

# FLOW-FIRING PROCESSES

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**ABSTRACT.** We consider a discrete non-deterministic *flow-firing* process for rerouting flow on the edges of a planar complex. The process is an instance of higher-dimensional chip-firing. In the flow-firing process, flow on the edges of a complex is repeatedly diverted across the faces of the complex. For non-conservative initial configurations we show this process never terminates. For conservative initial flows we show the process terminates after a finite number of rerouting steps, but there are many possible final configurations reachable from a single initial state. Finally, for conservative initial flows around a topological hole we show the process terminates at a unique final configuration. In this case the process exhibits global confluence despite not satisfying local confluence.

## 1. INTRODUCTION

We consider a discrete process for rerouting flow on the *edges* of a planar complex. The process is a form of discrete diffusion; a flow is repeatedly diverted according to a discrete Laplacian. It is also an instance of higher-dimensional chip-firing. In the flow-firing process considered here, flow is placed on the 1-dimensional cells of a complex and is rerouted across the 2-dimensional cells. This is compared to graphical chip-firing, where chips are placed on the vertices (0-dimensional cells) of a graph and redistributed across the edges (1-dimensional cells). Previous work on higher-dimensional chip-firing has considered algebraic structures defined for finite complexes. Here we consider the dynamics of higher-dimensional chip-firing and work with infinite complexes.

We focus on two important features of the flow-firing process – whether or not the system is terminating and whether or not the system is confluent. To this end, three settings are explored. We show that:

- For non-conservative initial configurations, the process does not terminate (Section 4).
- For conservative initial configurations, the process always terminates but does not have a unique terminating state. The final configuration depends on the choices made during the firing process (Section 5).
- For conservative initial configurations around a distinguished face (a topological hole), the process terminates in a unique state. The final configuration is always the same regardless of the choices made during the firing process (Section 6).

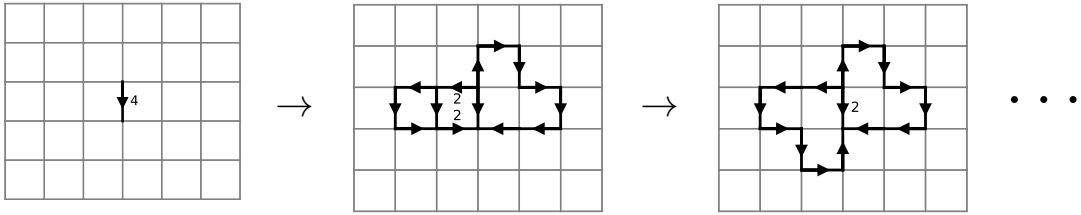
See Figure 1 for an illustration of the three different settings.

The third case is of particular interest. The uniqueness of the final configuration is an example of global confluence that does not follow from local confluence, thus adding to an active narrative in chip-firing, see Section 2.

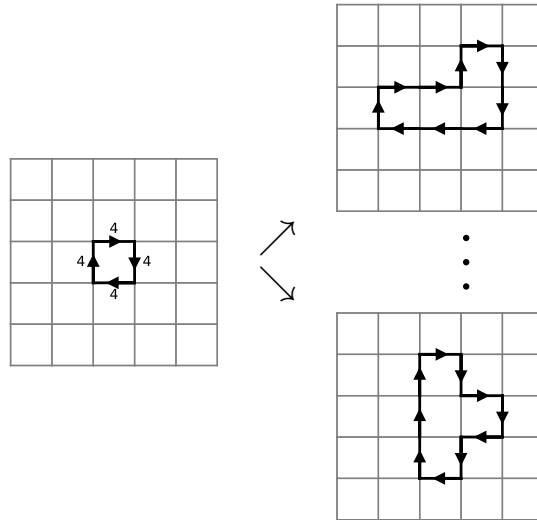
The special case of the 2-dimensional grid is treated throughout most of the paper for simplicity. Section 7 discusses extensions to more general cases including arbitrary planar graphs and higher dimensional polytopal decompositions.

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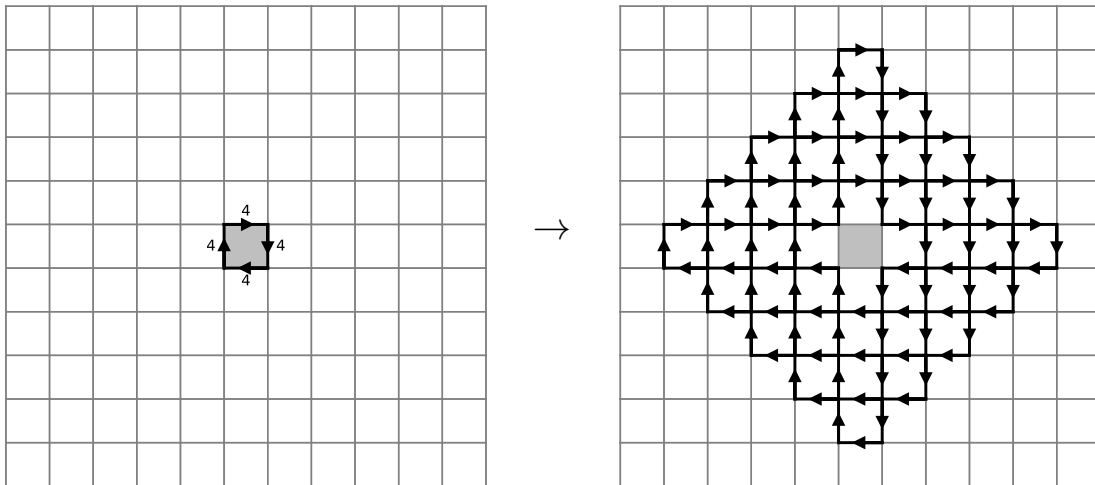
*Key words and phrases.* chip-firing, confluence, conservative flows.



(a) Non-conservative flow: non-terminating process.



(b) Circulation: terminating but non-unique final configuration.



(c) Circulation around a hole: terminating and unique final configuration

FIGURE 1. Flow-firing in three different settings.

