The Rectangle of Influence Drawability Problem

G. Liotta, A. Lubiw, H. Meijer
and S. H. Whitesides

Department of Computer Science
Brown University
Providence, Rhode Island 02912

CS-96-22
June 1996

The Rectangle of Influence Drawability Problem *

G. Liotta[†] A. Lubiw [‡] H. Meijer [§] S. H. Whitesides[¶]

Abstract

A proximity drawing of a graph is a straight-line drawing where adjacent vertices are represented by points that are deemed to be close according to some proximity measure. A rectangle of influence drawing is a proximity drawing where the measure is based on the concept of rectangle of influence. Given two points u and v, the rectangle of influence of u and v is the axis-aligned rectangle having u and v at opposite corners. The rectangle of influence drawing of a graph G is a proximity drawing of G such that (i) for each pair of adjacent vertices u, v of G, the rectangle of influence of the points representing u and v is empty (i.e. contains no point representing a vertex distinct from u and v), and (ii) for each pair of non-adjacent vertices u, v of G, the rectangle of influence of the points representing u and v is not empty. Two different representation models are possible depending on whether the rectangle of influence is an open or a closed set.

In this paper we study the drawability of several classes of graphs with respect to both the closed and the open model. We characterize, for each class and model, which graphs have a rectangle of influence drawing. For each class we show that testing whether a graph G has a rectangle of influence drawing can be done in $O(n)$ time, where n is the number of vertices of G. Furthermore, if the test for G is affirmative, we show how to construct a rectangle of influence drawing of G. All the drawing algorithms can be implemented so that they (1) produce drawings with all vertices placed at intersection points of an integer grid of size $O(n^2)$, (2) perform arithmetic operations on integers only, and (3) run in $O(n)$ time, where n is the number of vertices of the input graph.

*Research supported in part by NSERC and FCAR, by the National Science Foundation under grant CCCR-9423847, by the U.S. Army Research Office under grant 34990-MA-MUR, and by the NATO-CNR Advanced Fellowships Programme.

†Center fo Geometric Computing, Department of Computer Science, Brown University, 115 Waterman Street, Providence, RI 02912 U.S.A. gl@cs.brown.edu

‡Department of Computer Science, University of Waterloo, Waterloo Ontario N2L 3G1 Canada. alubiw@math.uwaterloo.ca

§Computing and Information Science Department, Queen's University, Kingston, Ontario K7L 3N6 Canada. henk@qucis.queensu.ca

¶School of Computer Science, McGill University, Montréal, Québec H3A 2A7 Canada. sue@cs.mcgill.ca

1 Introduction

The problem of drawing a graph in the plane has been receiving intensive investigation due to its importance in a large number of application areas. These areas include VLSI layout, algorithm animation, visual languages and CASE tools. Vertices are usually represented as points in the plane and edges as simple Jordan curves connecting pairs of points. Typically, graph drawing tools and algorithms adopt a given drawing convention (for example edges are represented by straight-line segments or orthogonal chains) and attempt to solve several optimization problems (e.g. minimizing the area of the drawing, maximizing the number of convex faces, minimizing the number of bends along the edges, etc.). For an extensive overview of graph drawing problems and algorithms the reader is referred to the bibliography by Di Battista, Eades, Tamassia and Tollis [6].

1.1 Proximity Drawings

Given a pair u, v of points in the plane, a *proximity region* of u and v is a portion of the plane, determined by u and v, that contains points relatively close to both of them. Many definitions for the proximity region of u and v have been proposed in the literature. Widely used definitions include: (i) closed and open disks (also called *Gabriel regions* [23] and *modified Gabriel regions* [4], respectively) having u, v as antipodal points; (ii) closed and open lunes (also called *relatively closest regions* [20], and *relative neighborhood regions* [29], respectively) obtained as the intersection of the two disks with centers u and v and radius the distance $d(u, v)$; and (iii) closed and open axis-aligned rectangles (also called *closed rectangles of influence* and *open rectangles of influence* [16]) having u, v at opposite corners.

A *straight-line drawing* (for example, see [14, 15, 18]) of a graph G is a mapping of each vertex v of G to a distinct point of the plane and of each edge (u, v) of G to a straight-line segment connecting the points that represent u and v.

A *proximity drawing* of a graph G is a straight-line drawing of G such that (i) for each pair of adjacent vertices u, v of G, the proximity region of the points representing u and v is empty (i.e. contains no point representing a vertex distinct from u and v) and (ii) for each pair of non-adjacent vertices u, v of G, the proximity region of the points representing u and v is not empty. An *open proximity drawing* is a proximity drawing where all proximity regions are open. Analogously, a *closed proximity drawing* is a proximity drawing where all proximity regions are closed. An *open rectangle of influence drawing* is an open proximity drawing where the proximity region is the open rectangle of influence. A *closed rectangle of influence drawing* is a closed proximity drawing where the proximity region is the closed rectangle of influence.

Figure 1(a) shows an open rectangle of influence drawing. The open rectangle of influence of vertices g and h is represented by a dotted rectangle in the figure. It is easy to verify the emptiness of all the open rectangles of influence determined by endpoints of edges in the drawing. Observe also that for any pair of non-adjacent vertices, the corresponding open rectangle of influence is non-empty; for example, it is easy to see that the open rectangle of influence defined by vertices d and b contains c. Figure 1(b) shows the closed rectangle of influence drawing whose set of vertices is the same set of points as the one of Figure 1(a).

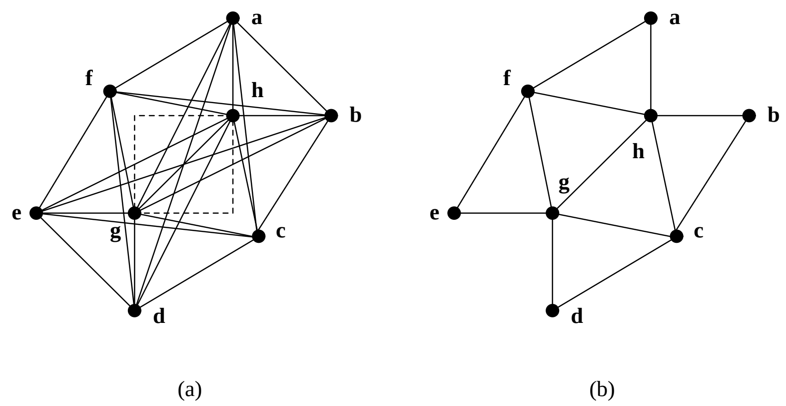


Figure 1: Examples of rectangle of influence drawings for (a) the closed model and (b) the open model.

Proximity drawings have been intensively studied in recent years because they arise in many areas as descriptors of the shape or *skeleton* of a set of points (for example, see [19], [27]). Examples of such areas include pattern recognition and classification, geographic variation analysis, geographic information systems, computational geometry, computational morphology, and computer vision. For a complete survey on the different types of proximity regions and their corresponding skeletons and on application areas, the reader is referred to the survey paper by Jaromczyk and Toussaint [17].

While techniques have been designed for the efficient computation of the skeleton of a given set of points, the problem of determining which graphs have proximity drawings has only just begun to be studied. The *proximity drawability testing problem* is to determine, for a given definition of proximity region, whether a graph admits a proximity drawing. Recent results in this new area of research include the following. In [22] it is proved that all biconnected outerplanar graphs have proximity drawings with the relative neighborhood proximity region and that all maximal outerplanar graphs have proximity drawings with the Gabriel proximity region. In [2] a complete characterization is given of those trees that admit a proximity drawing for each of the following types of proximity region: (i) relative neighborhood region, (ii) relatively closest region, (iii) modified Gabriel region, and (iv) Gabriel region. In [3], the authors studied the proximity drawability testing problem for trees for an infinite family of parametrized proximity regions, called *β-regions*, that include open and closed disks and lunes as special cases. The proximity drawability testing problem of trees in 3-dimensional space with 3-dimensional β-regions is studied in [21]. Results that are closely related to the proximity drawability testing problem concern the drawability of trees as minimum spanning trees [10], of triangulations as Delaunay triangulations [9, 8], and of graphs as nearest neighbor graphs [25, 11]. A survey on the proximity drawability testing problem is given by [7].

