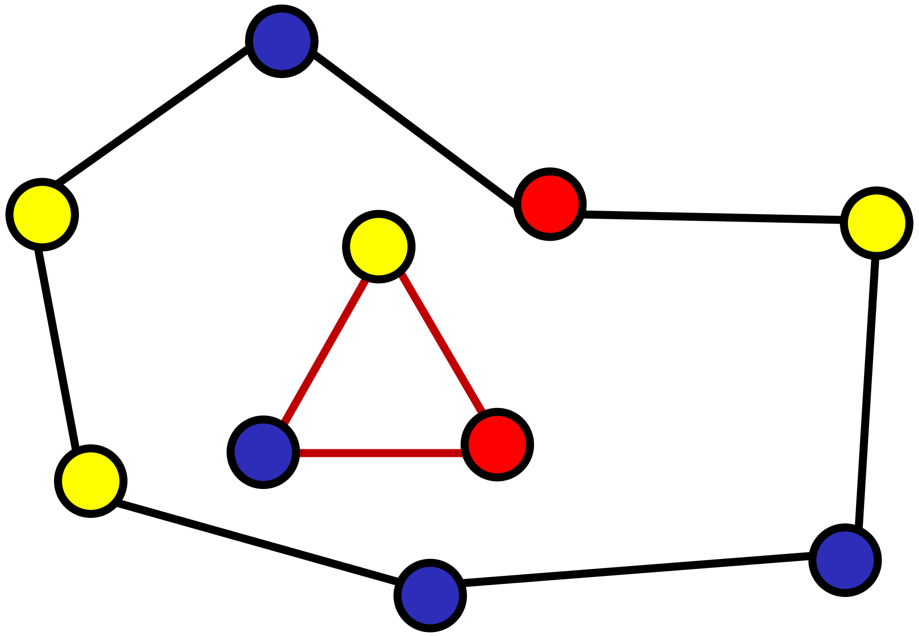

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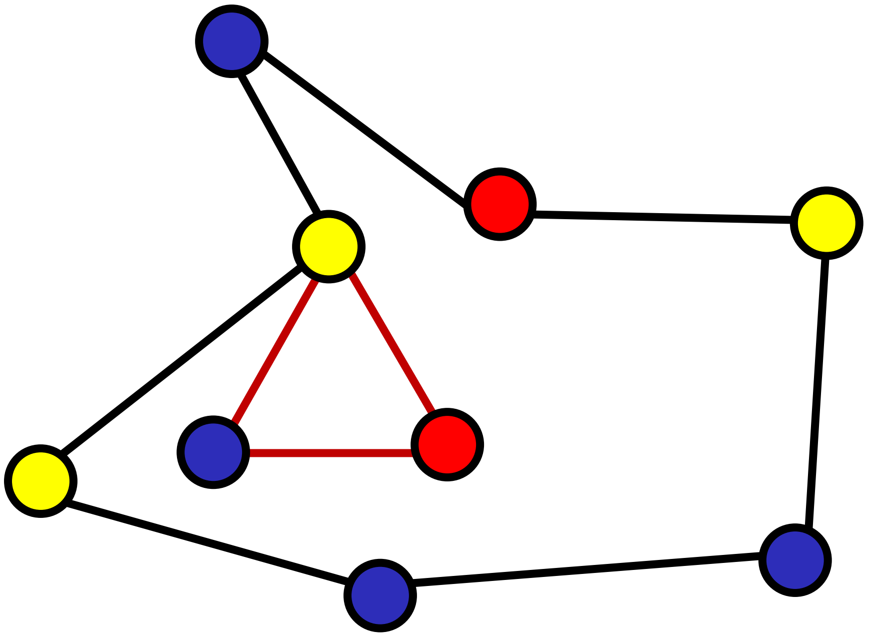

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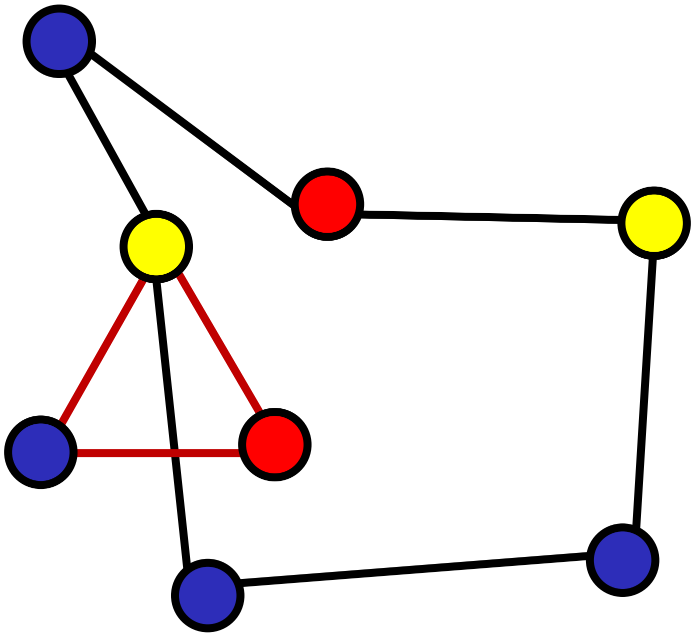

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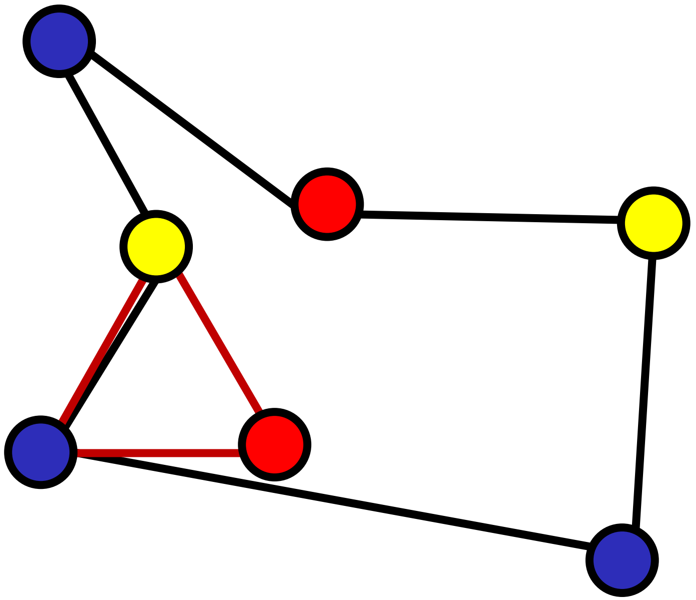

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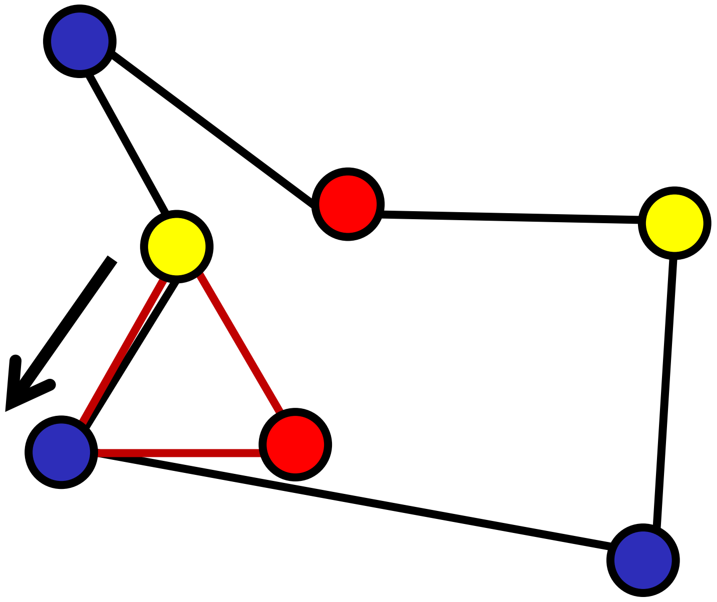

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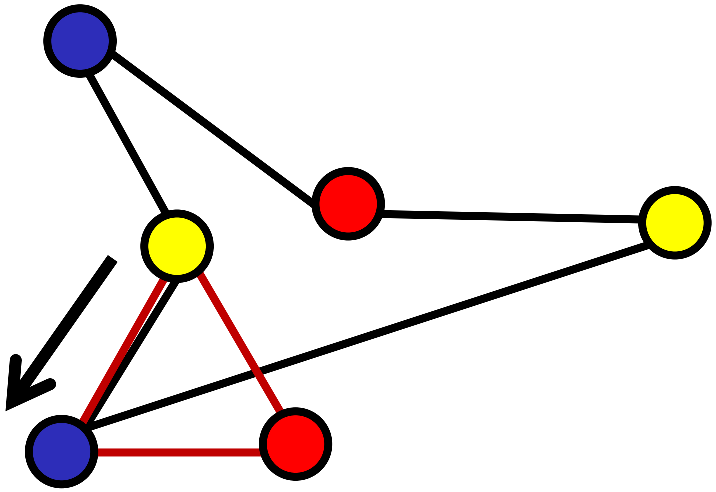

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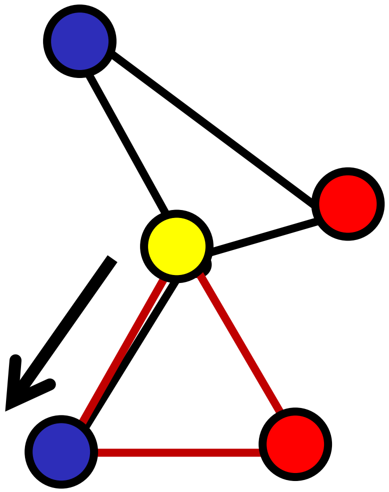

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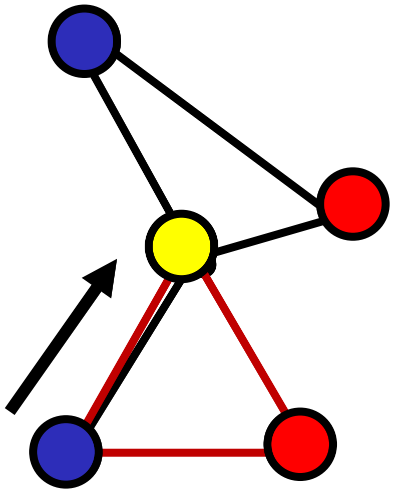