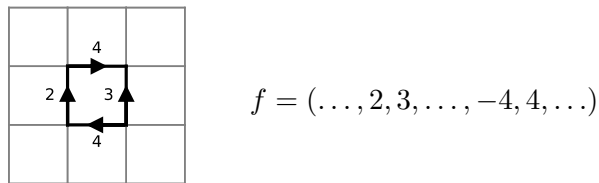


FIGURE 2. A flow configuration.

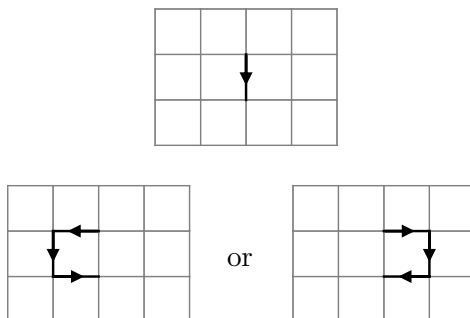


FIGURE 3. Rerouting a unit of flow. A unit of flow along an edge, as in the top, can reroute across a face to the left or to the right, resulting in one of the two configurations on the bottom.

## 2. FLOW-FIRING

Let  $G$  be the (infinite) grid graph embedded as  $\mathbb{Z}^2$ . For bookkeeping purposes, we orient each edge from South to North and West to East. The flow-firing process on  $G$  involves configurations of integral flow on the edges of the graph.

**Definition 1.** A flow configuration for  $G$  is an integer assignment  $f$  specifying an amount of flow on each edge. Negative values signify that the flow is oriented opposite that of the edge itself.

Figure 2 illustrates a flow configuration and the corresponding integer vector.<sup>1</sup>

Let  $e$  be an edge and  $\sigma$  a face (square) that contains  $e$ . Rerouting a unit of flow on  $e$  across  $\sigma$  replaces one unit of flow along  $e$  with one unit of flow along the alternate path formed by the other edges of  $\sigma$ , see Figure 3.

If an edge has two units of flow (in either direction) we can reroute one unit around each of the two faces containing  $e$ . We are now ready to define the flow-firing process.

### The flow-firing process

For the grid graph

At each step:

- Choose an edge  $e$  with 2 or more units of flow (in either direction).
- Fire  $e$  by rerouting 1 unit of flow around each of the two faces containing  $e$ .

<sup>1</sup>In our terminology a flow may or may not be conservative at each vertex. Other sources reserve the name “flow” for the more restricted case.

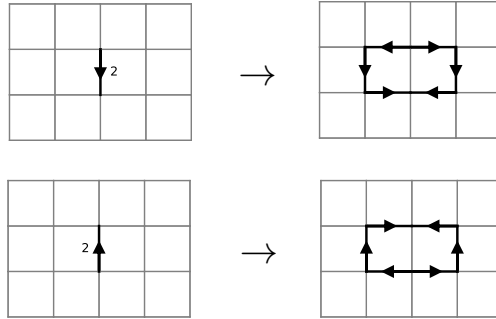


FIGURE 4. The flow-firing process. An edge can fire when it has at least two units of flow in either direction.

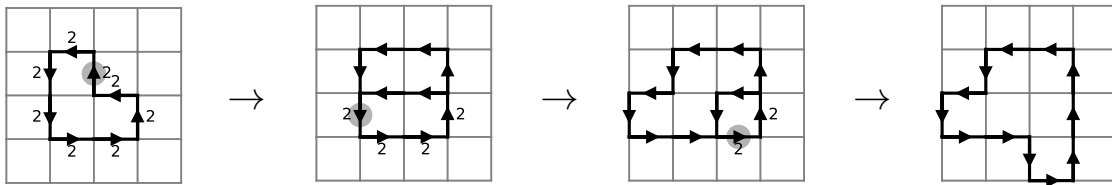


FIGURE 5. Example of the flow-firing process. In each step the highlighted edge *fires* and 2 units of flow are rerouted across two faces. The process terminates when no edge has 2 or more units of flow.

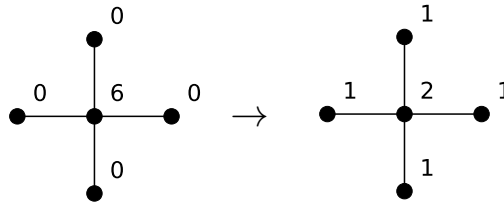


FIGURE 6. Graphical chip-firing.

Figure 4 shows the flow-firing process on an initial configuration consisting of 2 units of flow on a single *edge*. Figure 5 shows an example of the flow-firing process from a larger initial configuration.

The flow-firing process is a form of higher-dimensional chip-firing as introduced in [DKM13], see also [Kli18][Chapter 7]. In graphical chip-firing, a *chip configuration* is an integer assignment (of chips) to the *vertices* of a graph. The firing rule is that a vertex  $v$  can fire if the number of chips at  $v$  is at least  $\deg(v)$ . Firing  $v$  decreases the value at  $v$  by  $\deg(v)$  and increases the value at each neighbor of  $v$  by 1, see Figure 6.

Graphical chip-firing is 1-dimensional. Chips are on 0-dimensional vertices and move across 1-dimensional edges. Flow-firing is 2-dimensional. Flow is on 1-dimensional edges and moves across 2-dimensional faces. The degree of an edge  $e$  is the number of faces containing  $e$ . For the grid graph, the degree of each edge is 2, hence the need for 2 units of flow for an edge to fire. Two edges are neighbors if they are contained in a common face. When an edge  $e$  fires, flow is diverted from  $e$  to neighboring edges.

The result of firing a vertex in graphical chip-firing can be expressed in terms of the graph Laplacian,  $\Delta_1(G)$ . If  $c'$  is the configuration obtained from  $c$  after firing vertex  $i$ , then  $c' = c - \Delta_1(G)e_i$ . Similarly, flow-firing can be expressed in terms of a combinatorial Laplacian,  $\Delta_2(G)$ . The two-dimensional Laplacian of a complex reflects the degrees and incidence relations between faces of the complex. If  $f'$  is the configuration obtained from  $f$  after firing edge  $i$ , then  $f' = f \pm \Delta_2(G)e_i$ . The sign of the update depends on the orientation of the flow on edge  $i$  in  $f$ .