1.2 Overview of the Paper

In this paper, we study the drawability problem in both the open rectangle of influence model and in the closed rectangle of influence model. We say that a graph is *open rectangle of influence drawable* (or *open RID*) if it admits an open rectangle of influence drawing. Similarly, a graph is *closed rectangle of influence drawable* (or *closed RID*) if it admits a closed rectangle of influence drawing. For example Figure 2(a) shows a closed rectangle of influence drawing of a 4-cycle, which is not an open RID graph. Figure 2(b) is an open rectangle of influence drawing of K_5 (i.e., the complete graph on five vertices), which is not a closed RID graph.

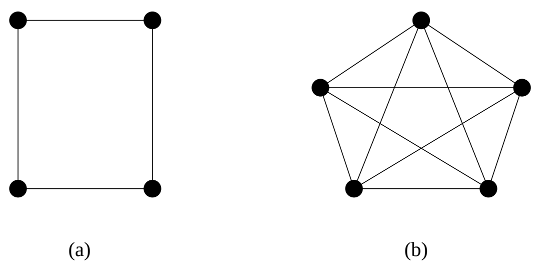


Figure 2: Examples of rectangle of influence drawings for (a) a 4-cycle and (b) K_5.

All the known algorithms for constructing proximity drawings of graphs assume the real RAM model of computation and a grid of exponential size. In the next theorem, we show that by contrast, rectangle of influence drawings do not require exponential area. Furthermore, the algorithms we will describe can be implemented so that all the numbers they deal with are integers only.

Theorem 1.1 *Any (open or closed) rectangle of influence drawable graph G with n vertices admits an (open or closed) rectangle of influence drawing on an integer grid of size $O(n^2)$.*

Proof: Let Γ be any (open or closed) rectangle of influence drawing of G. We show that Γ can always be transformed into a grid drawing of G that requires $O(n^2)$ area.

Let $x_0, x_1, \cdots, x_t$ denote the distinct x-coordinates of the points of Γ representing vertices of G in order from left to right, and let $y_0, y_1, \cdots, y_l$ denote their distinct y-coordinates sorted from bottom to top. Construct a grid drawing Γ' of G as follows. Transform each point (x_i, y_j) of Γ to point (i, j) on the grid. Since both the x- and the y-ordering of the points representing the vertices of G in Γ' is the same as the x-ordering and the y-ordering of the corresponding points in Γ, we can conclude the following.

1. If the (open or closed) rectangle of influence of points $p \equiv (x_i, y_j)$ and $q \equiv (x_k, y_h)$ in Γ is empty, then the (open or closed) rectangle of influence of $p' \equiv (i, j)$ and $q' \equiv (k, h)$ in Γ' is empty.

2. If the (open or closed) rectangle of influence of points $p \equiv (x_i, y_j)$ and $q \equiv (x_k, y_h)$ in Γ contains a point $a \equiv (x_r, y_s)$, then the (open or closed) rectangle of influence of $p' \equiv (i, j)$ and $q' \equiv (k, h)$ in Γ' contains a point $a' \equiv (r, s)$.

Thus, Γ' is an (open or closed) rectangle of influence drawing of G with all the vertices placed at integer grid points. Because of the choice of the points representing the vertices of G in Γ', the size of the minimum axis-aligned rectangle covering Γ' is $O(n^2)$. □

Our characterization results are as follows.

Open rectangle of influence drawable graphs .

1. Cycles and Wheels: Every wheel is open rectangle of influence drawable. No cycle consisting of more than three vertices is open rectangle of influence drawable (Subsection 4.1).
2. Trees: A trees is open rectangle of influence drawable if and only if it is a path; furthermore, this result can be generalized to all triangle-free graphs (Subsection 4.2).
3. Outerplanar graphs: A biconnected outerplanar graph is open rectangle of influence drawable if and only if it is maximal and its dual is a path (Subsection 4.3).
4. Cliques: A clique is open rectangle of influence drawable if and only if it has at most eight vertices (Subsection 4.4).

Closed rectangle of influence drawable graphs .

1. Cycles and Wheels: Every cycle and every wheel is closed rectangle of influence drawable (Subsection 5.1).
2. Trees: A tree is closed rectangle of influence drawable if and only if it has at most four leaves (Subsection 5.2).
3. Outerplanar graphs: Any outerplanar graph whose dual is a tree with at most three leaves is closed rectangle of influence drawable. No outerplanar graph whose dual is a tree with more than four leaves is closed rectangle of influence drawable. Some outerplanar graphs whose dual has four leaves are closed RID and some are not. (Subsection 5.3).
4. Cliques: A clique is closed rectangle of influence drawable if and only if it has at most four vertices (Subsection 5.4).

A consequence of the above characterizations is that, given one of the classes of graphs considered in this paper, the test to determine whether a graph G of the class is an open (closed) RID graph can be done in $O(n)$ time, where n is the number of vertices of G. Furthermore, if the test for G is affirmative, a proximity drawing of G can be constructed in $O(n)$ time. Since the analyses of the time complexity of the algorithms presented in the paper are entirely straightforward, we have omitted them.

Besides graph drawing applications, our research is motivated by questions about *rectangular visibility* between points (for example, see [24], [5]). Given a set of distinct points in the plane, two points of the set are said to be *rectangularly visible* if their rectangle of influence is empty. Much attention has been given to rectangular visibility over the past years because of its importance in several computational geometry problems (for example the enclosure problem of n points in the plane, the problem of finding the shortest Manhattan path among planar obstacles, and art gallery problems). Since the edges of a rectangle of influence drawing relate pairs of points that are rectangularly visible, the results of the present paper answer the question of recognizing, for various classes of graphs, which graphs of the class can describe rectangular visibility relations between points in the plane.

The paper is organized as follows. Basic definitions are in Section 2. Properties of both open and closed rectangle of influence drawable graphs are in Section 3. Classes of open rectangle of influence drawable graphs and closed rectangle of influence drawable graphs are studied in Sections 4 and 5, respectively. Conclusions and open problems are in Section 6.

2 Preliminaries

We start by reviewing some definitions on graphs. Then we introduce the geometric terminology that will be used in the paper. See also [1, 26]

2.1 Classes of Graphs

A *graph* $G = (V, E)$ consists of a finite non-empty set $V(G)$ of *vertices* and a set $E(G)$ of undirected *edges*. An edge $e \in E(G)$ joining vertices u and v is denoted by $e = (u, v)$; u and v are called the *endpoints* of e and are said to be *adjacent* vertices or *neighbors*. Edge e is also said to be *incident* with v. The *degree* of a vertex $v \in V(G)$, denoted by $deg_G(v)$, or just $deg(v)$, is the number of edges of $E(G)$ that have v as an endpoint. A graph is *complete* if all pairs of vertices are adjacent. A *subgraph* of G is a graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. An *induced subgraph* $H = (V', E')$ of G is the maximal subgraph of G with vertex set V'. A *clique* of a graph is a complete subgraph. A clique with n vertices is denoted by K_n.

A *path* in a graph G is a finite non-empty sequence $P = v_1 v_2 \dots v_k$ where the vertices $v_1, v_2, \dots, v_k$ are distinct and (v_i, v_{i+1}) is an edge for each $i = 1, \dots, k-1$. The vertices v_1 and v_k are called the *endpoints* of the path. A *cycle* is a path whose endpoints are adjacent. A graph consisting of a cycle on n vertices is denoted by C_n. A *wheel* consists of a cycle along with a *center* vertex adjacent to all the cycle vertices. A wheel with n vertices is denoted by W_n. An *acyclic* graph is a graph that contains no cycles. A graph is *connected* if, for each pair of vertices $u, v \in V$, there is a path from u to v. A graph is *biconnected* if, for each pair of vertices $u, v \in V$, there are two paths from u to v that have only their endpoints in common. A *tree* T is a connected acyclic graph. A vertex $v \in V(T)$ such that $deg_T(v) = 1$ is a *leaf* of T. A *branch* of a tree is a path formed by vertices of the tree that have degree at most 2 in T.