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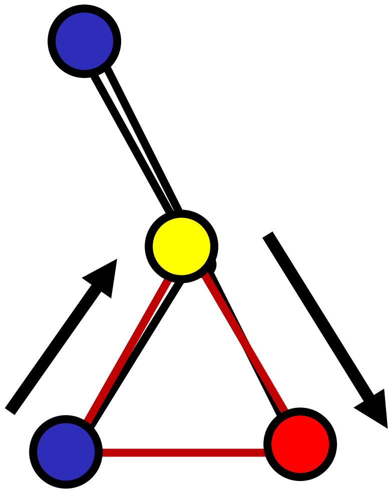
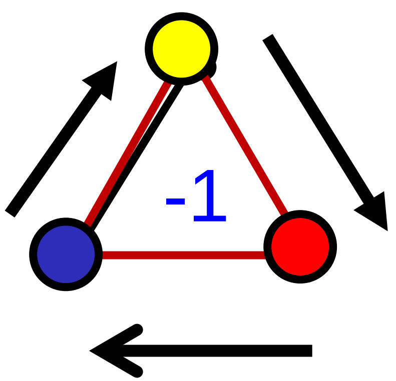


Think of the index as the number of times the boundary of $\mathcal{K}$ is wrapped around the boundary of Δ^{2n}