There are important differences between 1-dimensional and 2-dimensional chip-firing. Graphical chip-firing on the infinite grid always terminates if started from a chip configuration with finite support. This is not the case in flow-firing. In Section 5 we show that the flow-firing process always terminates if started from a *conservative* flow (a circulation) with finite support.

Another important difference is that in graphical chip-firing, the total number of chips is conserved. In the flow-firing process, the quantity

$$\text{inflow}(v) - \text{outflow}(v)$$

is conserved (at each vertex) instead.

Note that in the flow-firing process a rerouting operation can lead to cancellation when flow runs in opposite directions across an edge, see Figure 5.

The cancellation of flow, as seen in Figure 5, cannot happen in graphical chip-firing. When a vertex  $v$  fires, the number of chips at vertices other than  $v$  can only increase. This simple observation leads to an important property of graphical chip-firing known as local confluence. Local confluence refers to the fact that from a fixed configuration  $c$ , if two different states  $c_1$  and  $c_2$  can be reached after a single step, then there is a common state reachable from both  $c_1$  and  $c_2$  after a single step. A system that satisfies this property is also said to satisfy the diamond lemma. Local confluence in terminating systems implies *global confluence* [New42]. In graphical chip-firing, this fact tells us that, if the chip-firing process from an initial configuration terminates, then it terminates in a unique final configuration regardless of the choices made at each step.

Because of cancellation of flow, the flow-firing process does not satisfy the diamond lemma, see Figure 7. In fact, we show that in the flow-firing process there can be many terminating states from a single initial configuration, see Section 5.

In Section 6 we consider a modification of flow-firing which displays global confluence despite not satisfying local confluence. Global confluence without local confluence has recently been observed in other chip-firing contexts. *Labeled chip-firing*, see [HMP17], is an example. Labeled chips are fired along the path graph, larger to the right and smaller to the left. Depending on the parity of the initial number of chips, the labeled chip-firing process terminates in a unique final configuration (with the chips sorted) even though the process does not satisfy the diamond lemma.

The chip-firing processes on root systems introduced in [GHMP17a] and [GHMP17b] generalize labeled chip-firing. Again, the root systems processes do not satisfy the diamond lemma, nonetheless many cases do display global confluence.

In general, establishing global confluence without local confluence has proved difficult. For the flow-firing process we introduce a topological hole to the grid (remove a square) and show that starting from a conservative flow around the hole, the flow-firing process satisfies global confluence despite not satisfying the diamond lemma, see Theorem 9.

### 3. THE PULSE

In graphical chip-firing, an important example is the *pulse* on the infinite grid. The pulse configuration consists of  $n$  chips at the origin and 0 chips elsewhere in  $\mathbb{Z}^d$ . For  $d = 1$ , this is chip-firing from a single stack of chips on a line. The properties of chip-firing from a single stack on the line were studied in detail in [Spe86] and [ALS<sup>+</sup>89]. The final configuration is independent of

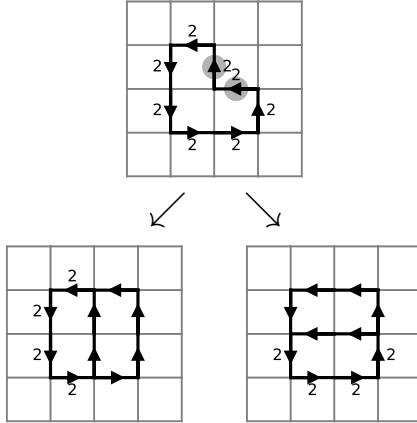


FIGURE 7. Flow-firing does not satisfy the diamond property. In the top configuration the two highlighted edges can fire but after one of the edges fires the other one can no longer fire.

the firing choices made throughout the process and consists of a single chip in each position of an interval centered at the origin; the origin itself has zero chips if the initial number of chips is even.

For  $d = 2$ , chip-firing from a stack of chips at the origin yields a well known fractal pattern associated with chip-firing; see, e.g., [Pao14], [Kli18][Chapter 5]. The final configuration is again independent of the firing choices made throughout. Yet this final configuration, resulting from the pulse on the graph of the 2-dimensional grid, has proved very difficult to understand. Much work has gone into studying its properties; see, e.g., [LBR02], [LP09], [PS13], [LPS17].

The current paper can be seen as a first step in understanding the basic properties of fundamental initial configurations (pulses of flow) on higher dimensional spaces.

#### 4. FLOW ON A SINGLE EDGE / NON-TERMINATING

Following the graphical chip-firing examples of the pulse, one might naturally consider an initial flow configuration consisting of a large amount of flow on a single edge. However, the flow-firing process from such an initial configuration does not terminate.

Figure 8 shows an example of such an initial state and the configurations resulting from the flow-firing process after several steps.

**Proposition 2.** *The flow-firing process on the grid does not terminate from any initial configuration which has a vertex  $v$  with  $|\text{inflow}(v) - \text{outflow}(v)| > 4$ .*

*Proof.* Suppose  $|\text{inflow}(v) - \text{outflow}(v)| > 4$ . Since  $\text{deg}(v) = 4$ , by the pigeonhole principle, some edge touching  $v$  must have at least 2 units of flow and can fire. Since  $\text{inflow}(v) - \text{outflow}(v)$  is conserved by rerouting there will always be an edge touching  $v$  that can fire.  $\square$

#### 5. CONSERVATIVE FLOWS / TERMINATING

**Definition 3.** *A flow configuration is conservative if for each vertex  $v$ ,  $\text{inflow}(v) - \text{outflow}(v) = 0$ .*

In this section we prove that the flow-firing process initiated at a conservative flow always terminates in a finite number of steps. First note that if the initial flow is conservative, it remains conservative throughout the process.