A drawing of a graph $G = (V, E)$ is called a *planar drawing* if no curve representing an edge intersects itself or any other curve except possibly at a common endpoint with that other curve. A planar drawing divides the plane into topologically connected regions called *faces*; each face is identified by the circular list of the vertices and the edges of its boundary. The unbounded region is referred to as the *external face*; the bounded regions are referred to simply as faces. A graph is said to be a *planar graph* if it admits a planar drawing. The *dual* graph of a planar graph G is a graph that has a vertex for each face (bounded) of G and an edge between two vertices u and v if and only if the faces corresponding to u and v share an edge in G.

An *outerplanar graph* is a planar graph whose vertices all belong to the external face. The edges on the external face are called *boundary edges*, and the remaining edges are called *chordal edges*. In what follows we restrict our attention to outerplanar graphs that are biconnected. Observe that the dual graph of a (biconnected) outerplanar graph is a tree. A *maximal outerplanar graph* is an outerplanar graph such that no edge can be added without losing outerplanarity. A *triangulation* is a planar graph where each face (except possibly the external face) is bounded by three edges. A *triangle-free graph* is a graph that does not have a complete subgraph K_3.

2.2 Geometric Notation

Given two distinct points u, v of the plane, we denote by $R(u, v)$ the *open rectangle of influence of* u, v (i.e. the axis-aligned open rectangle having u and v at opposite corners), and we denote by $R[u, v]$ the *closed rectangle of influence of* u, v (i.e. the axis-aligned closed rectangle having u and v at opposite corners).

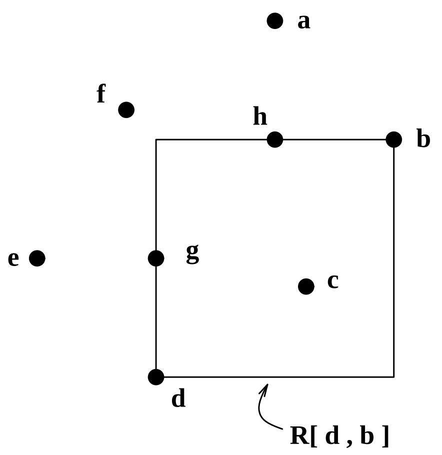
Note that if u and v determine either a horizontal or a vertical line, then $R(u, v)$ and $R[u, v]$ become degenerate rectangles. Since we aim at constructing *readable* [28] drawings of graphs, we want to disallow an edge to go through a vertex in the drawing. Consequently, we adopt the following convention to handle degenerate rectangles.

Convention 2.1 *If u, v, w are points on the same horizontal (vertical) line and if v lies between u and w, then we say that $R(u, w)$ and $R[u, w]$ contain v.*

Let P denote a set of distinct points in the plane. We denote by $RIG[P]$ the graph G that has vertices corresponding to the points of P, with an edge joining two distinct vertices if and only if their corresponding points u and v determine a closed rectangle of influence that is empty, i.e. $R[u, v] \cap (P - \{u, v\}) = \emptyset$. For example, Figure 3 gives a set P and the adjacency list of $RIG[P]$. In Figure 3(a), the rectangle of influence $R[d, b]$ is highlighted; since it contains points other than d and b, vertices d and b are not adjacent in $RIG[P]$. Notice that computing $RIG[P]$ on a given set of points is a different problem than computing a closed rectangle of influence drawing of a given graph. For a solution to the first problem see [24].

Clearly, $RIG[P]$ is a closed RID graph, and a closed rectangle of influence drawing of $RIG[P]$ can be obtained by connecting with straight-line segments the points of P that correspond to pairs of adjacent vertices in $RIG[P]$.

Similarly, $RIG(P)$ denotes the graph that has vertices corresponding to the points of P, with an edge joining two distinct vertices if and only if their corresponding points u and v determine an open rectangle of influence that is empty, i.e. $R(u, v) \cap P = \emptyset$. In case P contains three or more collinear points lying on a horizontal or a vertical line, Convention 2.1 applies: a pair of these points generates an edge in $RIG(P)$ if and only if the two points lie adjacent to each other on the line. As for the previous definition, $RIG(P)$ is an open



(a)

a: f, h

b: h, c

c: b, d, h, g

d: c, g

e: g, f

f: e, g, h, a

g: d, c, h, f, e

h: g, c, b, a, f

(b)

Figure 3: (a) A set of points P, and (b) the graph $RIG[P]$.

RID graph, and an open rectangle of influence drawing of $RIG(P)$ can be obtained by connecting with straight-line segments the points of P that correspond to pairs of adjacent vertices in $RIG(P)$.

To simplify the notation, we use $RIG[P]$ to denote both the graph G and its closed rectangle of influence drawing with vertex set P. Similarly for $RIG(P)$. Consequently, we use the terms *point* and *vertex* interchangeably, and we use the terms *straight-line segment* and *edge* interchangeably as well.

2.3 Minimum Spanning Trees and Gabriel Graphs

A *minimum spanning tree* of P, denoted $MST(P)$, is a connected straight-line drawing with vertex set P that minimizes the total edge length (clearly such a drawing is a tree). In general, a set P may have many minimum spanning trees (for example, P might consist of the vertices of a regular polygon). We adopt the same convention as in the previous section and denote by $MST(P)$ both the tree and a drawing of it with vertex set P.

The *Gabriel graph* of P, denoted $GG(P)$, is a straight-line drawing with vertex set P that has an edge between two distinct vertices $u, v \in P$ if and only if $d^2(u,v) < d^2(u,w) + d^2(v,w)$ for all $w \in P, w \neq u, v$. That is, u, v are adjacent if and only if the closed disk having u, v as antipodal points does not contain any other vertex except u, v. $GG(P)$ is connected and planar [23]. Again, $GG(P)$ denotes both the graph and a drawing of it with vertex set P.

3 Properties of Open and Closed RID Graphs

First we establish a relation between open and closed RID graphs, minimum spanning trees, and Gabriel graphs; next we show a geometric property of open and closed rectangle of influence drawings; finally, we present a theorem that makes it possible to define (open or closed) RID subgraphs of (open or closed) RID graphs.

Lemma 3.1 *Let P be a set of distinct points of the plane. Then*

$$MST(P) \subseteq GG(P) \subseteq RIG[P] \subseteq RIG(P).$$

Proof: The relation $MST(P) \subseteq GG(P) \subseteq RIG[P]$ has been proved in [16]. We prove here that $RIG[P] \subseteq RIG(P)$. For every pair of adjacent vertices $u, v \in RIG[P]$, we have that $R[u, v] \cap P = \{u, v\}$. Since $R(u, v) \subset R[u, v]$, $R(u, v)$ is empty. □

One consequence of the above lemma is that both open and closed rectangle of influence drawable graphs are connected. A second consequence is stated in the following corollary.

Corollary 3.1 *Let P be a set of points such that $RIG(P)$ is a tree. Then $MST(P) = GG(P) = RIG[P] = RIG(P)$, and the minimum spanning tree of P is unique. Similarly, if $RIG[P]$ is a tree, then $MST(P) = GG(P) = RIG[P]$ and the minimum spanning tree of P is unique.*

Lemma 3.2 *Let u, v be any two vertices in a (open or closed) rectangle of influence drawing, and let S be the set of vertices contained in the (open or closed) rectangle of influence of u and v. Then there is a path Π connecting u and v such that all the vertices of Π belong to S.*

Proof: The proof is by induction on the number of vertices in the rectangle of influence R of u and v. The lemma is clearly true if R is empty. Suppose the lemma holds when R contains k points. If R contains $k + 1$ points, let w be a point in R closest to u. Observe that (u, w) is an edge in the drawing. The rectangle of influence of w and v contains at most k points of S and, by the inductive hypothesis, there is a path connecting w and v using only vertices of S. This completes the proof. □

Corollary 3.2 *Let P be a set of points in the plane, let r be a horizontal (vertical) line, and let $P_1 \subseteq P$ be the subset of points lying in one of the two open half-planes defined by r. If x, y is any pair of points of P_1, then either $RIG(P)$ ($RIG[P]$) has the edge (x, y), or $RIG(P)$ ($RIG[P]$) contains a path Π from x to y such that all the vertices of Π belong to P_1.*