Road Map

An Upper Bound: 2*n*+1 Names

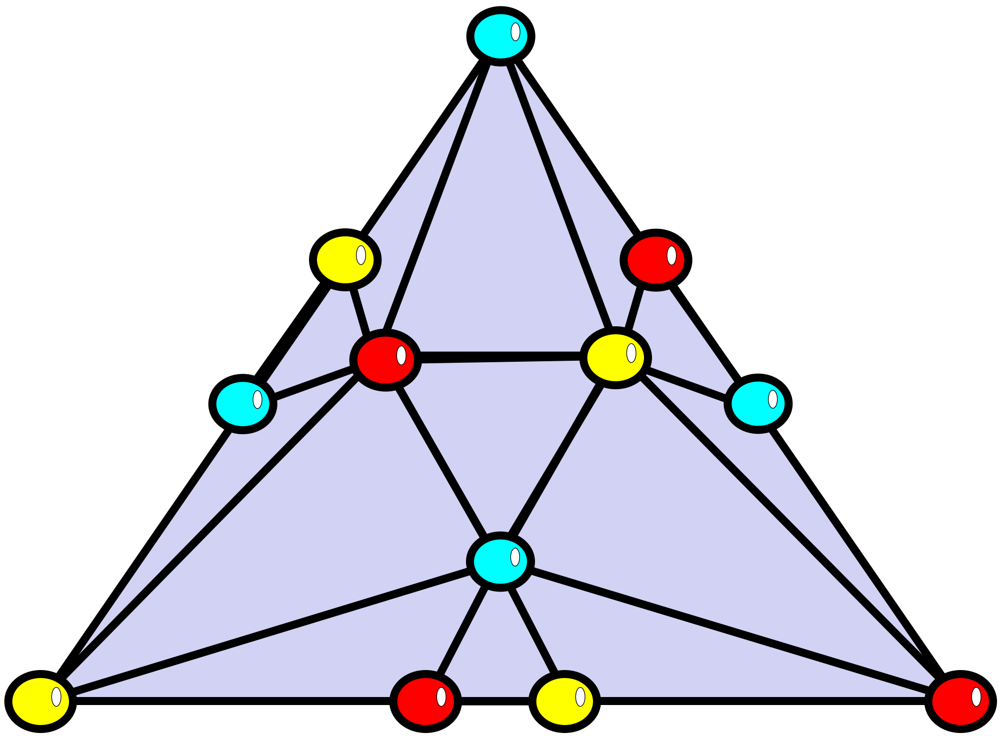
Weak Symmetry-Breaking

The Index Lemma

Binary Colorings

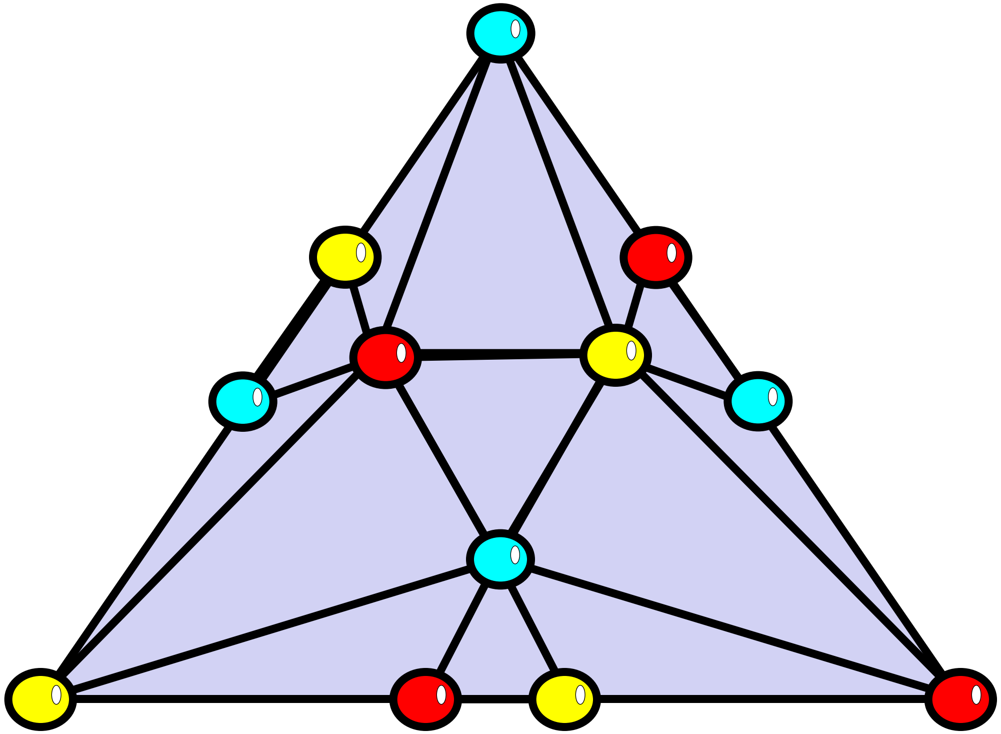
A Lower Bound for 2*n*-Renaming

Strategy

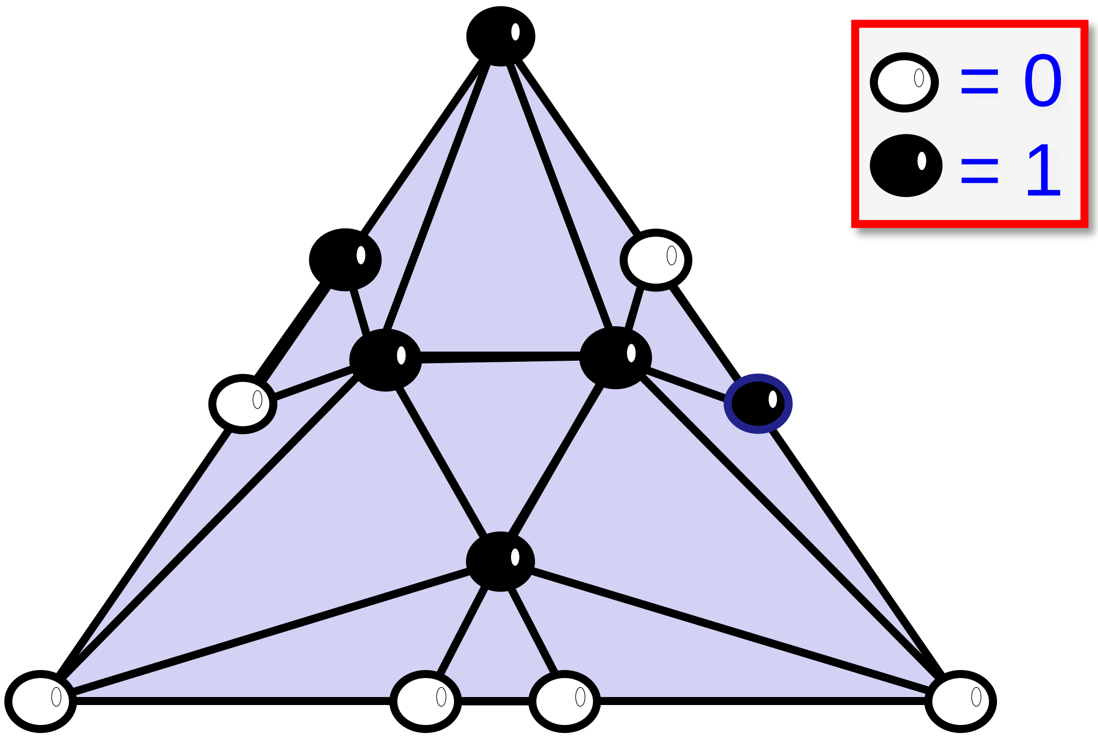


$Ch^N(\sigma)$ = WF immediate snapshot protocol complex

Protocol Complex

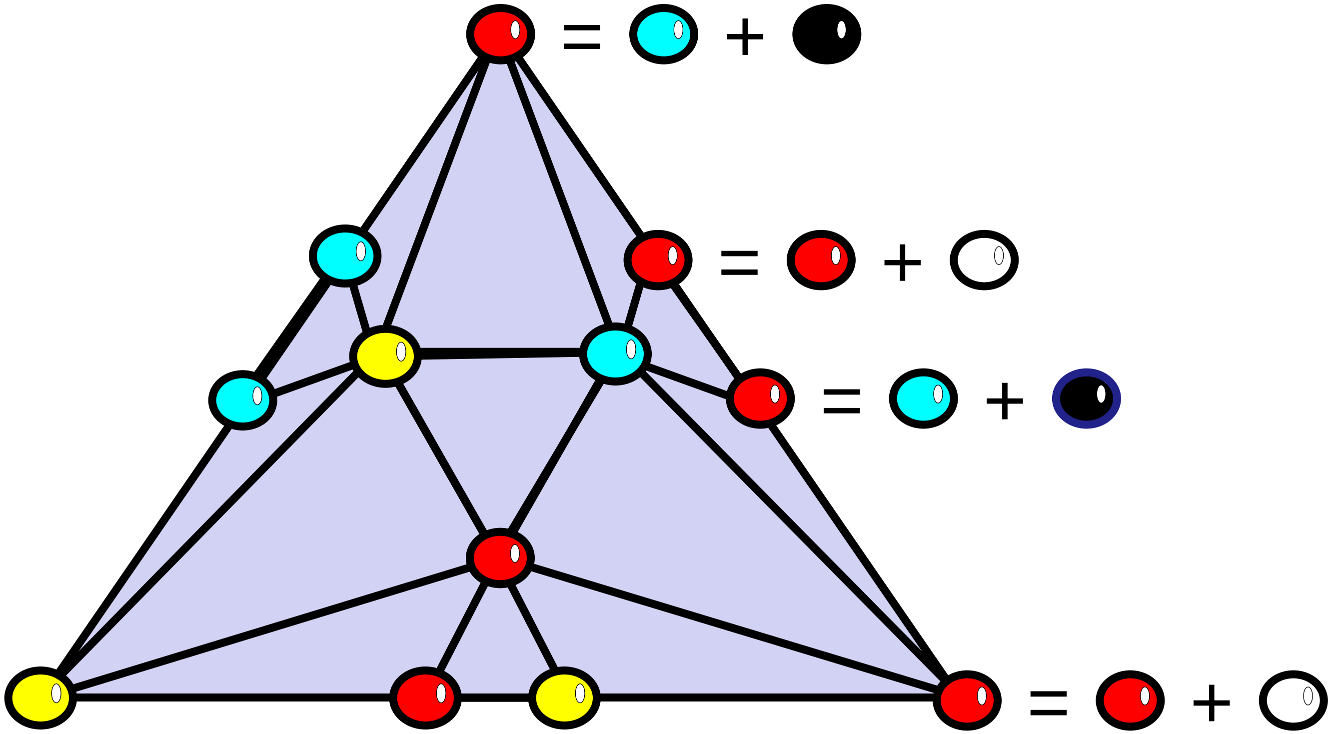


Every simplex properly colored by process name

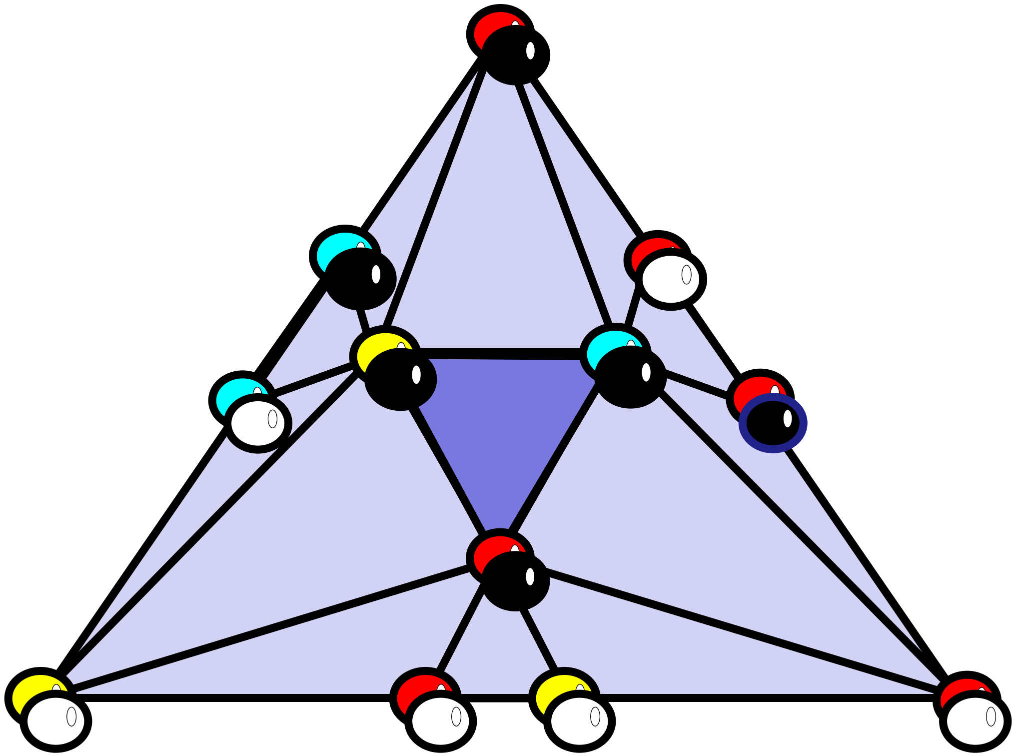


Every vertex colored by binary decision value

Every vertex colored by (name + value) mod n+1

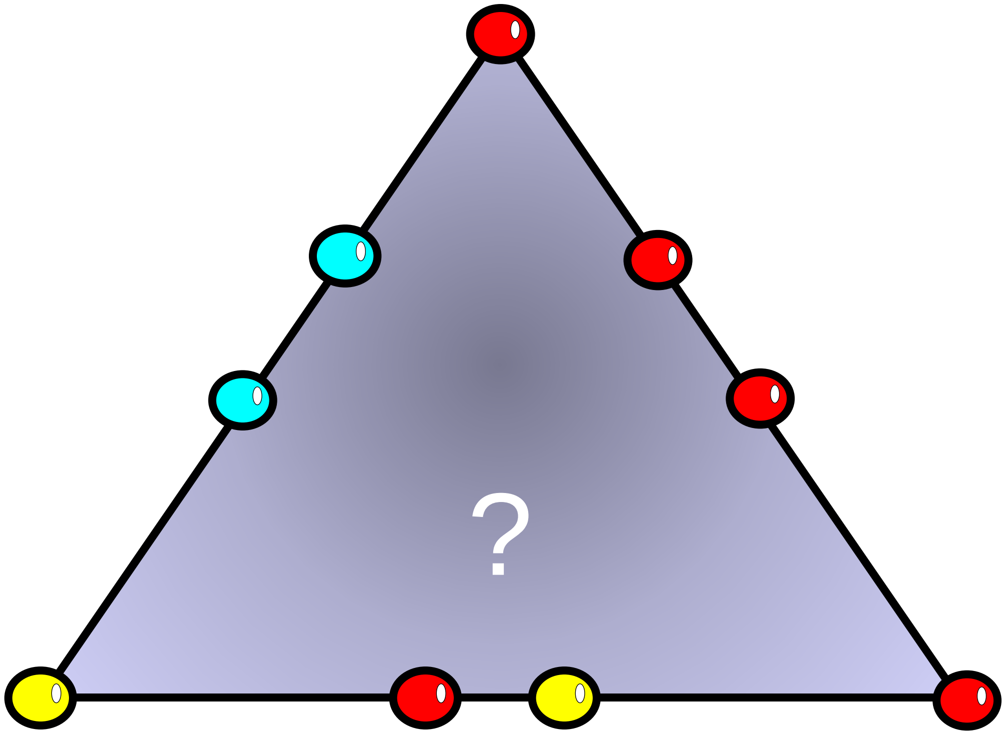


Every vertex colored by (name + value) mod n+1



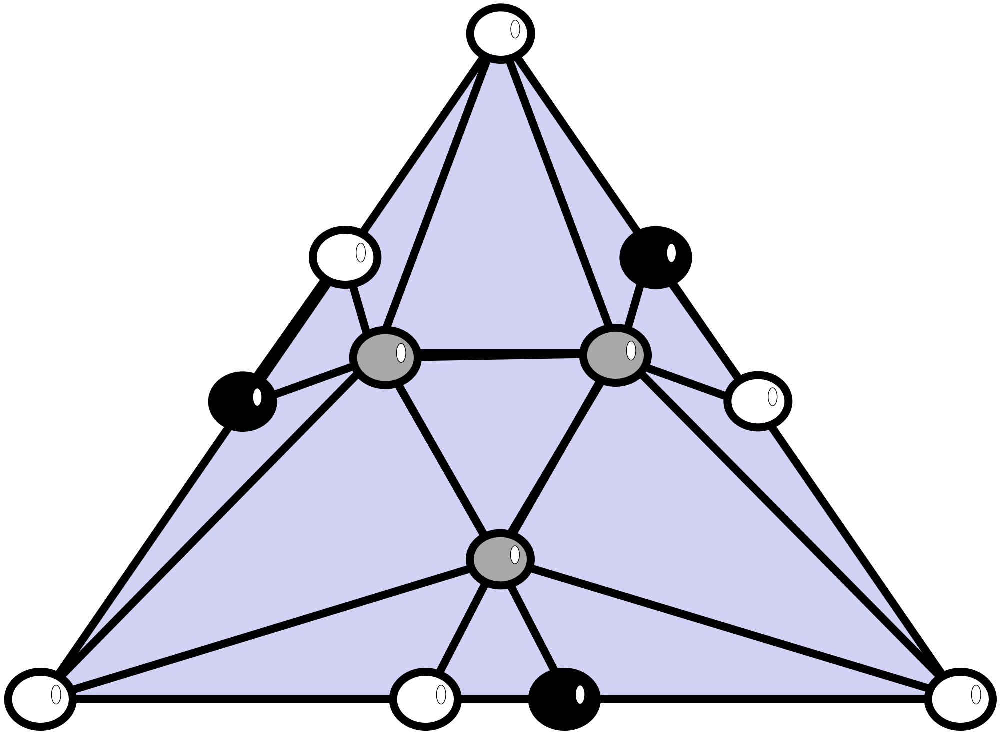
Properly colored ⇔ monochrome

Number of monochromatic n-simplexes ...