Importantly, conservative flows allow for a dual representation consisting of flow on faces. For this representation, associate an integer value to each face of the grid instead of each edge. A

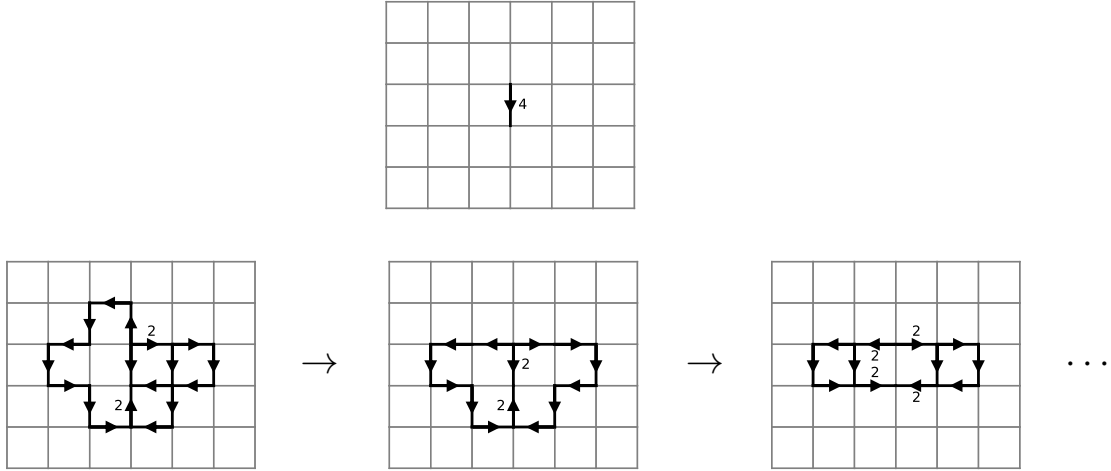


FIGURE 8. Some intermediate configurations reachable from a single edge flow. This flow-firing process never terminates.

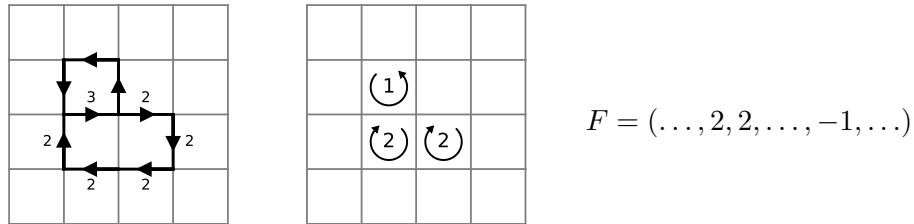


FIGURE 9. A flow configuration and the corresponding face representation.

positive value is interpreted as a local clockwise circulation. A negative value is interpreted as a local counter-clockwise circulation.

A flow configuration on the faces induces a flow configuration on the edges. For each edge  $e$ , the flow on  $e$  is the sum of the flows implied by the circulations around the two faces containing  $e$ . Furthermore, any *conservative* flow on the edges is induced from some face configuration. This follows from the fact that every conservative flow is a sum of cycles and the boundaries of the faces of a planar graph span the cycle space of the graph (see, e.g., [FF74, KV12]).

Figure 9 shows an example of a conservative flow and the corresponding face representation.

Note that the conservative condition is necessary. A configuration with flow on a single edge, as considered the previous section, cannot be represented by a configuration of face circulations.

In the flow-firing process an edge  $e$  can fire if it has at least two units of flow. In the face representation this means that the values on the two faces containing  $e$  differ by at least two. Using a face representation  $F$  the flow-firing process can be equivalently defined as follows.

**The flow-firing process (face representation)**

For the grid graph and conservative initial configuration

At each step:

- Choose two neighboring faces  $a$  and  $b$  with  $F_a \geq F_b + 2$ .
- Fire  $a$  and  $b$  by decreasing  $F_a$  by 1 and increasing  $F_b$  by 1,

$$F_a \rightarrow F_a - 1,$$

$$F_b \rightarrow F_b + 1.$$

The definition of the flow-firing process using the face representation is perhaps more natural in comparison to graphical chip-firing. One can picture stacks of “circulation chips” on the faces of  $G$ . A firing move sends a chip from one stack to a neighboring smaller stack. A significant difference from graphical chip-firing is that in flow-firing with the face representation, circulation chips move to a single neighbor not to all neighbors at the same time.

**Theorem 4.** *The flow-firing process on the grid starting from a finite conservative flow terminates after a finite number of steps.*

*Proof.* Let  $f$  be a finite conservative flow on the edges of  $G$ . Let  $F$  be the corresponding face representation. Define the potential function

$$\phi(F) = \sum F_\sigma^2,$$

which is an infinite sum over all faces of  $G$  with finite non-zero support.

Suppose that neighboring faces  $a$  and  $b$  fire and that  $F_a > F_b$ . Call the resulting configuration  $F'$ . We have,

$$F'_a = F_a - 1,$$

$$F'_b = F_b + 1,$$

and  $F'_c = F_c$  at all other faces  $c$ . The difference in potential is:

$$\begin{aligned} \phi(F) - \phi(F') &= [(F_a)^2 + (F_b)^2] - [(F_a - 1)^2 + (F_b + 1)^2] \\ &= (F_a)^2 + (F_b)^2 - (F_a)^2 - (F_b)^2 + 2F_a - 2F_b - 2 \\ &= 2F_a - 2F_b - 2 \\ &= 2(F_a - F_b - 1) \\ &\geq 2. \end{aligned}$$

The last inequality follows from the fact that  $F_a - F_b \geq 2$  or else faces  $a$  and  $b$  could not fire.