Theorem 3.1 *Let G be open (closed) RID. Suppose the set of vertices of G can be partitioned into three subsets A, B and K (A or B may be empty), such that K is a clique and there are no edges from A to B. Then the subgraphs induced by $A \cup K$, $B \cup K$ and K are* *open (closed) RID. Furthermore, an open (closed) rectangle of influence drawing of $A \cup K$, $B \cup K$ and K can be obtained from an open (closed) rectangle of influence drawing of G by deleting all the points representing vertices of G not in these subgraphs together with their incident line segments.*

Proof: Let Γ be an open (closed) rectangle of influence drawing of G. Without loss of generality assume that A is non-empty. Suppose the deletion of the points representing A in Γ creates a new edge (x, y) in the drawing. Then $R(x, y)$ ($R[x, y]$) contains only vertices from A ($A \cup \{x, y\}$). At least one of the vertices of A is in the rectangle $R(x, y)$ ($R[x, y]$) and x and y are not in A. By Lemma 3.2, there is a path from x to y containing only vertices from A. Hence by the definition of B, neither x nor y can represent a vertex of B. It follows that $x, y \in K$. Therefore (x, y) is not a new edge, because K is a clique. □

In other words, if K is a clique in an open or closed RID graph G, and if A is such that all edges with an endpoint in A have the other endpoint in $K \cup A$, then A can be removed from G and the resulting graph is still open or closed RID. This implies that the result of removing any vertex whose neighbors induce a clique is again RID. Also, if an open or closed RID graph G has several biconnected components, then each of these components is open or closed RID.

4 Classes of Open RID Graphs

4.1 Wheels

Theorem 4.1 *Every wheel is an open RID graph.*

Proof: Let W_n denote a wheel with n vertices. We present an algorithm to compute an open rectangle of influence drawing of W_n. Let v be the center of the wheel. Draw v at the origin. If n is 4, draw the three vertices of the external face at points $(-1, -1)$, $(1, -1)$, and $(0, 1)$ (see Figure 4(a)). For larger values of n, again place v at the origin. Then place three vertices at points $(n-2, 1)$, $(-1, 1)$ and $(-1, -n+2)$. Place the remaining $n-4$ vertices of W_n at the points with integer coordinates lying on the closed line segment with endpoints $(n-4, -1)$ and $(1, -n+4)$ (see Figure 4(b)). □

4.2 Trees

The class of trees that are open RID graphs coincides with the class of triangle-free open RID graphs. Notice that trees are in general a subclass of triangle-free graphs.

Theorem 4.2 *A triangle-free graph is open RID if and only if it is a simple path.*

Proof: We prove first that if a graph G is open RID and contains no 3-cycles, then G is a simple path. Let $RIG(P)$ denote an open rectangle of influence drawing of G. First,

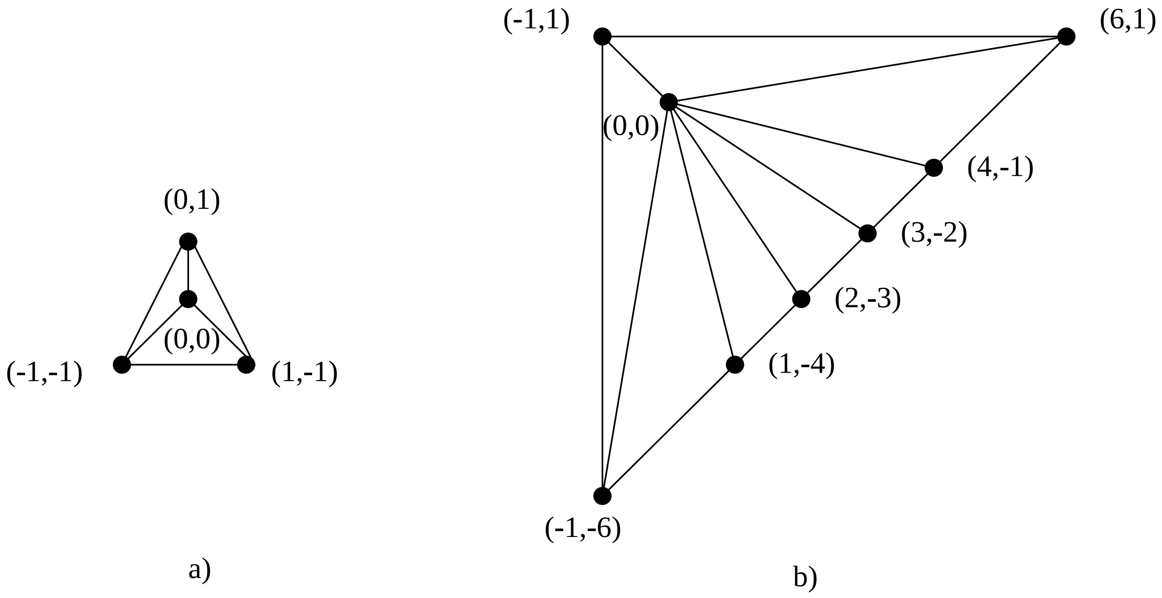


Figure 4: Illustration for Theorem 4.1 for a) $n = 4$ and b) $n = 8$.

suppose that there exists an edge (u, v) of $RIG(P)$ along some horizontal or vertical line L. Suppose P contains a point not on L. Let p be a point in P whose distance to L is minimum. Then $R(p, v)$ and $R(p, u)$ are empty and p, u, v is a 3-cycle in G, a contradiction. Hence all points of P lie on the line L, and G is a simple path.

Secondly, suppose that no horizontal or vertical line contains more than one point of P. Let (u, v) be an edge of $RIG(P)$, and consider where the other points of P can lie relative to the horizontal and vertical lines through u and v (see Figure 5). Without loss of generality, assume that both the x- and y-coordinates of v are strictly greater than the corresponding coordinates of u. Clearly $R(u, v)$ must be empty. Hence $R[u, v]$ is also empty. Let P' denote the subset of P consisting of points with x-coordinate smaller than that of v and y-coordinate larger than that of u. If P' is not empty, then any point $p' \in P'$ whose distance to $R(u, v)$ is minimum determines a 3-cycle with u and v in G, a contradiction. Hence P' is empty. Similarly, the set P'' of points with x-coordinate larger than that of u and y-coordinate smaller than that of v is empty.

Because P' and P'' are empty, any vertex w_i adjacent to u and distinct from v must be located at a point whose x- and y-coordinates are less than those of u; otherwise the rectangle $R(w_i, u)$ would contain v, which is impossible. Let W denote the set $\{w_i\}$ of such vertices. We claim that $|W| \leq 1$. If W is not empty, let w_1 denote the vertex in W with the largest x-coordinate. Then by the above argument with u and v replaced by w_1 and u, respectively, any remaining vertices $w_i \in W$, $w_i \neq w_1$, must have x- and y-coordinates smaller than those of w_1. Hence if W contains a second vertex w_2, the interior of $R(w_2, u)$ contains w_1, which is impossible. Hence $|W| \leq 1$. But u represents any vertex of degree at least 1 in G. It follows that G is a path.

The proof is completed by observing that any path is open RID. In fact, an open rectangle of influence drawing of a path Π can be obtained by representing the vertices of Π as collinear points. $\square$

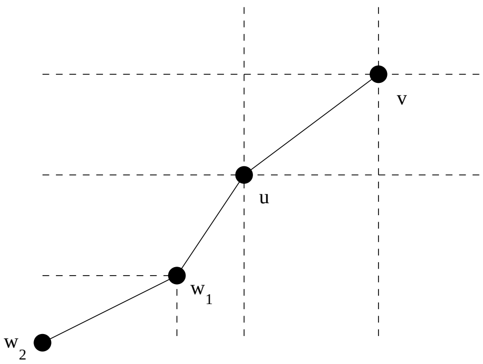


Figure 5: Illustration for Theorem 4.2.

Corollary 4.1 *A tree is open RID if and only if it is a simple path.*

Besides the characterization of open RID trees and open RID triangle-free graphs, Theorem 4.2 implies the following.

Corollary 4.2 *A bipartite graph is open RID if and only if it is a simple path.*

Corollary 4.3 *No cycle C_k such that $k \geq 4$ is open RID.*

Note that C_3 is open RID, as any equilateral triangle is an open rectangle of influence drawing of it.