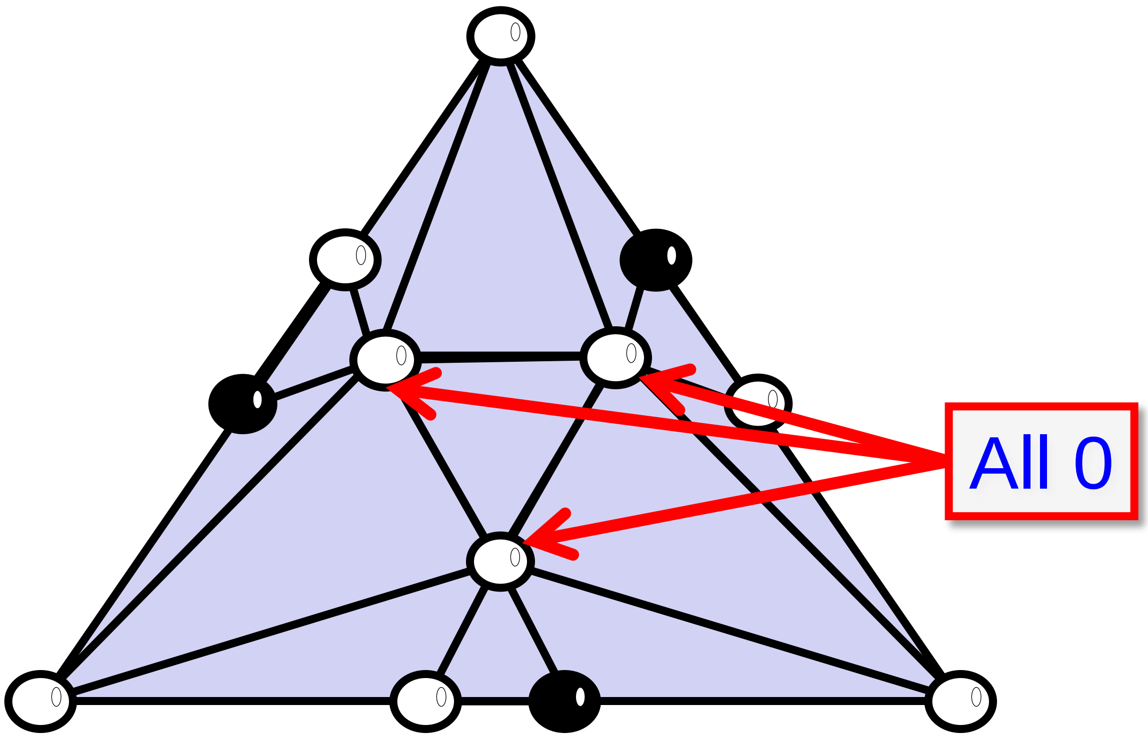
Determined by coloring on *boundary*!

If number of monochromatic simplexes is determined by boundary …



We can color interior vertexes any way we want!

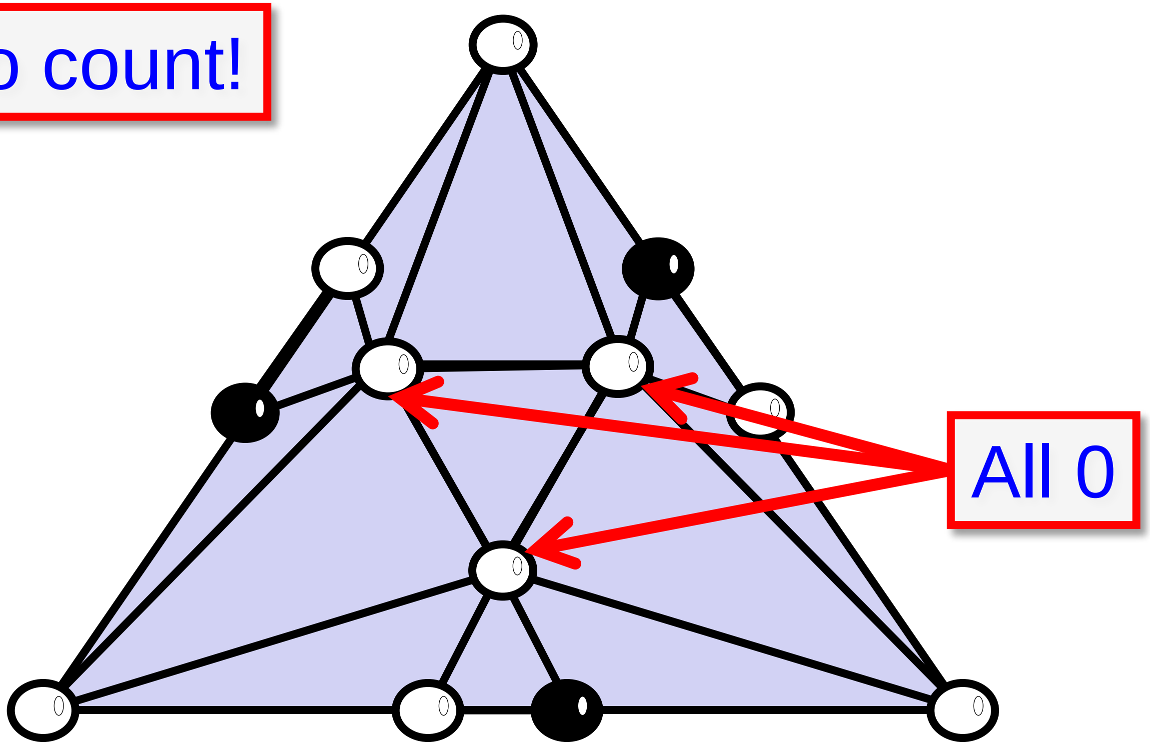
If number of monochromatic simplexes is determined by boundary …



We can color interior vertexes any way we want!

Only 0-monochromatic simplexes …

Easier to count!



Road Map

An Upper Bound: 2*n*+1 Names

Weak Symmetry-Breaking

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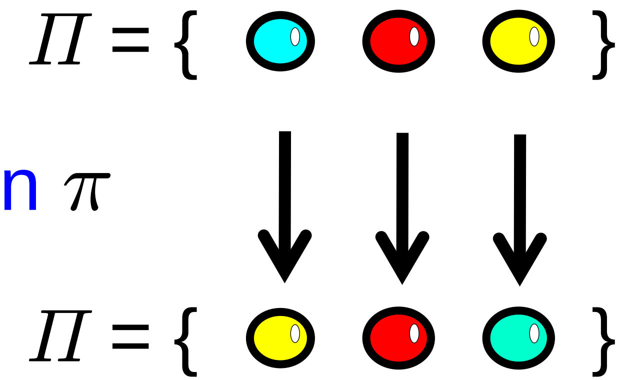
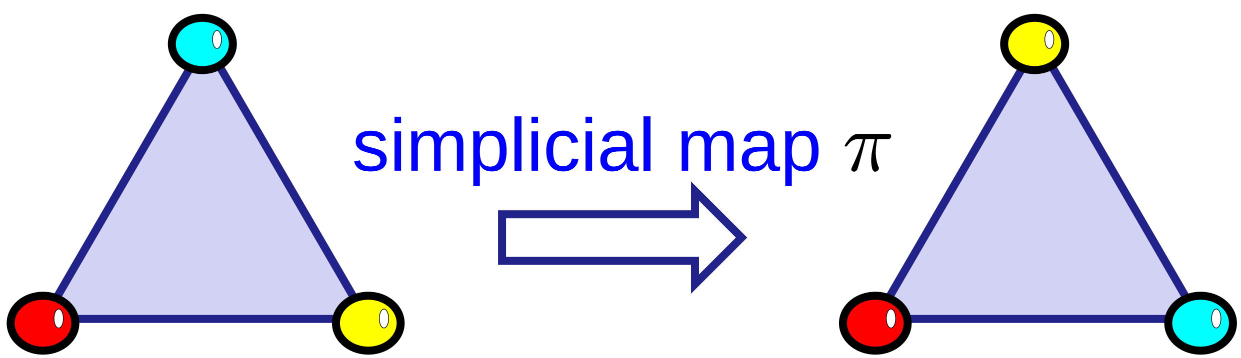
A Lower Bound for 2*n*-Renaming