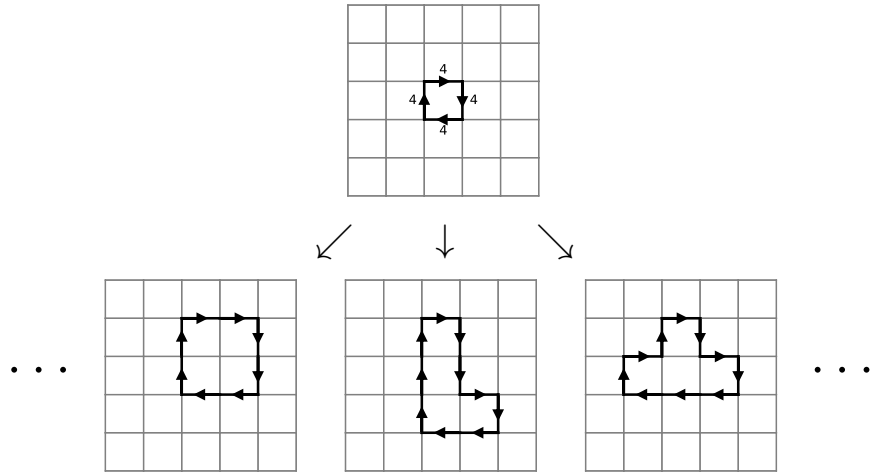
The potential function  $\phi$  is non-negative and strictly decreases with each flow-firing step. Therefore, starting from any configuration with finite potential, the process must terminate in a finite number of steps.  $\square$

Again, note that this argument does not apply to non-conservative flows, such as the configurations with flow on a single edge considered in the last section. Non-conservative flows do not afford a face representation and therefore the potential function  $\phi$  cannot be defined.

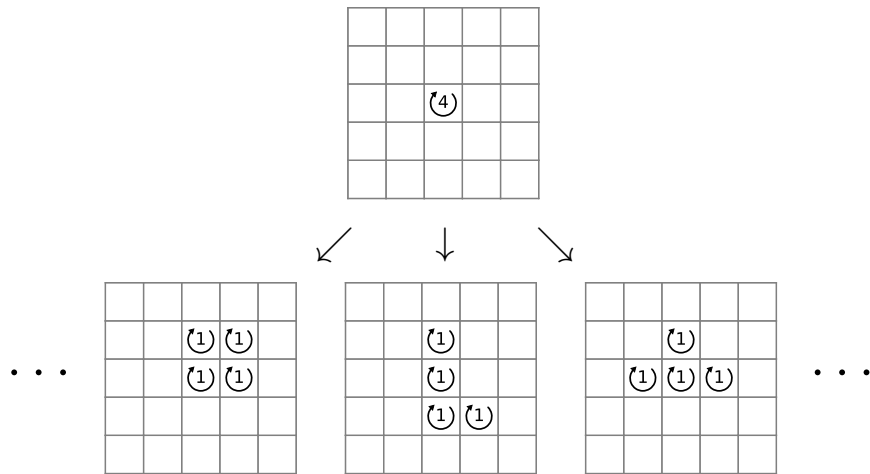
Within the class of conservative flows a possible analog to the pulse is a configuration with a large circulation around a single face. In terms of the face representation this corresponds to a large stack of positive circulation chips on a single face.

**Corollary 5.** *The flow-firing process starting from  $k$  units of flow around a single face terminates after a finite number of steps.*





(a) Edge representation.



(b) Face representation.

FIGURE 10. Starting with  $k = 4$  units of flow around a face always terminates but there are many possible final configurations.

Figure 10 illustrates the result of the flow-firing process starting from  $k = 4$  units of flow around a face. While the process always terminates, there are many possible final configurations.

## 6. CONSERVATIVE FLOWS AROUND A BOUNDARY / CONFLUENT

We next consider the grid graph with a distinguished face (square). In terms of flow rerouting, the distinguished face behaves like an obstruction or hole – flow on an edge incident to the distinguished face cannot divert across the distinguished face. In terms of the face representation (for conservative flows), the distinguished face behaves like a source or sink – flow on adjacent faces determines which behavior is seen. A value at the distinguished face can be thought of as a boundary condition for the flow-firing process.

Formally, let  $G$  be the grid graph again embedded as  $\mathbb{Z}^2$  and let  $\sigma$  be a fixed face of  $G$ . Define the following process.

**The flow-firing process (edge representation)**

For the grid graph with distinguished face  $\sigma$

At each step:

- Choose an edge  $e$ .
- If  $e \not\subset \sigma$  and  $e$  has 2 or more units of flow (in either direction) then 1 unit of flow is rerouted around each of the two faces containing  $e$ .
- If  $e \subset \sigma$  and  $e$  has 1 or more units of flow (in either direction) then 1 unit of flow is rerouted around the unique face not equal to  $\sigma$  containing  $e$ .

For conservative flows we have the following equivalent description of the process using the face representation introduced in Section 5.

**The flow-firing process (face representation)**

For the grid graph with distinguished face  $\sigma$  and conservative initial configuration

At each step:

- Choose two neighboring faces  $a$  and  $b$ .
- If  $a \neq \sigma, b \neq \sigma$  and  $F_a \geq F_b - 2$ , decrease  $F_a$  by 1 and increase  $F_b$  by 1,

$$F_a \rightarrow F_a - 1,$$

$$F_b \rightarrow F_b + 1.$$

- If  $b = \sigma$  and  $F_\sigma > F_a$ , increase  $F_a$  by 1,

$$F_a \rightarrow F_a + 1.$$

- If  $b = \sigma$  and  $F_\sigma < F_a$ , decrease  $F_a$  by 1,

$$F_a \rightarrow F_a - 1.$$

In the first case (when  $a$  and  $b$  are both not equal to  $\sigma$ ) we say a circulation chip moves from  $a$  to  $b$ . In the second case we say a circulation chip is created at  $a$ . In the third case we say a circulation chip is deleted from  $a$ .

Write  $(G, \sigma)$  for the grid graph with distinguished face  $\sigma$ .

**Proposition 6.** *Under the flow-firing process with the face representation for  $(G, \sigma)$ :*

- (1) *The maximum value over all faces does not increase.*
- (2) *The minimum value over all faces does not decrease.*
- (3) *The value at  $\sigma$  does not change.*

*Proof.* For (1), note that all moves that increase the value of a face involve a face of greater value, therefore the maximum value in a configuration cannot increase. Part (2) is analogous. Statement (3) is the observation that the flow-firing rules never alter the value of  $F_\sigma$ .  $\square$

From Proposition 6 part (2) we see that starting from a configuration of positively oriented face circulations we can only ever generate configurations of positively oriented face circulations.

