Online algorithms: Ski rental

Assume that renting skis costs 1 per day and buying skis costs b units. Every day $x \geq 1$ you have to decide, in an online fashion, whether you will continue renting skis for one more day or buy a pair of skis. The online adversary, during the course of some unknown future day D, is going to break your leg. You would like to minimize the cost of skiing.

With foresight, you would either rent every day in $[1, D]$ or buy skis right away, thus incurring a cost of $\min\{D, b\}$. That's the optimal offline cost. A deterministic online algorithm chooses some day t, rents for up to $t-1$ days and buys skis on the morning of day t if the adversary has not yet cut your skiing career short. The algorithm's cost will be t if $D < t$ and $t - 1 + b$ if $D \geq t$. The adversary knows t and will break your leg on day t at the latest, since waiting further could only increase the optimal offline cost and would not hurt the algorithm's cost. Choosing $D = t$ yields competitive ratio

$$\frac{t-1+b}{\min(t,b)} = 1 + \frac{\max(t,b)}{\min(t,b)} - \frac{1}{\min(t,b)}.$$

Choosing $D < t$ yields competitive ratio $\frac{D}{\min(D,b)}$, a monotone non-decreasing function of D, so the adversary's best choice in that case would be $D = t - 1$ to yield competitive ratio $\frac{t-1}{\min(t-1,b)}$, which is less than or equal to $t/\min(t,b)$, and so the adversary will prefer $D = t$. Thus the deterministic algorithm has competitive ratio exactly

$$1 + \frac{\max(t,b) - 1}{\min(t,b)},$$

which is minimized for $t = b$, yielding optimal deterministic competitive ratio of $2 - 1/b$.

A randomized algorithm chooses some day T at random according to some distribution, rents for up to $T - 1$ days and buys skis on the morning of day T if the adversary has not yet broken your leg. Given the distribution of T, the adversary chooses D to maximize the expected cost of the algorithm divided by the optimal cost.

In order to defeat the adversary, the randomized algorithm thus seeks a c-competitive strategy for the distribution $(p_t)_t$ of T, that is:

$$\inf c \text{ s.t. for every } D,$$

$$bp_1 + (1+b)p_2 + (2+b)p_3 + \cdots + (D-1+b)p_D + D\sum_{j>D} p_j \leq c\min(D,b).$$

The constraints defining the problem form an infinite linear program which we now want to solve.

For simplicity, consider the case $b = 4$. The algorithm wants:

$$\inf c \text{ s.t.}$$

$$\begin{cases} 4p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + \cdots & \leq & c & (D = 1) \\ 4p_1 + 5p_2 + 2p_3 + 2p_4 + 2p_5 + 2p_6 + 2p_7 + \cdots & \leq & 2c & (D = 2) \\ 4p_1 + 5p_2 + 6p_3 + 3p_4 + 3p_5 + 3p_6 + 3p_7 + \cdots & \leq & 3c & (D = 3) \\ 4p_1 + 5p_2 + 6p_3 + 7p_4 + 4p_5 + 4p_6 + 4p_7 + \cdots & \leq & 4c & (D = 4) \\ 4p_1 + 5p_2 + 6p_3 + 7p_4 + 8p_5 + 5p_6 + 5p_7 + \cdots & \leq & 4c & (D = 5) \\ 4p_1 + 5p_2 + 6p_3 + 7p_4 + 8p_5 + 9p_6 + 6p_7 + \cdots & \leq & 4c & (D = 6) \\ \vdots & \leq & 4c & \end{cases}$$

Step 1. Now observe that for $D \geq 4$, the right hand side stays fixed while all the coefficients of the variables on the left hand side vary monotonically when going from one row to the next; thus, the row $D = 4$ is dominated by the subsequent rows and can be deleted from the set of constraints. In the remaining system, consider the column of p_4 and the column of p_5, and observe that the coefficient of p_4 is always less than or equal to the coefficient of p_5. Thus, if $p_5 \neq 0$, the algorithm can always construct a solution which is at least as good by setting $p'_5 = 0$, $p'_4 = p_4 + p_5$, and $p'_i = p_i$ otherwise. Thus, without loss of generality we can assume that $p_5 = 0$ and remove column 5, leading to the following equivalent LP:

$$\inf c \text{ s.t.}$$

$$\begin{cases} 4p_1 + p_2 + p_3 + p_4 + p_6 + p_7 + \cdots & \leq & c & (D = 1) \\ 4p_1 + 5p_2 + 2p_3 + 2p_4 + 2p_6 + 2p_7 + \cdots & \leq & 2c & (D = 2) \\ 4p_1 + 5p_2 + 6p_3 + 3p_4 + 3p_6 + 3p_7 + \cdots & \leq & 3c & (D = 3) \\ 4p_1 + 5p_2 + 6p_3 + 7p_4 + 5p_6 + 5p_7 + \cdots & \leq & 4c & (D = 5) \\ 4p_1 + 5p_2 + 6p_3 + 7p_4 + 9p_6 + 6p_7 + \cdots & \leq & 4c & (D = 6) \\ \vdots & \leq & 4c & \end{cases}$$

Iterating, we can argue that for any $i > 4$, without loss of generality we have $p_j = 0$ for every $j \in (4, i]$. We will apply this for $i = 2b/\epsilon$.