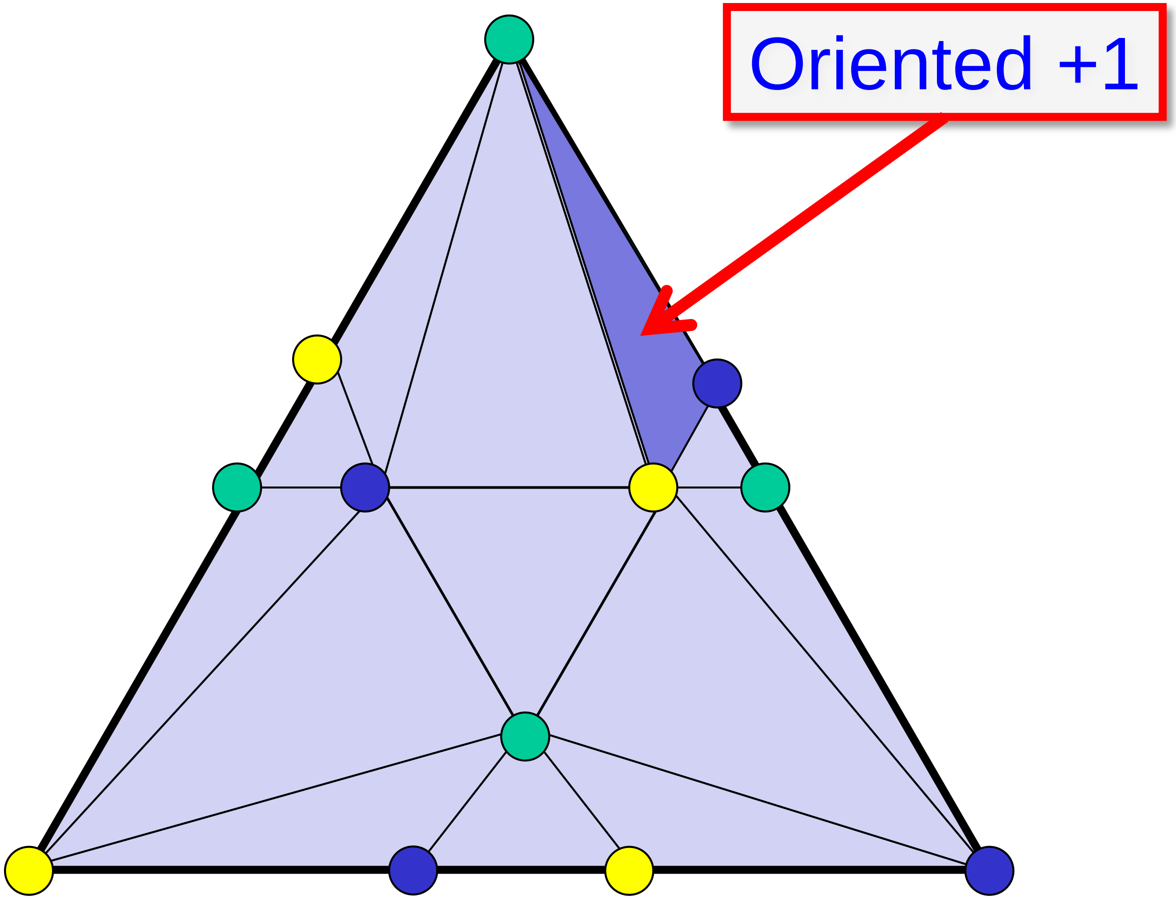
Anonymity & Symmetry



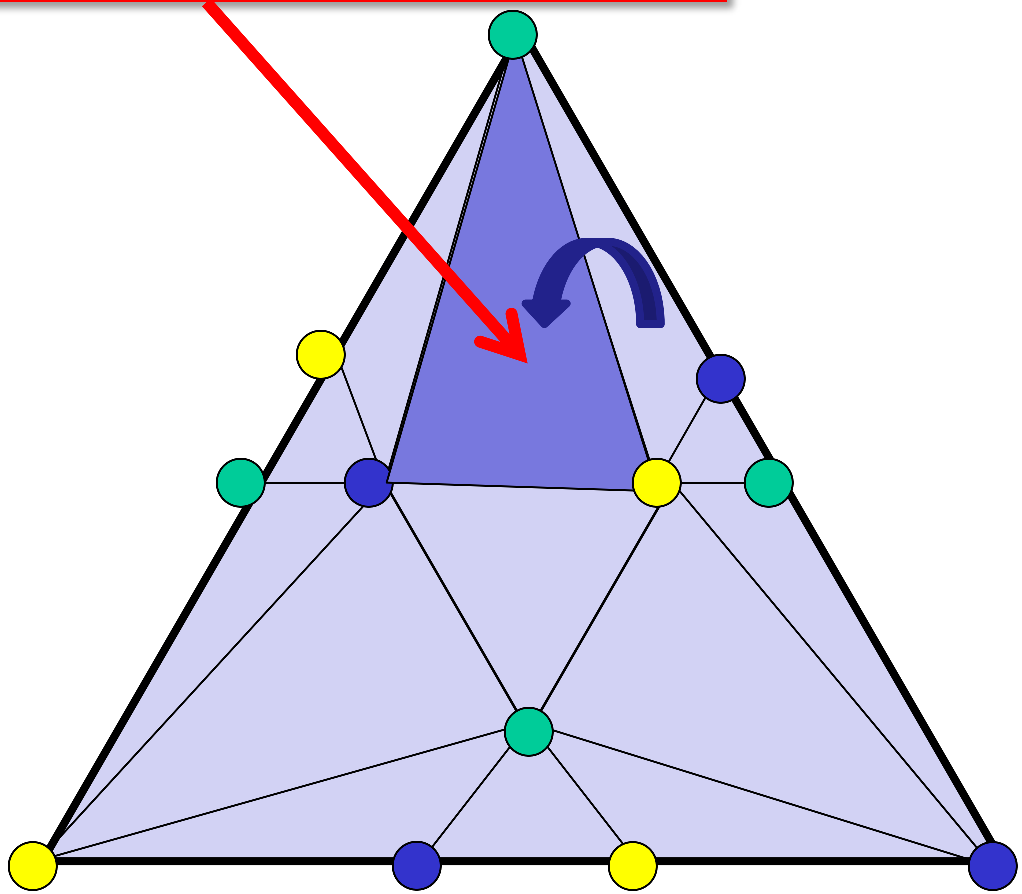
permutation π

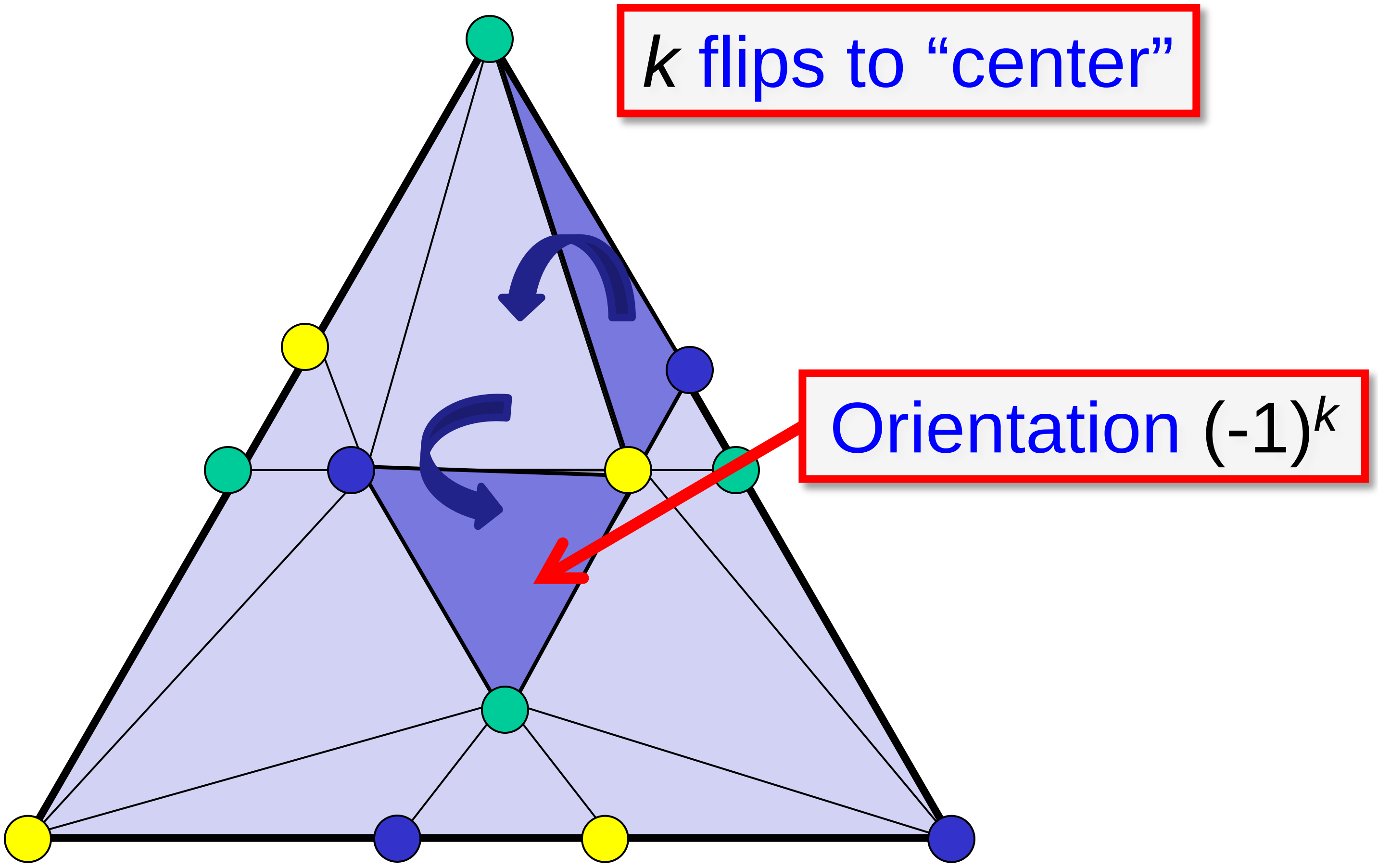


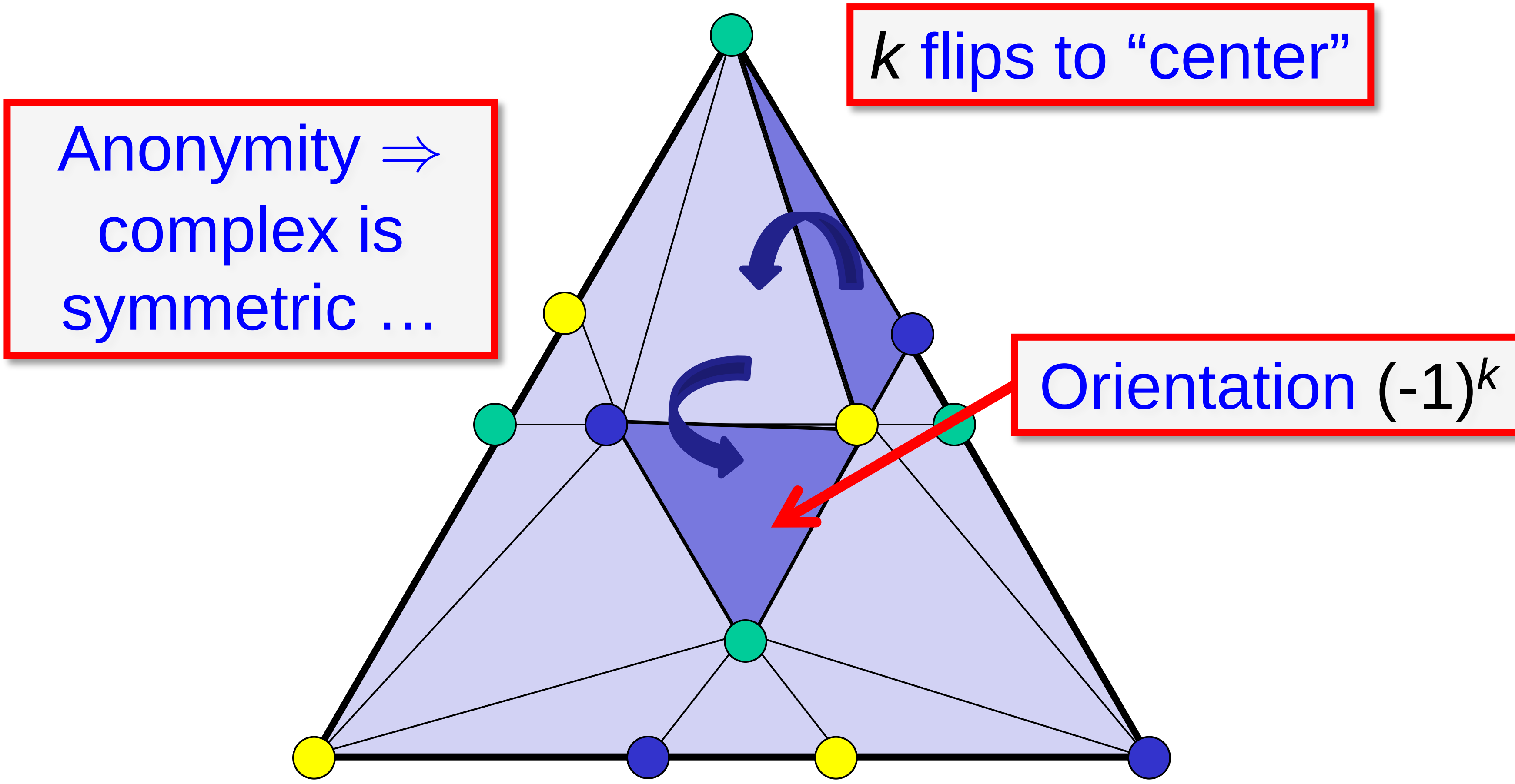
Orientations of symmetric simplexes?