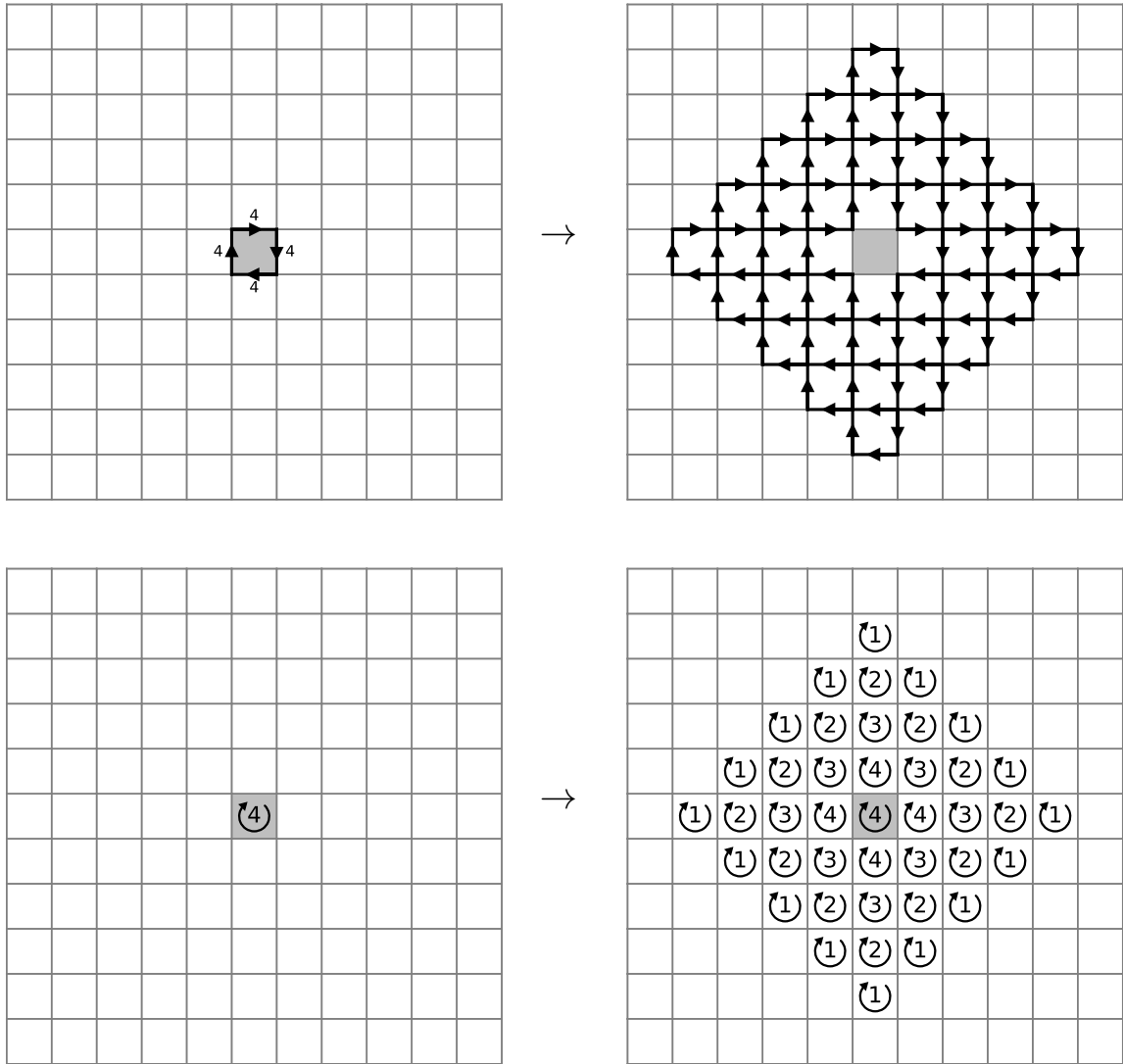


FIGURE 11. Flow-firing starting with a configuration of  $k = 4$  units of flow around a distinguished face (a hole). The top shows the edge representation and the bottom shows the face representation of the initial and final configurations.

For the remainder of the section, we consider the specific initial configuration consisting of  $k$  units of flow around  $\sigma$ . The face representation,  $K$ , for this configuration is,

$$K_\sigma = k \text{ and } K_\tau = 0 \text{ for all } \tau \neq \sigma.$$

Figure 11 shows the result of flow-firing starting from  $K$ . Surprisingly, as we show next, there is a unique final configuration in this case.

Define  $\text{dist}(\sigma, \tau)$  to be the distance from  $\sigma$  to  $\tau$  in the dual graph of  $G$ . For the grid graph this is the Manhattan distance.

**Lemma 7.** *Let  $K^*$  denote any configuration reachable from  $K$  via the flow-firing process for  $(G, \sigma)$ . Then for all faces  $\tau \neq \sigma$  of  $G$ ,*

$$K_\tau^* \leq \max\{0, k - \text{dist}(\sigma, \tau) + 1\}.$$

*Proof.* We proceed by induction on  $\text{dist}(\sigma, \tau)$ .

Base case: When  $\text{dist}(\sigma, \tau) = 1$  the result follows from the fact that the maximum value in  $K$  is  $k$  and the maximum value cannot increase.

Induction step: Suppose the claim holds for all faces with distance at most  $d - 1$  from  $\sigma$ . Let  $A = \{a \mid \text{dist}(\sigma, a) \geq d\}$  be the set of faces with distance at least  $d$  from  $\sigma$ . Initially,  $K_a = 0$  for all  $a \in A$ . Suppose  $K_a^* \not\leq \max\{0, k - d + 1\}$  for some  $a \in A$ . Consider, in particular, the first time that  $K_a^* > \max\{0, k - d + 1\}$  for some  $a \in A$ . The face  $a$  must have just received a circulation chip from a neighboring face  $b$  with  $K_b^* > \max\{0, k - d + 1\} + 1$  before the last step. Since this is the first time  $K_a^* > \max\{0, k - d + 1\}$  for  $a \in A$ , the face  $b$  cannot be in  $A$ . Since  $b$  is a neighbor of a face in  $A$  and not in  $A$ , it must be that  $\text{dist}(\sigma, b) = d - 1$ . But, by induction, the value at  $b$  must be at most  $\max\{0, k - d + 2\} \leq \max\{0, k - d + 1\} + 1$  which is a contradiction.  $\square$

The main result of this section, Theorem 9, shows that starting from the initial configuration  $K$ , the flow-firing process always terminates at the configuration achieving equality for all bounds in Lemma 7. First we need the following observations.