4.3 Outerplanar Graphs

We first characterize maximal outerplanar open RID graphs. Then we show that biconnected non-maximal outerplanar graphs are not open RID.

Lemma 4.1 *A maximal outerplanar graph whose dual is a simple path is open RID.*

Proof: Let G be a graph whose dual is a simple path. If G has fewer than five vertices, a drawing can be found easily (for example if G has four vertices, three of them can be represented as horizontally collinear and the fourth one as vertically collinear with one of the first three), so assume that G has at least five vertices.

Observe that G has exactly two vertices of degree 2. Label these vertices $left$ and $right$. Divide the remaining vertices into two chains from $left$ to $right$. Let a_1 and a_2 be neighbors of $left$ and $right$, respectively, on one of the chains. We will call this chain the *top* chain. Let b_1 and b_2 be neighbors of $left$ and $right$, respectively, on the second chain, called the *bottom* chain. Note that one of a_2 or b_2 has degree 3, but not both. Without loss of generality assume that b_2 has degree 3.

An open rectangle of influence drawing of G can be constructed as follows. Make a list of the vertices of the top chain, ordered from a_1 to a_2. Make a list of vertices of the bottom chain, ordered from b_1 to b_2. Merge these two lists as follows. Begin by placing all the vertices of the top chain in the merged list. Then, for each vertex b from b_1 to (but not including) b_2, place b in the merged list such that it appears immediately before its last neighbor in the top chain. Finally, place b_2 at the end of the list. Draw vertex $left$ at the point (0,1). Draw the ith vertex of the merged list at x-coordinate i and on the line $y = x/c + 1$, if it is a vertex of the top chain and on line $y = -x/c$ otherwise, where c is a suitably defined positive constant. Draw vertex $right$ on line $y = x/c + 1$ to the right of b_2 (see Figure 4).

It can be verified that this algorithm draws the vertices in such a way that $R(u, v)$ is empty if and only if (u, v) is an edge of G. $\square$

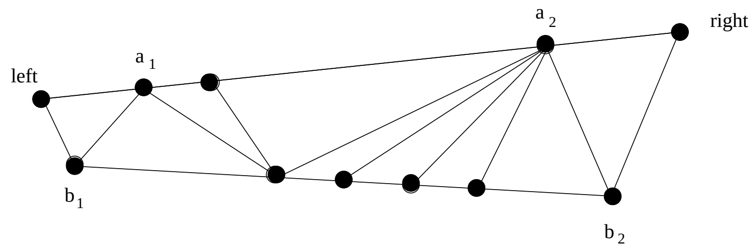


Figure 6: How to draw a maximal outerplanar open RID graph.

Lemma 4.2 *A maximal outerplanar graph whose dual consists of a vertex of degree* 3 *adjacent to three vertices of degree* 1 *is not open RID.*

Proof: Let G be such a graph. Number the vertices of G from 0 to 5 such that vertex 0 has degree 4 and the vertices 0, 1, 2, 3, 4 and 5 form a Hamiltonian cycle (see Figure 7). Thus vertices 0, 2 and 4 form a 3-cycle. Since not all three points representing vertices 0, 2 and 4 can lie on the same vertical or horizontal line in an open rectangle of influence drawing of G, either two such points lie on the same horizontal or vertical line, or none do. Without loss of generality we may assume that the points representing vertices 0, 2 and 4 have one of the placements in Figure 8. From Theorem 3.1 we deduce that if we have an open rectangle of influence drawing of G, we can remove any of the vertices 1, 3 and 5 and still have a valid drawing. Using this fact, we will show that all four placements are impossible by trying to add the remaining vertices one at the time.

In placements (a), (b), and (c), vertex 5 has to be placed below the line through 0 and 4, so it is not possible to place vertices 1 or 3 on the line through 0 and 4. Placement (a) is not possible since point 1 cannot be placed such that only triangle 012 is formed. In the second triangle (placement(b)), vertex 1 has to be on the vertical line through vertex 4. Then it is not possible to place vertex 3 such that only triangle 234 is formed. In placement(c), vertices 1 and 3 have to be placed in the open areas indicated in the figure, which is impossible without creating the edge $(1, 3)$. Figure 8(d) shows the last possible placement for the triangle 024 and the open areas that have to contain vertices 1, 3 and 5. Again, it can be seen that this is impossible without creating either the illegal edge $(1, 5)$ or the illegal edge $(3, 5)$. □

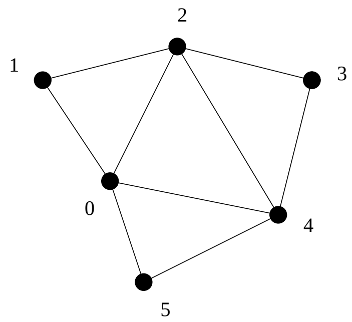


Figure 7: A non-open RID outerplanar graph.

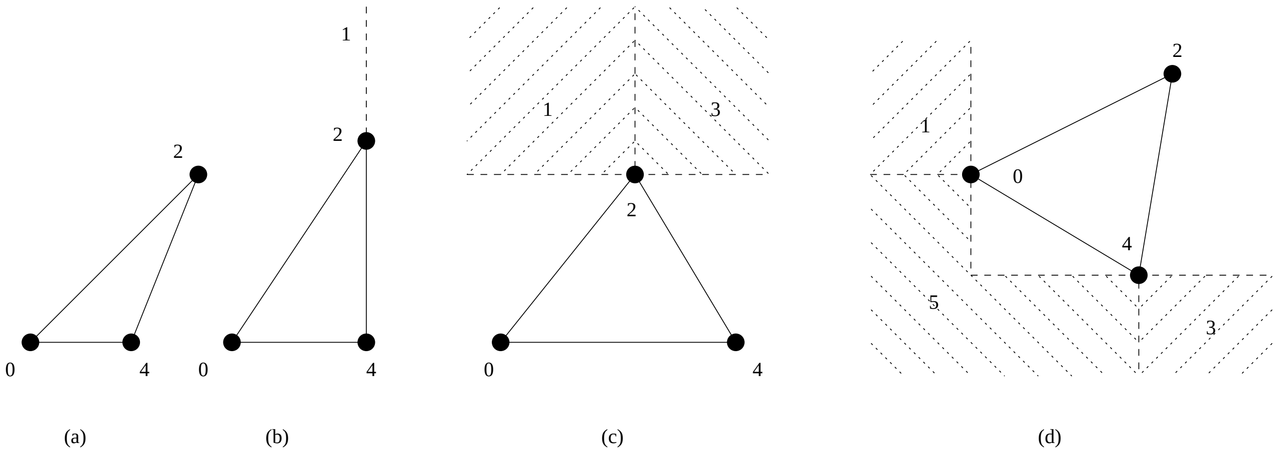


Figure 8: Illustration for Lemma 4.2.

The result of the previous lemma can be generalized as follows.

Lemma 4.3 *Any maximal outerplanar graph whose dual has a vertex of degree* 3 *is not open RID.*

Proof: Suppose G is a maximal outerplanar graph whose dual has a vertex of degree 3. Assume that G is open RID. Each face of G that corresponds to a leaf in the dual tree of G

is a triangle with one vertex of degree 2. By Theorem 3.1, this vertex may be deleted from G and the resulting graph remains open RID. Therefore, the smallest maximal outerplanar graph whose dual has a vertex of degree 3 (see Figure 7) is open RID. This contradicts Lemma 4.2. □

Lemma 4.4 *Any biconnected component of an outerplanar open RID graph is maximal outerplanar.*

Proof: To obtain a contradiction, let G be a biconnected component of an outerplanar RID graph having a face F with more than three vertices. We will remove vertices from G such that it remains open RID until only face F remains.

If G has more than one face, then there is an edge in G that F shares with another face. The two vertices of this edge form a clique cut-set K of size 2. This cut-set partitions the set of vertices of G into subsets A, B and K. Assume that the vertices of F are in $K \cup B$. By Theorem 3.1, the vertices in A can be deleted and the resulting graph remains open RID. If $K \cup B$ still has more than one face, we choose another edge of F and repeat the procedure until only F remains. Since F is a simple cycle with more than three vertices and only cycles of length 3 are open RID by Corollary 4.3, it follows that G has no interior faces with more than three vertices. Hence G is maximal outerplanar. □

The following theorem summarizes the results of this section.