“Flip” reverses orientation

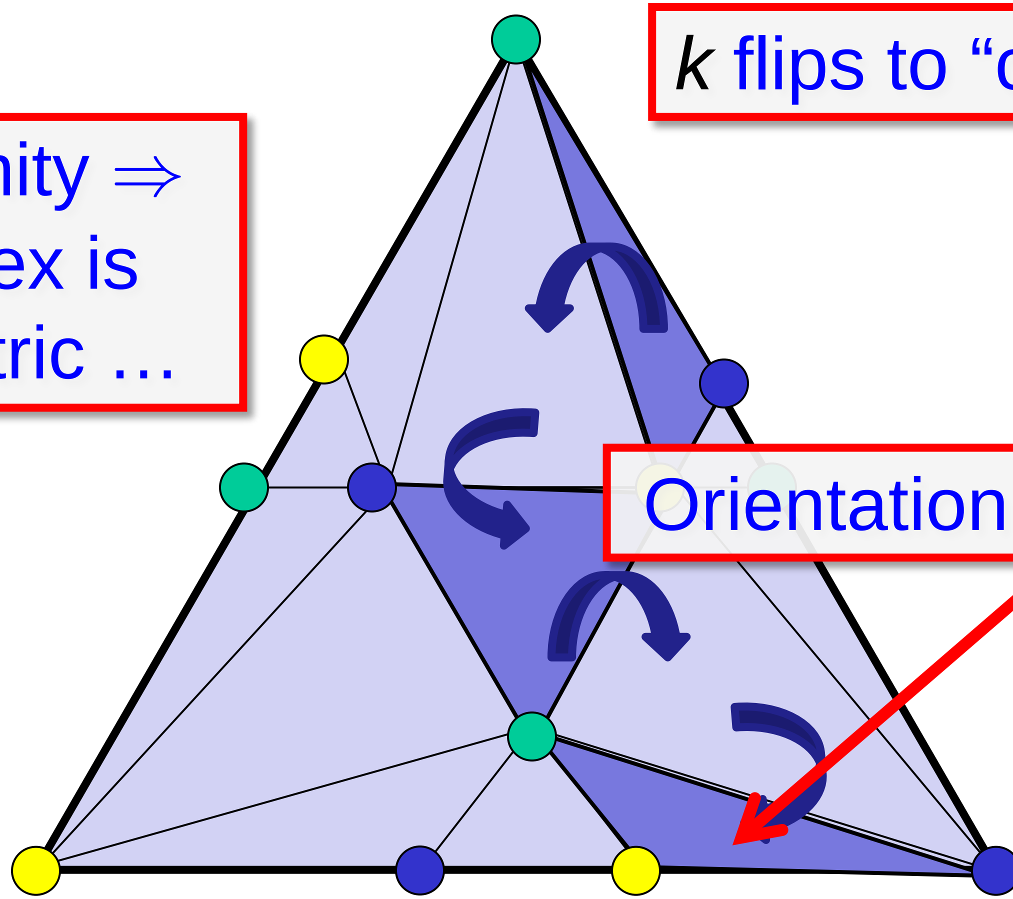



k flips to "center"
Orientation $(-1)^k$


Anonymity ⇒ complex is symmetric …
k flips to “center”
Orientation $(-1)^k$

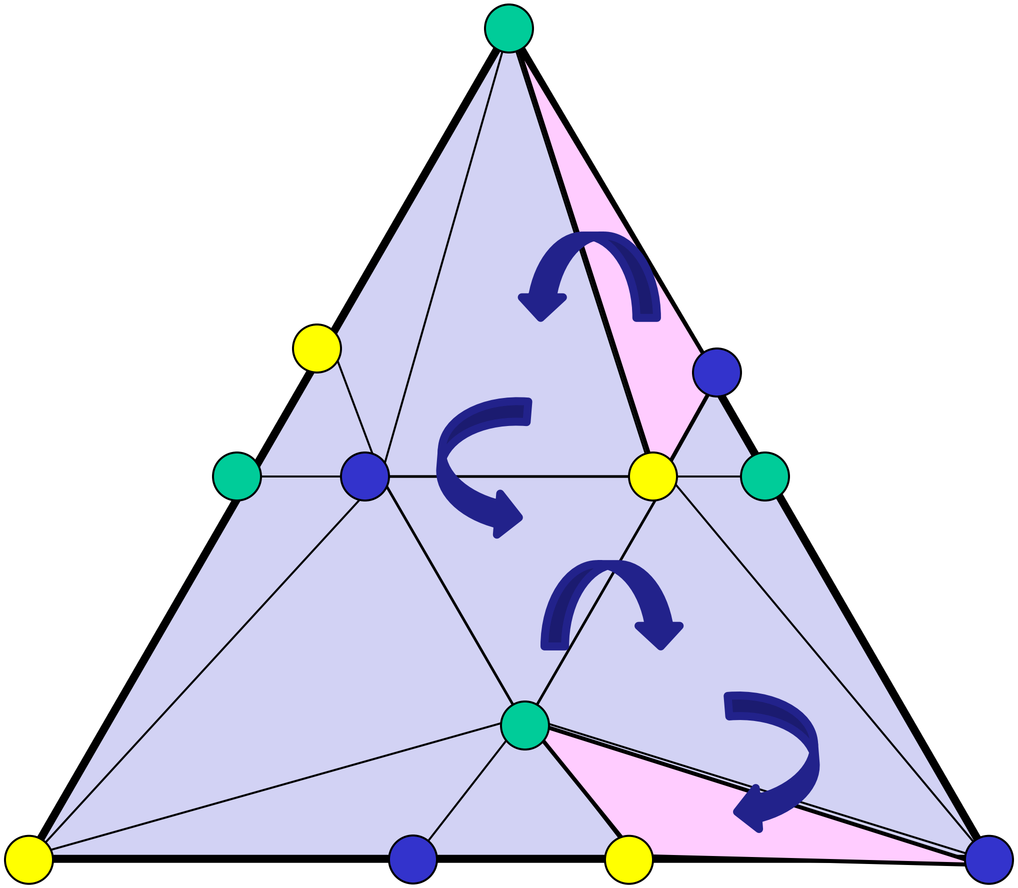
Anonymity ⇒ complex is symmetric …

k flips to “center”