Step 2. Now, note that for every $N \geq 2b$, it must be that $\sum_{t \geq N} p_t < 2b/N$, since otherwise the adversary could just set $D = N$ to force competitive ratio at least $(2b/N)N/b \geq 2$, which would be worse than the simple deterministic algorithm described above. (In particular, the probability p_∞ that we keep renting forever is 0.) Applying this remark to $N = 2b/\epsilon$ yields $\sum_{t > 2b/\epsilon} p_t < \epsilon$. Now, consider the solution such that $p'_t = 0$ for $t > 2b/\epsilon$, $p'_1 = p_1 + \sum_{t > 2b/\epsilon} p_t$, and $p'_t = p_t$ otherwise. The left hand side of any constraint increases by at most 3ϵ. Letting $c' = c + 3\epsilon$ creates a feasible solution to the LP.

Thus we have proved that without loss of generality, $p_t = 0$ for every $t \geq 5$, and have reduced ourselves to solving a finite problem (where the inf now becomes a min by compactness):

$$\min c \text{ s.t.}$$

$$\begin{cases} 4p_1 + p_2 + p_3 + p_4 & \leq & c & (D=1) \\ 4p_1 + 5p_2 + 2p_3 + 2p_4 & \leq & 2c & (D=2) \\ 4p_1 + 5p_2 + 6p_3 + 3p_4 & \leq & 3c & (D=3) \\ 4p_1 + 5p_2 + 6p_3 + 7p_4 & \leq & 4c & (D \geq 4) \end{cases}$$

Step 3. We now prove the "principle of equality", which claims that in the optimal solution, every constraint is exactly tight.

Indeed, take a solution which achieves the minium c and assume, for a contradiction, that one of the inequalities has slack, for example the inequality for $D=3$. Then we can increase p_3 and decrease p_4 until that inequality is tight. This does not affect inequalities $D=1, D=2$ and creates slack in the inequality $D=4$. We then increase p_1 a little bit and decrease p_4 a little bit to create slack in every constraint, which then enables us to decrease c, a contradiction. Thus we have:

$$\min c \text{ s.t.}$$

$$\begin{cases} 4p_1 + p_2 + p_3 + p_4 & = & c & (D=1) \\ 4p_1 + 5p_2 + 2p_3 + 2p_4 & = & 2c & (D=2) \\ 4p_1 + 5p_2 + 6p_3 + 3p_4 & = & 3c & (D=3) \\ 4p_1 + 5p_2 + 6p_3 + 7p_4 & = & 4c & (D \geq 4) \end{cases}$$

Step 4. To solve this simple linear system, substract each row from the row below to make the system upper triangular, then substract each row from the row above to make the system near diagonal, then substitute. We go back to general b to give the explicit form of the general solution. Subtracting once:

$$\begin{cases} bp_1 + p_2 + p_3 + \cdots + p_b & = & c & (D=1) \\ bp_2 + p_3 + \cdots + p_b & = & c & (D=2) \\ bp_3 + \cdots + p_b & = & c & (D=3) \\ \vdots & & & \\ bp_b & = & c & (D=b) \end{cases}$$

Subtracting again:

$$\begin{cases} bp_1 - (b-1)p_2 & = & 0 & (D=1) \\ bp_2 - (b-1)p_3 & = & 0 & (D=2) \\ bp_3 - (b-1)p_4 & = & 0 & (D=3) \\ \vdots & & & \\ bp_{p-1} - (b-1)p_b & = & 0 & (D=b-1) \\ bp_b & = & c & (D=b) \end{cases}$$

We obtain:

$$p_i = (\frac{b-1}{b})^{b-i}\frac{c}{b}.$$

Since the p_i's sum to 1, this yields the value of c, the optimal randomized competitive ratio, and using the fact that $(1-1/b)^b$ is always at most $1/e$ and tends to $1/e$ at infinity:

$$c = \frac{b}{\sum_{0 \leq i \leq b-1} (\frac{b-1}{b})^i} = \frac{b}{\frac{1-(1-1/b)^b}{1-(1-1/b)}} = \frac{1}{1-(1-1/b)^b} \approx \frac{e}{e-1}.$$

Step 4, alternate approach. Alternatively, one can go to the continuous setting (p_t) over real numbers t and (with some kind of justification presumably needed in order to make this rigorous) replace the system of equations by the following:

$$\forall x \in [0,b], \quad \int_0^x (b+t)p_t dt + x\int_x^b p_t dt = cx.$$

Differentiating:

$$(b+x)p_x + \int_x^b p_t dt + x(p_b - p_x) = c.$$

Differentiating again and noting that by continuity we should have $p_b = 0$:

$$p_x + (b+x)p'_x - p_x - xp'_x - p_x = 0.$$

In other words, $p'_x/p_x = (1/b)$, hence $p_x = Ke^{x/b}$. Since $\int_0^b p_t dt = 1$, it follows that $Kb(e-1) = 1$ and $K = 1/(b(e-1))$. Finally, using the equation giving c for $x = 0$:

$$c = bp_0 + \int_0^b p_t dt = \frac{1}{e-1} + 1 = \frac{e}{e-1}.$$