Theorem 4.3 *A biconnected outerplanar graph is open RID if and only if it is maximal and its dual is a path.*

4.4 Cliques

In this subsection, we consider which cliques are open RID graphs or subgraphs of such graphs.

Let P be a set of points such that $RIG(P)$ is a clique and let R denote the smallest closed, axis-aligned rectangle that contains $RIG(P)$. We call this the *bounding rectangle* of P. We call a set S of points *non-aligned* if no two points of S are on the same horizontal or vertical line.

Lemma 4.5 *If a, b, c are non-aligned points of P ordered by x-coordinate, then they cannot also be in order (increasing or decreasing) by y-coordinate.*

Proof: The vertex b would be in $R(a, c)$. □

Lemma 4.6 *If P' is a non-aligned subset of P, then $|P'| \leq 4$.*

Proof: Suppose $|P'| = 5$. Order P' by increasing x-coordinate. We apply a result of Erdös and Szekeres [13] that in a sequence of more than jk distinct integers, there is either an ascending (not necessarily contiguous) subsequence of length $j + 1$ or a descending subsequence of length $k + 1$. In our case $j = k = 2$. Thus there are three points from this ordered list with increasing or with decreasing y-coordinates, which contradicts Lemma 4.5. □

Theorem 4.4 *A clique K_n is open RID if and only if $n \leq 8$.*

Proof: The sufficiency of the condition can be shown by construction. There are many ways to define a point set P such that $RIG(P)$ is K_8. For example, see Figure 9.

An open rectangle of influence drawing for each $n < 8$ can be obtained from the drawing in Figure 9 by removing the appropriate number of vertices, since this destroys no edges. This completes the proof that the condition suffices.

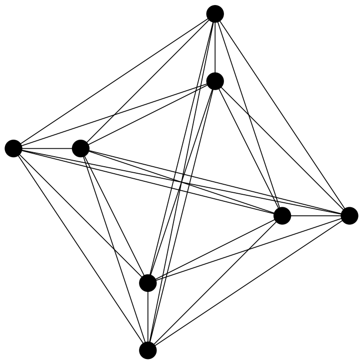


Figure 9: An open rectangle of influence drawing of K_8.

Now we establish the necessity of the condition. Let P' be a maximum size non-aligned subset of P. Lemma 4.6 implies that $|P'| \leq 4$. If $|P'| \leq 3$, then, since each point of P' can be aligned horizontally with at most one point of $P - P'$ and can be aligned vertically with at most one point of $P - P'$, P has at most 9 points. The only way to get $|P| = 9$ would be to have $|P'| = 3$, and for each point of P', to have two other points aligned with it and not aligned with any other point of P'. Suppose that $x \in P'$ and that y and z are aligned with x. Then $P' - \{x\} \cup \{y, z\}$ is a larger non-aligned set, a contradiction.

Now consider the case $|P'| = 4$. Let a, b, c, d be the points of P' ordered from top to bottom. Assume without loss of generality that b is left of a (otherwise flip the points about a vertical line through a). We cannot have c left of b; otherwise c, b, a would be in order by x- and y-coordinates, contrary to Lemma 4.5. Having c between b and a in the x-ordering does not leave room for d: d left of c would make d, c, a violate Lemma 4.5; and d right of c would make d, c, b violate Lemma 4.5. Thus we must have c to the right of a.

Now d can be neither left of b nor right of c by the same argument as above. We are left with two cases: (1) d between b and a in the x-ordering and (2) d between a and c in the

x-ordering. Any remaining points must be aligned vertically or horizontally with at least one of a, b, c, d. We will show that there are at most four such remaining points.

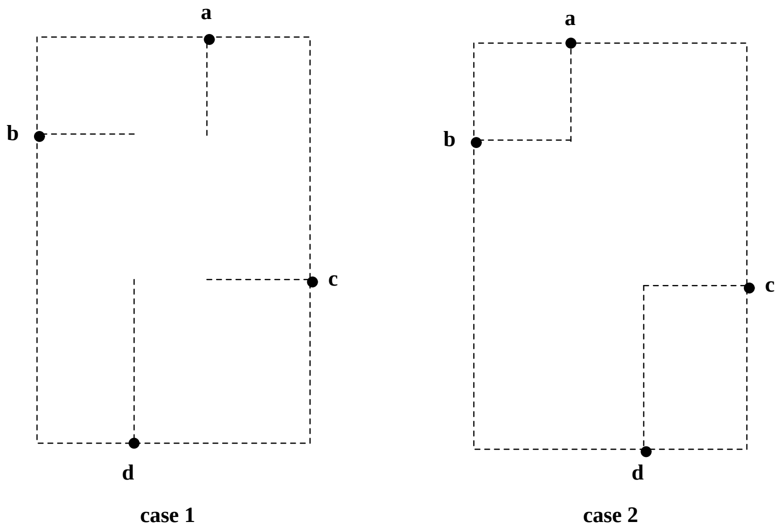


Figure 10: Figure for Theorem 4.4.

Suppose there is a point x higher than a. We cannot have x left of a; otherwise c, a, x would violate Lemma 4.5. Nor can x be right of a; otherwise b, a, x would violate Lemma 4.5. If x is vertically aligned with a, we replace a by x in P'. We can make a similar argument for points left of b, points right of c, and points below d.

Thus we may assume no points of P are outside the closed rectangle R that is the smallest rectangle containing P'. Points of P may lie on the boundary of R, or on a few closed line segments inside R: the vertical line segment from a down to the y-coordinate of b, the horizontal line segment from b on the left to the minimum of the x-coordinates of a and d; the horizontal line segment from c on the right to the maximum of the x-coordinates of a and d, and the vertical line segment from d up to the y-coordinate of c. Other possibilities are ruled out because the open rectangles determined by each pair of points of P' must be empty. See Figure 10.

Any point of $P - P'$ must be aligned with at least one of a, b, c, d. It is possible that a point of $P - P'$ is aligned with more than one of a, b, c, d. We will assign points of $P - P'$ to a unique aligned point of P' to simplify the ensuing case analysis. A point vertically aligned with a and horizontally aligned with d should be assigned to d. In case 1, the point vertically aligned with a and horizontally aligned with c should be assigned to c. Do a symmetric assignment for the other points. Any remaining points that are aligned with two of a, b, c, d may be assigned arbitrarily. Note that the set of possible locations for the points assigned to a is a connected set in the form of a "T"; similarly for the possible locations for points assigned to b, c, or d, respectively.

Now the only possible way to get $|P| > 8$ is to have two points of P assigned to some point of P'.

Consider case 1 first. The dashed line in Figure 10 shows the possible locations for points in $P - P'$. Having a point assigned to a and vertically aligned with a precludes having a point horizontally aligned with a to the right or to the left of a because of d and c respectively. A similar argument applies to the other points. Thus we can have at most one point assigned to each point of P', giving a total of at most eight points.

In case 2 it is possible to have a point u assigned to a and vertically aligned with a, and to have a point v horizontally aligned with a to the right of a, but only by placing u horizontally aligned with b. In this case we cannot have a point vertically aligned with b above or below b, because of u, c or u, v, respectively. Thus the two points a and b can have in total only two assigned points, and since a similar argument can be applied to the other points, P has at most eight points. □

The next theorem uses the results of this subsection to determine which cliques can be proper subgraphs of open RID graphs.

Theorem 4.5 *K_n may appear as a proper subgraph of an open RID graph if and only if $n \leq 8$.*

Proof: To establish the necessity of the condition, note that any clique K_n that is a proper subgraph of an open RID graph G must itself be open RID. This is because a drawing for K_n can be obtained from a drawing for G by removing all points not representing vertices of K_n, by Theorem 3.1. Hence by Theorem 4.4, $n \leq 8$.