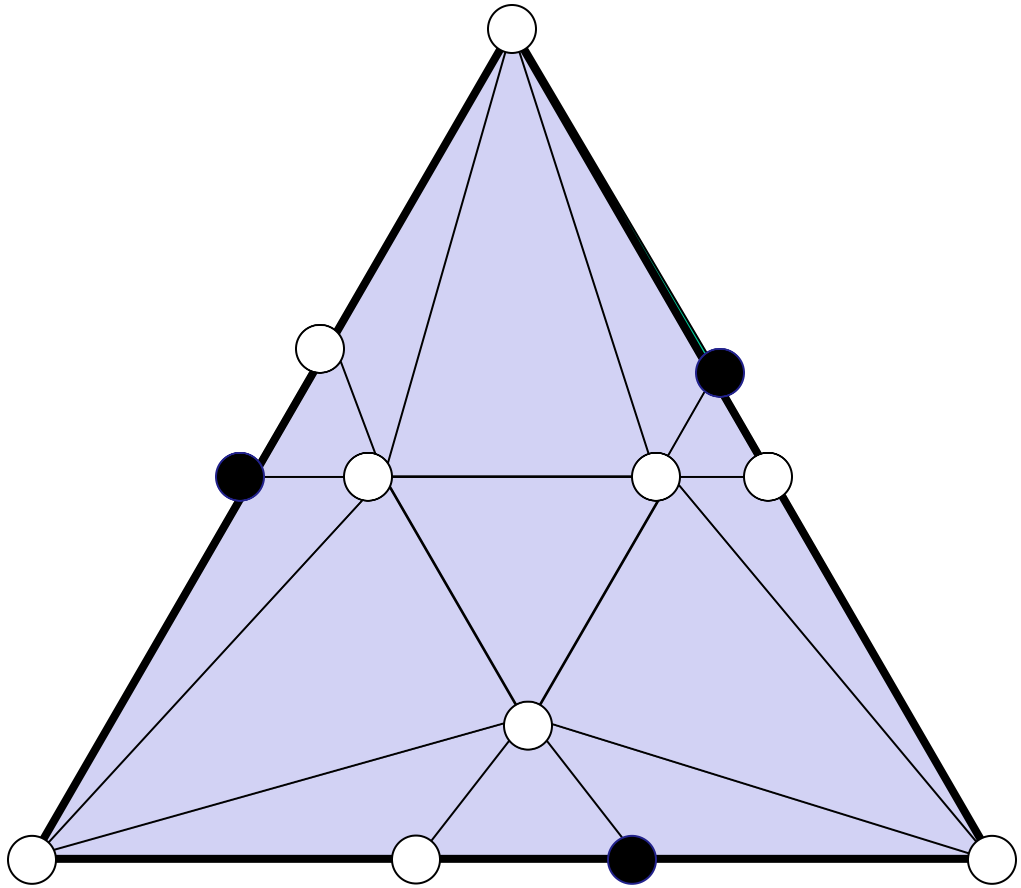
Orientation $(-1)^{2k} = +1$

k flips to symmetric edge



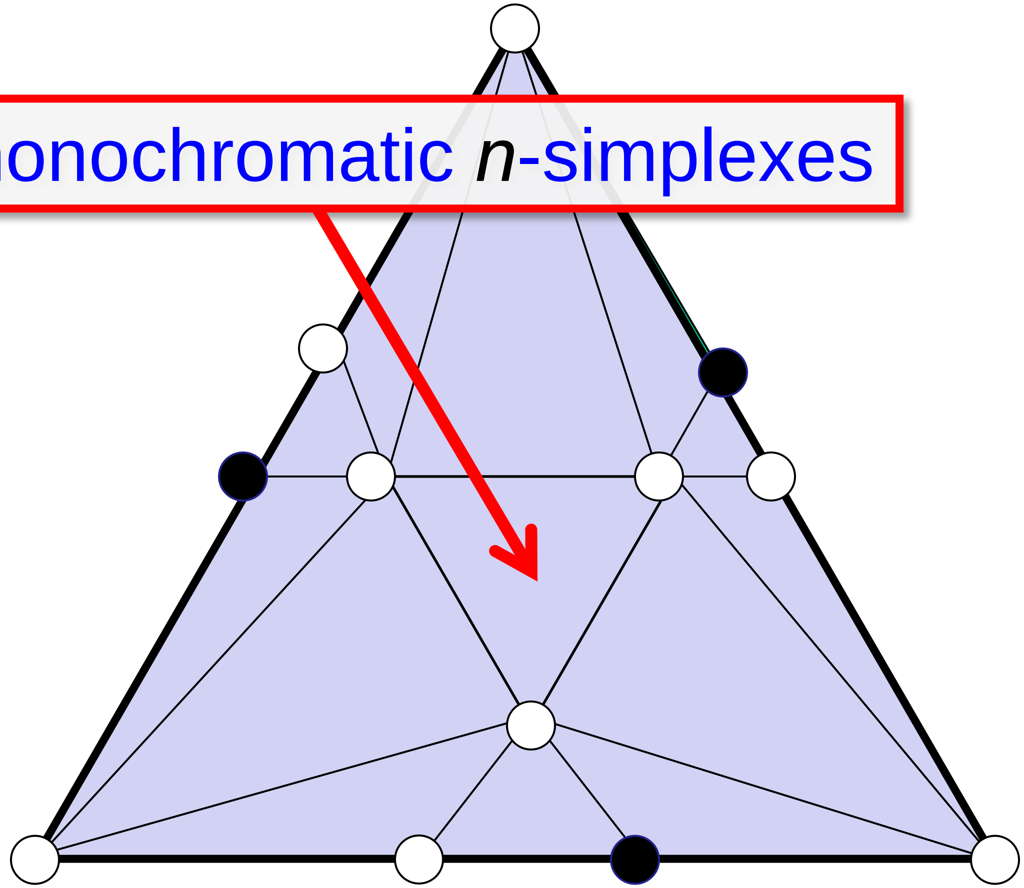
Symmetric boundary simplexes have same orientation

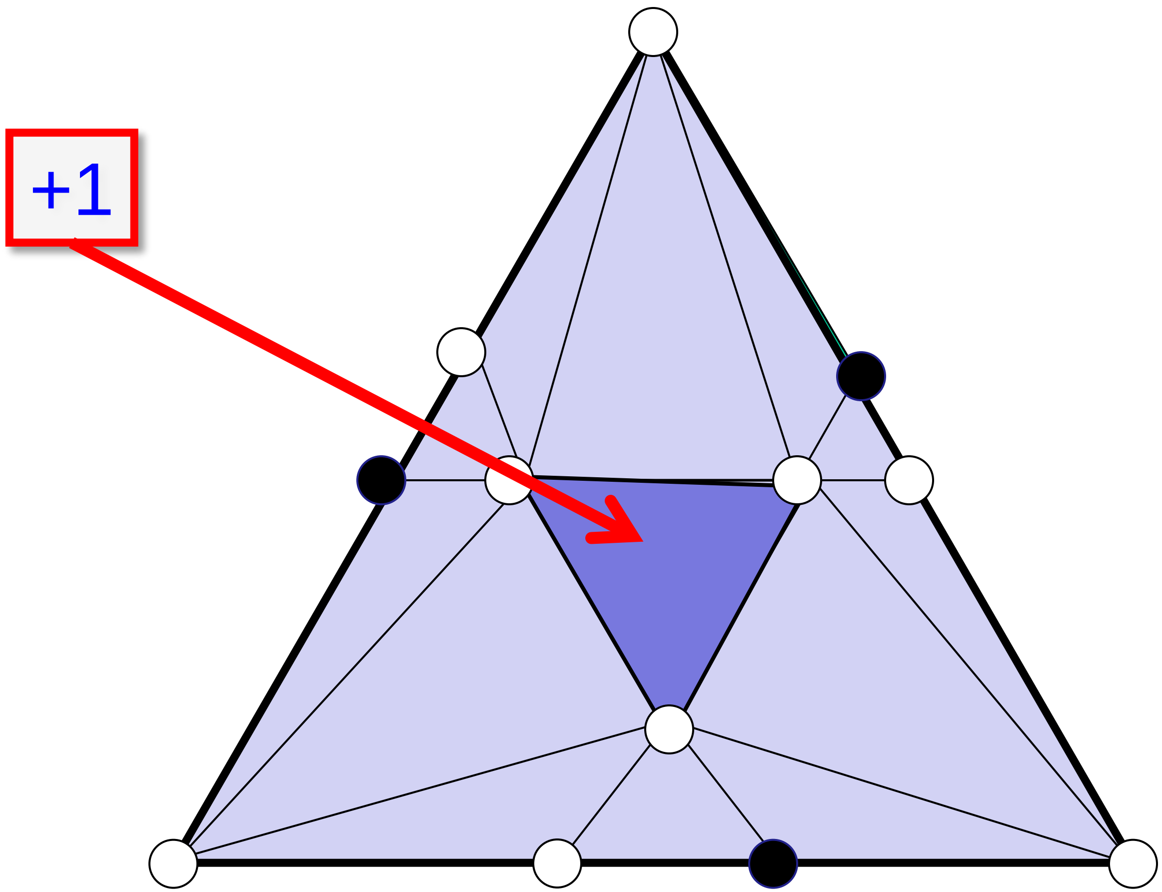
How many monochromatic simplexes?



How many monochromatic *n*-simplexes?

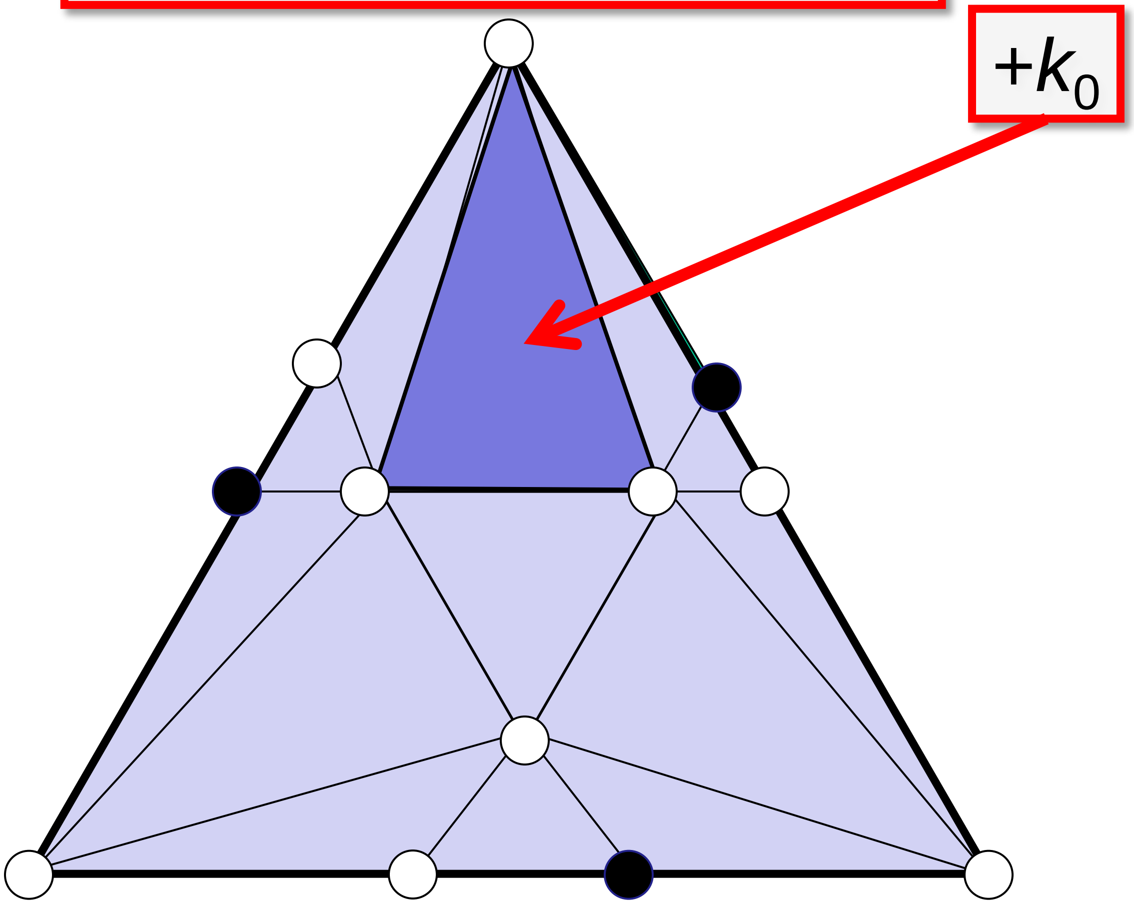
No 1-monochromatic *n*-simplexes





How many 0-monochromatic *n*-simplexes?