**Proposition 8.** *Let  $K^*$  denote any configuration reachable from  $K$ . Then*

- (1)  $K_\tau^* \geq 0$  for all  $\tau$ .
- (2) *The total number of circulations chips,  $\sum K_\tau^*$ , is bounded and non-decreasing over time.*

*Proof.* (1) This follows from Proposition 6 part (2) since all values are non-negative in the initial configuration.

(2) For any reachable configuration,  $K_\sigma^* = k$  and  $K_\tau^* \leq \max\{0, k - \text{dist}(\sigma, \tau) + 1\}$  for  $\tau \neq \sigma$ , thus the total sum is bounded. The sum is non-decreasing because no circulation chips are ever deleted. Neighbors of  $\sigma$  always have value at most  $k$  by Lemma 7, and  $\sigma$  always has value  $k$ . Therefore neighbors of  $\sigma$  never have value larger than  $\sigma$ .  $\square$

**Theorem 9.** *The flow-firing process on  $(G, \sigma)$  with initial configuration  $K$  terminates at a unique configuration  $K^\bullet$  after a finite number of steps. The final configuration has face representation*

$$K_\sigma^\bullet = k \text{ and } K_\tau^\bullet = \max\{0, k - \text{dist}(\sigma, \tau) + 1\} \text{ for all } \tau \neq \sigma.$$

*Proof.* First, we prove that the process stops. Let  $K^*$  be a configuration reachable from  $K$ . Define the potential function

$$\psi(K^*) = \sum_{\tau} (k - K_\tau^*)^2,$$

where the sum is over all faces with distance at most  $k+1$  from  $\sigma$ . Note that this function is bounded from below, i.e.  $\psi(K^*) \geq 0$ . Moreover,  $\psi(K^*)$  is finite for the initial configuration  $K^* = K$ . Each flow-firing step decreases  $\psi(K^*)$  by at least one:

For a step that creates a circulation chip at a face  $\tau$  neighboring  $\sigma$ ,  $K_\tau^*$  is always at most  $k$ . Therefore adding a circulation at  $\tau$  can only decrease  $(k - K_\tau^*)^2$ .

For a step that moves a circulation chip from  $\tau$  to  $\gamma$ : Let  $F$  be the configuration before the step and  $G$  be the configuration after the step. Then

$$\begin{aligned}
\psi(F) - \psi(G) &= [(k - F_\tau)^2 + (k - F_\gamma)^2] - [(k - F_\tau - 1)^2 + (k - F_\gamma + 1)^2] \\
&= [k^2 + F_\tau^2 - 2kF_\tau + k^2 + F_\gamma^2 - 2kF_\gamma] \\
&\quad - [k^2 + (F_\tau - 1)^2 - 2k(F_\tau - 1) + k^2 + (F_\gamma + 1)^2 - 2k(F_\gamma + 1)] \\
&= [F_\tau^2 - 2kF_\tau + F_\gamma^2 - 2kF_\gamma] \\
&\quad - [F_\tau^2 + 1 - 2F_\tau - 2kF_\tau + 2k + F_\gamma^2 + 1 + 2F_\gamma - 2kF_\gamma - 2k] \\
&= 2(F_\tau - F_\gamma) - 2 \\
&\geq 2,
\end{aligned}$$

where the final inequality follows from the fact that  $F_\tau - F_\gamma \geq 2$  for a circulation chip to move from  $\tau$  to  $\gamma$ .

Next,  $K_\sigma^\bullet = k$  since the value at  $\sigma$  never changes. To see that  $K_\tau^\bullet = \max\{0, k - \text{dist}(\sigma, \tau) + 1\}$  for  $\tau \neq \sigma$ , we argue by induction on  $\text{dist}(\sigma, \tau)$ .

Base case: When  $\text{dist}(\sigma, \tau) = 1$  we have that  $\tau$  is a neighbor of  $\sigma$ . Because of the allowable firing steps the process can only terminate if  $K_\tau^\bullet = K_\sigma^\bullet = k$ .

Induction step: Suppose  $\text{dist}(\sigma, \tau) = d > 1$ . Let  $\gamma$  be a neighbor of  $\tau$  with  $\text{dist}(\sigma, \gamma) = d - 1$ . By induction  $K_\gamma^\bullet = \max\{0, k - d + 2\}$ . Because of the allowable firing steps the process can only terminate if  $K_\tau^\bullet$  is in  $\{K_\gamma^\bullet - 1, K_\gamma^\bullet, K_\gamma^\bullet + 1\}$ . By Lemma 7 it must be that  $K_\tau^\bullet \leq \max\{0, k - d + 1\}$ . Considering the two possible values for  $K_\gamma^\bullet$  and the three possible values for  $K_\tau^\bullet$  directly gives that  $K_\tau^\bullet$  must equal  $\max\{0, k - d + 1\}$ . □

Figure 11 shows a pulse with  $k = 4$  units of flow and the resulting final configuration. In terms of the face representation, the final configuration is an ‘‘Aztec pyramid’’. The number of circulation chips at  $\sigma$  and neighbors of  $\sigma$  is  $k$ . The number of the circulation chips decreases linearly with the  $\ell_1$  distance from  $\sigma$  until reaching zero. In terms of the edge representation, the final configuration has exactly one unit of flow on every edge not in  $\sigma$  that is in a face within a  $\ell_1$ -ball of radius  $k$  centered at  $\sigma$ . The remaining edges have no flow.

## 7. EXTENSIONS

As mentioned in the introduction, we study the grid for simplicity but the results described here can be extended to more general settings.