To see that K_n can appear as a proper subgraph of a larger, open RID graph whenever $n \leq 8$, take an open rectangle of influence drawing $RIG(P)$ of K_n and add to P a point z lying outside the bounding rectangle of $RIG(P)$. We have $RIG(P) \subset RIG(P \cup \{z\})$. □

5 Classes of closed RID Graphs

5.1 Wheels and Cycles

Theorem 5.1 *Every wheel is a closed RID graph.*

Proof: Every wheel except $W_4 = K_4$ can be drawn by following the algorithm in the proof of Theorem 4.1. To construct a closed rectangle of influence drawing of K_4, one can map the four vertices of K_4 to points $(-1, 0)$, $(1, 0)$, $(0, -1)$, and $(0, 1)$. □

Notice that it is not possible to construct a planar closed rectangle of influence drawing of K_4, since this would require having a vertex inside a triangle and outside the rectangle of influence of any of the edges in the triangle, which is impossible.

Theorem 5.2 *Every cycle is a closed RID graph.*

Proof: C_3 is a closed RID graph, as the vertices of C_3 can be represented by the vertices of a triangle with all angles acute. Consider any other C_k, where $k \geq 4$. A closed rectangle of influence drawing of C_k can be constructed by placing four of its vertices at the corners of an axis-aligned square and by placing the remaining $k-4$ vertices along one of the four edges of the square. □

5.2 Trees

The following lemma is a consequence of Corollary 3.1 and Lemma 5.2 of [2], which proves that whenever the Gabriel graph $GG(P)$ of a set of points P is a tree, then the angle between any two incident edges is greater than or equal to $\pi/2$.

Lemma 5.1 *Let P be a set of points such that $RIG[P]$ is a tree. Then the angle between any two consecutive edges is greater than or equal to $\pi/2$.*

Corollary 5.1 *A closed RID tree has vertices with degree at most 4.*

Lemma 5.1 together with Corollary 3.2 have important implications for the shape of the closed rectangle of influence drawing of a tree, as the following lemma states.

Lemma 5.2 *Assume $RIG[P]$ is a tree and let x be a vertex of $RIG[P]$.*

1. *If $deg(x) = 3$, at least two of the edges incident with x are axis-aligned.*
2. *If $deg(x) = 4$ all edges incident with x are axis-aligned.*

Proof: One cannot have two edges from x in the same horizontal or vertical half-plane through x, as by Corollary 3.2 there would be a path joining the other end points of the edges but not containing x. □

Theorem 5.3 *A tree is a closed RID graph if and only if it has at most four leaves.*

Proof: Consider a closed rectangle of influence drawing $RIG[P]$ of a tree T. Let v be a leaf of $RIG[P]$ and let u be the neighbor of v. Let H be a closed orthogonal (vertical or horizontal) half-plane that bisects (v, u) and that contains v but not u. Let w be another vertex in H. By Lemma 3.2 there is a path from v to w not through u. This is impossible. Therefore the only point of P in H is v. Hence for each leaf v of T, there exists a closed orthogonal half-plane containing only v. Since there can be at most four such half-planes, it follows that T has at most four leaves.

To complete the proof we have to show that if a tree T has at most four leaves, then it is a closed RID graph. Observe that only one of the two following cases can occur:

1. T has one vertex of degree 4 and all the other vertices of T have degree at most 2.
2. T has no vertex of degree 4 and at most two vertices of T have degree at most 3. All the other vertices of T have degree at most 2.

When case 1 occurs, the construction of a closed rectangle of influence drawing of T can be done by visiting T and placing each vertex at a grid point and its neighbors at distinct grid points at distance 1, so that two consecutive edges form an angle $\pi/2$ only if they are incident with a vertex of degree greater than 2. If T has two vertices u and v of degree 3, the same approach as for case 1 can be followed, with the additional constraint that no adjacent vertices of u and v are represented in the same horizontal or vertical open half-planes defined by the line through the points representing u and v. □

5.3 Outerplanar Graphs

This subsection studies the closed rectangle of influence drawability of biconnected outerplanar graphs in terms of the duals of these graphs. As previously mentioned, the dual of a biconnected outerplanar graph G is always a tree. We prove that if this tree has at most three leaves, then G has a closed rectangle of influence drawing. Then we prove that if the dual of G is a tree that has more than four leaves, then G cannot be closed RID. We conclude by showing that if the dual of G is a tree that has exactly four leaves, then G may or may not be closed RID. In particular, we exhibit two outerplanar graphs having four leaves in their dual trees; one of such graphs is closed RID , while the other is not closed RID. We leave as open problems the closed RID characterization, recognition and construction problems for bisonnected outerplanar graphs with four leaves in the dual.

First we give some notation for our constructive proof that every biconnected outerplanar graph with at most three leaves in its dual is closed RID.

Let s be a straight-line segment with endpoints $u = (u_x, u_y)$ and $v = (v_x, v_y)$, where $u_y > v_y$ and $u_x \leq v_x$. Hence s is either vertical or has negative slope. Consider the open, horizontal strip bounded above and below by the lines $y = u_y$ and $y = v_y$, respectively. Rectangle $R[u, v]$ splits up this strip into pieces. Let $S_{west}(s)$ and $S_{east}(s)$ denote the half-infinite open strips to the left and right, respectively, of $R[u, v]$. See Figure 11. For a segment s that either is horizontal or has positive slope, we define strips $S_{north}(s)$ and $S_{south}(s)$ in a similar manner.

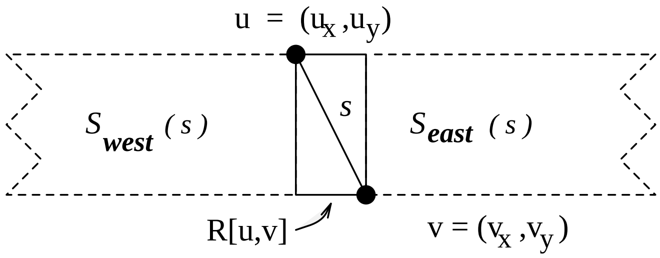


Figure 11: Strips for segment s.

Lemma 5.3 *Let G be an outerplanar graph whose dual is a path, and let e be a boundary edge in a face corresponding to a leaf of the dual. Let s be a line segment that either is* *vertical or has negative slope. Then graph G has a closed rectangle of influence drawing such that e is represented by segment s and the remaining vertices are contained in $S_{west}(s)$.*

Proof: The proof is constructive. To begin, place the endpoints of e at the endpoints of s. If G has only one face (i.e., one bounded face), place the vertices that are not the endpoints of e on any vertical line segment in $S_{west}(s)$ to obtain the desired drawing.

Assume now that G has at least two faces. We give a 3-step algorithm that constructs the desired closed rectangle of influence drawing of G. The first step is to construct a straight-line drawing of G that is not, in general, a closed rectangle of influence drawing. Its purpose is to facilitate the assignment in step 2 of labels to the chordal edges of G. Step 3 uses these labels to construct the desired drawing of G.

Assume that G has t chordal edges and hence $t+1$ faces. Let F_0 and F_t be the faces corresponding to the leaves in the dual, which by assumption is a path. Suppose that e belongs to face F_t. Denote the remaining faces by F_i, $0 < i < t$, such that face F_i has neighbours F_{i-1} and F_{i+1}. Let c_i denote the chordal edge shared by faces F_{i-1} and F_i.

Step 1 constructs a planar, straight-line drawing of G by placing one endpoint of each chordal edge c_i on the line $y=0$ and the other endpoint on the line $y=1$ as shown in Figure 12. The vertices of F_0 that are not endpoints of chordal edge c_1 are drawn with y coordinates between 0 and 1 on a vertical line segment on the extreme left of the drawing. Similarly, the vertices of F_t that are not endpoints of c_t are drawn with y coordinates between 0 and 1 on a vertical line on the extreme right. For $1 \le i \le t$, F_{i-1} is drawn to the left of F_i.

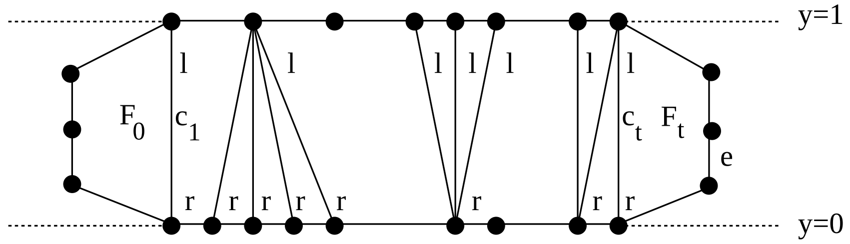


Figure 12: Outerplanar graph G.