WLOG “corner” color is 0

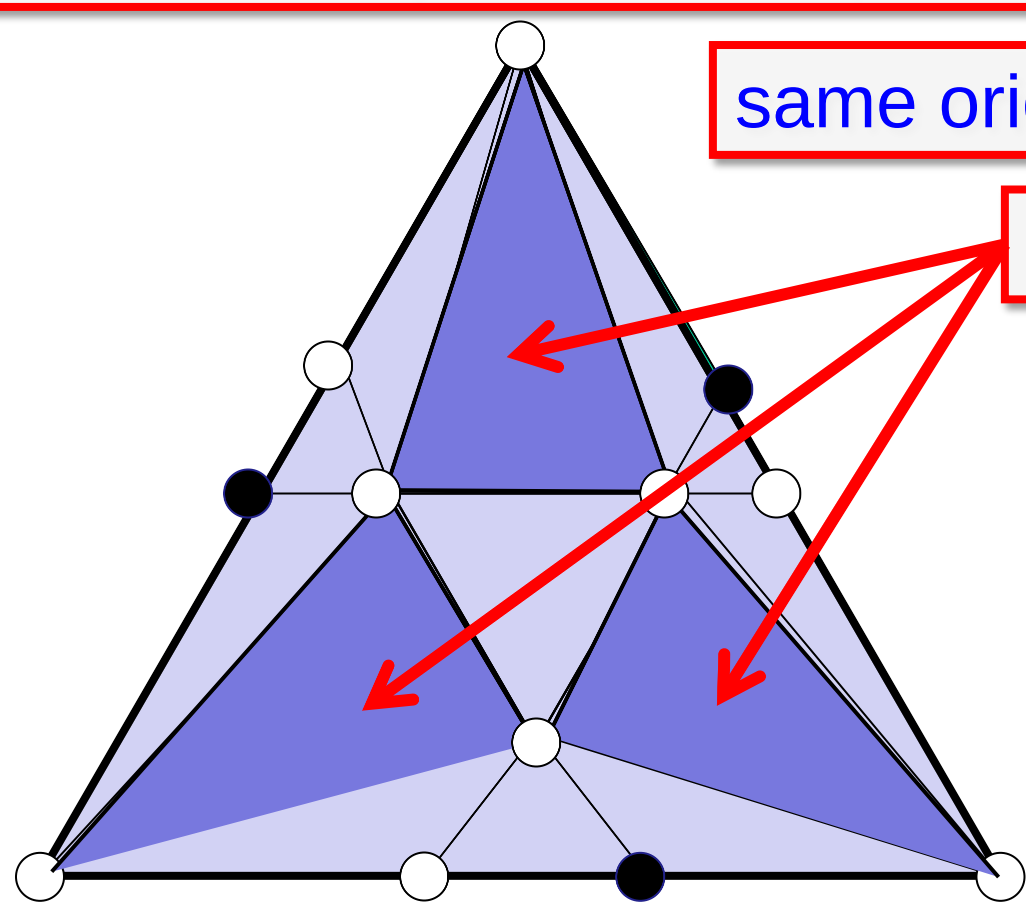


How many 0-monochromatic *n*-simplexes?

There are n+1 symmetric simplexes …

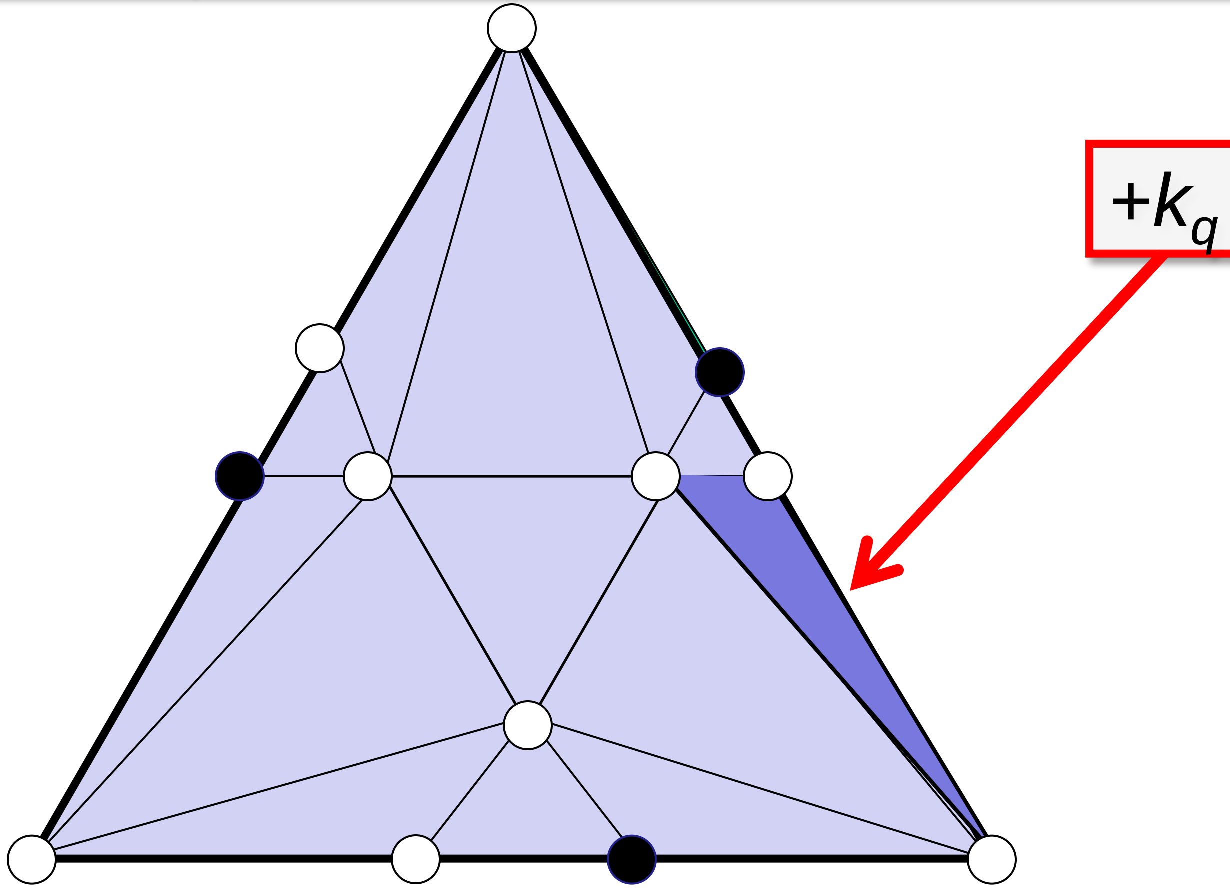
same orientation.

$+(n+1)k_0$



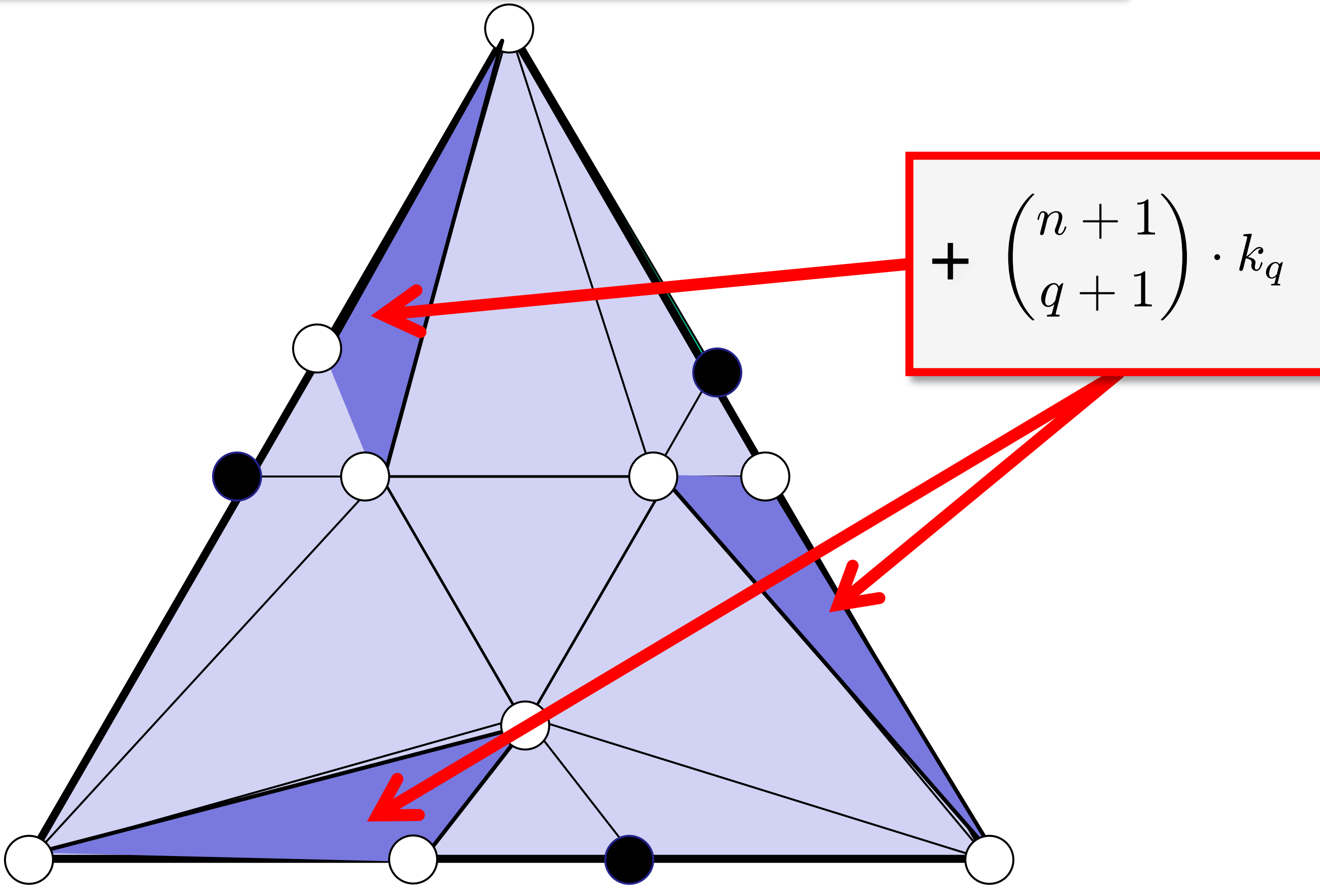
How many 0-monochromatic n-simplexes?

q-face has k_q 0-monochromatic simplexes …



How many 0-monochromatic n-simplexes?

There are $\binom{n+1}{q+1}$ symmetric faces...



How many 0-monochromatic *n*-simplexes?

Total number of monochromatic simplexes …

Counted by orientation …

$$1 + \sum_{i=0}^{n-1} \binom{n+1}{i+1} k_i$$

Integers k_i …

WSB requires this number to be zero …

This sum cannot be zero if …

$$1 + \sum_{i=0}^{n-1} \binom{n+1}{i+1} k_i$$

$$\left\{ \binom{n+1}{1}, \ldots, \binom{n+1}{n} \right\}$$

Binomial coefficients have a common factor!

Fact

$$\left\{ \binom{n+1}{1}, \ldots, \binom{n+1}{n} \right\}$$

Binomial coefficients have a common factor if and only if n+1 is a prime power

Lower Bound

2*n*-Renaming is impossible if ...

$$\left\{ \binom{n+1}{1}, \ldots, \binom{n+1}{n} \right\}$$

n+1 is not a prime power

n=5 smallest *n* for which impossibility fails …

Possible to prove that an algorithm exists …

But no explicit constriction known …

Yet.