**7.1. Planar graphs.** The results for the grid carry over essentially unchanged to any infinite planar graph.

Proposition 2 follows as: If there is a vertex  $v$  with

$$|\text{inflow}(v) - \text{outflow}(v)| > \text{deg}(v)$$

then the flow-firing process does not terminate.

Theorem 4 follows unchanged: If the initial configuration is a finite conservative flow then the flow-firing process terminates after a finite number of steps.

Theorem 9 also follows unchanged: If the initial configuration is a circulation around a topological obstruction  $\sigma$  then the flow-firing process terminates in a finite number of steps at a *unique* final configuration, see Figure 12.

The final configuration in Theorem 9 stated in terms of the face representation is the same for planar graphs. But in the edge representation of the final configuration for a general planar graph not every edge within some radius of the distinguished face will terminate with exactly one unit of

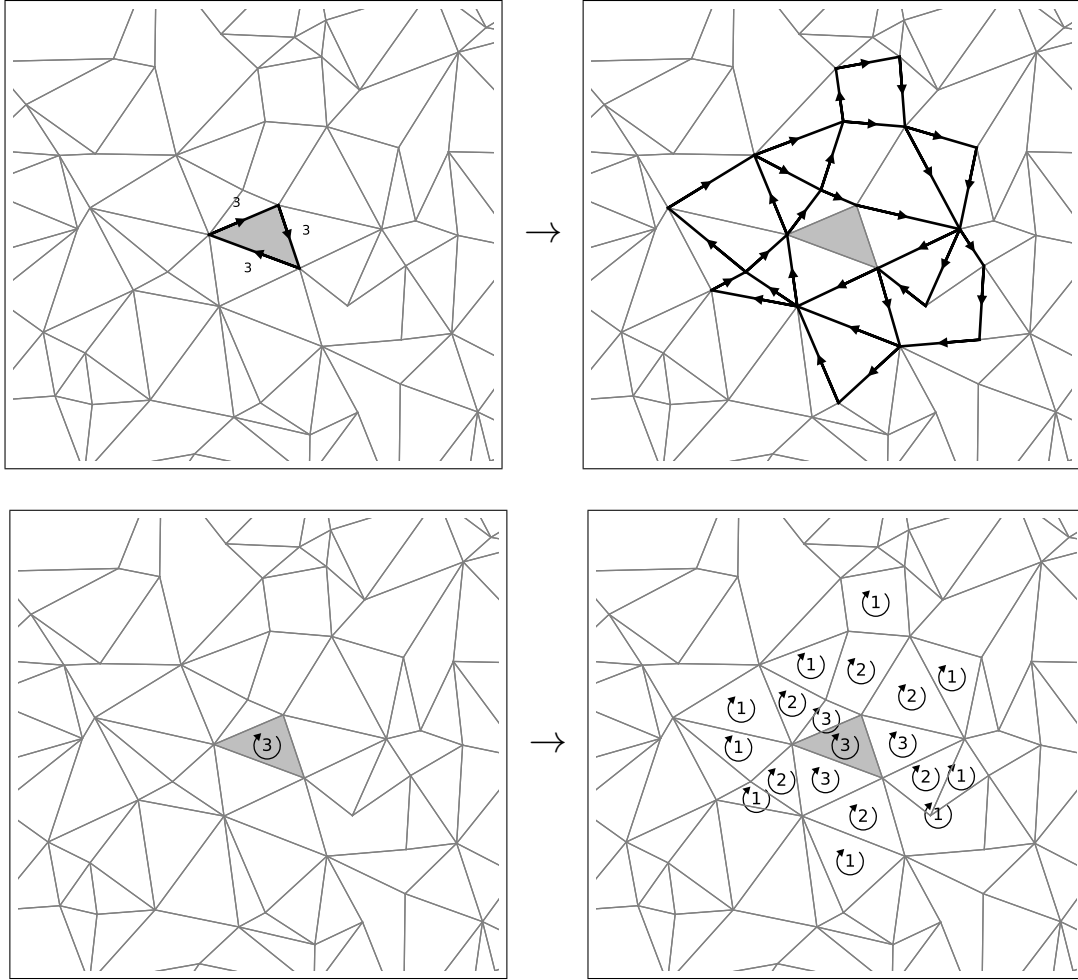


FIGURE 12. The “pulse” (of height 3) on a planar graph with a hole. The top shows the edge representation and the bottom shows the face representation of the initial and final configurations.

flow. If two neighboring faces have the same distance to  $\sigma$  then the edge between them will have zero flow in the final configuration (see Figure 12).

The results described above also hold for *finite* planar graphs. In this case the external face is included in the underlying complex.

**7.2. Higher dimensional complexes.** The flow-firing process on the grid (or a planar graph) is a form of two-dimensional chip-firing.

More generally, one can work over the  $n$ -dimensional grid (or a polytopal decomposition of  $n$ -dimensional space) and define a ridge-firing process. A ridge configuration is an integer assignment to the  $(n - 1)$ -dimensional faces of an  $n$ -dimensional complex. The ridge-firing process “reroutes” the value of a ridge to neighboring ridges along common facets.

Conservative configurations will, by definition, afford facet representations. The conservation requirement is a natural topological condition. To be conservative, the ridge configuration must be in the image of a boundary operator on facets. In particular, the boundary of a single facet (edges of a square, squares of a cube, etc.) is a conservative ridge configuration.

The same boundary operator is used to define the combinatorial Laplacian for the complex. The combinatorial Laplacian in turn dictates the rerouting rules of higher-dimensional chip-firing. For the  $n$ -dimensional grid (or polytopal decomposition), every ridge is contained in exactly two facets and the ridge-firing process in terms of the face representation is precisely the same as in the 2-dimensional (flow-firing) case. If two neighboring facets differ by 2 or more units of flow, they can fire to balance out.

Theorem 4 follows unchanged: If the initial state is a finite conservative ridge configuration then the ridge-firing process terminates after a finite number of steps.

Theorem 9 also follows unchanged: If the initial state is a conservative ridge configuration around a topological obstruction  $\sigma$  then the ridge-firing process terminates in a finite number of steps at a *unique* final configuration with the prescribed face representation.

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