Step 2 of the algorithm labels the chordal edges of G as follows. For each vertex of G on the line $y=1$ that is incident with one or more chordal edges, it labels the rightmost of these chordal edges with the letter l. For each vertex of G on the line $y=0$ incident with one or more chordal edges, it labels the rightmost of these chordal edges with the letter r.

Notice that this process assigns to each chordal edge at least one label and that c_t is assigned both labels l and r.

If c_t and e share a vertex on the line $y=1$, the label l is dropped from c_t; hence the label of c_t becomes simply r. If c_t and e share a vertex on the line $y=0$, the label r is dropped from c_t, whose label becomes simply l. If c_t and e do not share a vertex, c_t retains both labels l and r.

Step 3 constructs the desired closed rectangle of influence drawing as follows. All chordal edges with label l are drawn with negative slope, all chordal edges with label r are drawn with positive slope, and the remaining chordal edges are drawn vertically. How face F_0 is drawn depends on how its chordal edge c_1 is labeled. Figure 13 shows how to draw F_0. If c_1 is labeled with one of l and r, and if F_0 has three or more vertices, then the face is drawn as in Figure 13(a) or (b), respectively. If c_1 is labeled with both l and r, and if F_0 consists of three vertices, then the face is drawn as in Figure 13(c); otherwise, it is drawn as in Figure 13(d).

All faces F_i for $0 < i < t$ have two chordal edges c_i and c_{i+1}. Figure 14 shows how the these faces are drawn when c_i and c_{i+1} are labeled, respectively, with l and l, with l and r, with l and lr, with lr and lr, and with lr and r. The remaining possibilities for the labels of these chordal edges are r and r, r and l, r and rl, and lr and l. These cases are handled by drawing the faces in a manner analogous to the first, second, third and fifth drawings in Figure 14, respectively.

Face F_t has only one chordal edge, c_t. Figure 15 shows how to draw face F_t for each of the six possible situations that can arise: (a) c_t and e share a vertex on the line $y = 1$ and s has a negative slope, (b) c_t and e share a vertex on the line $y = 1$ and s is vertical, (c) c_t and e share a vertex on the line $y = 0$ and s has a negative slope, (d) c_t and e share a vertex on the line $y = 0$ and s is vertical, (e) c_t and e do not share a vertex and s has a negative slope, and (f) c_t and e do not share a vertex and s is vertical.

It is easy to verify that the resulting drawing is a closed rectangle of influence drawing of G such that all vertices except for the endpoints of e lie in the open strip $S_{west}(s)$. $\square$

Figure 16 shows the closed rectangle of influence drawing of the graph of Figure 12 that would be produced by applying the algorithm of Lemma 5.3 with segment s vertical.

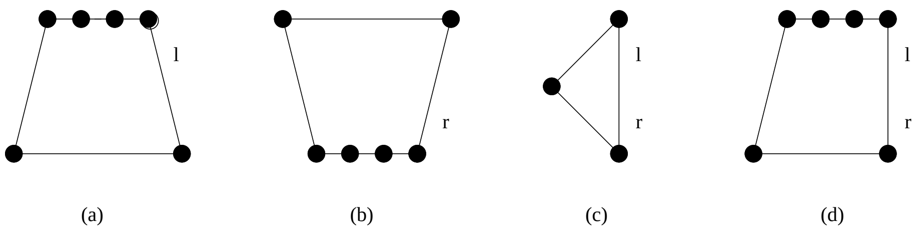


Figure 13: Drawings for face F_0.

Lemma 5.4 *Any biconnected outerplanar graph whose dual has at most three leaves is closed RID.*

Proof: Let G be an outerplanar graph with n vertices whose dual is a tree with at most three leaves. If the dual tree of G has less than three leaves, select an arbitrary boundary edge from a face corresponding to a leaf of the tree and draw it as a vertical segment s. Now place the remaining vertices of G by using the algorithm in the proof of Lemma 5.3.

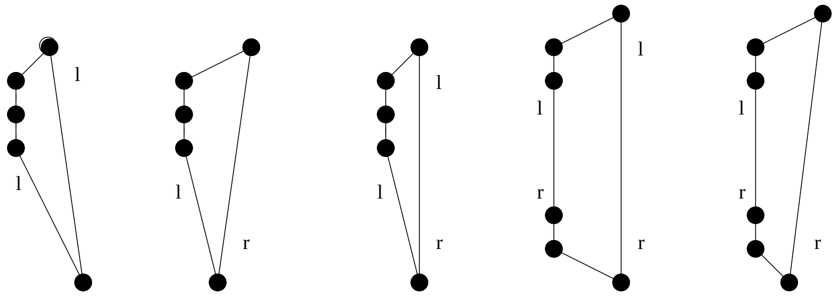


Figure 14: Drawings for face F_i for $0 < i < t$.

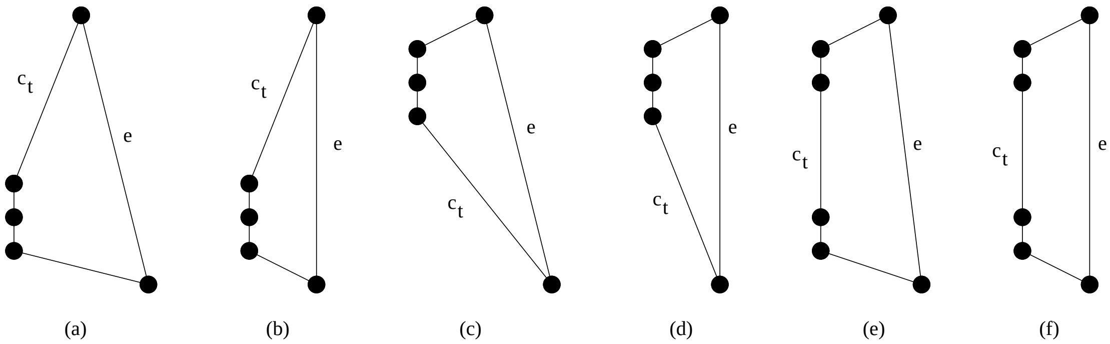


Figure 15: Face F_t.

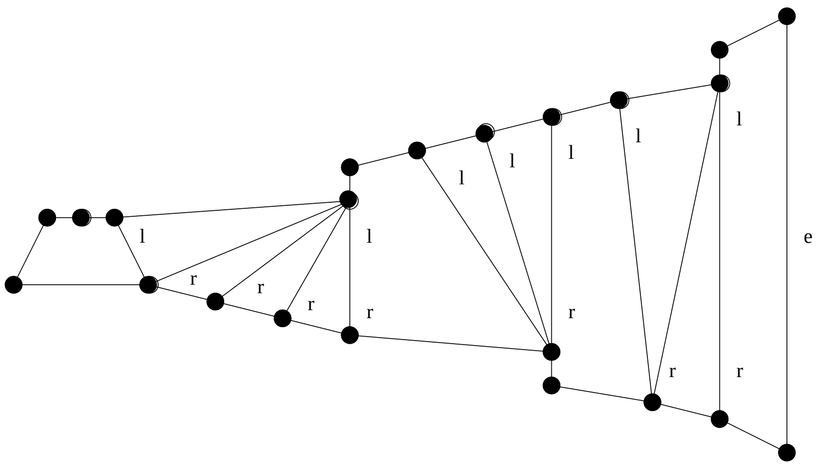


Figure 16: Closed rectangle of influence drawing of the graph in Figure 12.

Suppose that the dual tree of G has exactly three leaves. Let F be the face of G corresponding to the unique node of degree 3 in the tree. Partition the remaining faces of G into three sets A, B and C so that two faces of G go to the same set of the partition if and only if they correspond to two nodes in the same branch of the dual tree.

Let $e = (u, v)$ be a chordal edge of F separating F from a face H in the set C of the partition of faces, and let d be a boundary edge of H incident with u. Draw e with u above and to the left of v, and draw d to the right of u on the horizontal line through u. See Figure 17.

Traverse the edges of F beginning with e and proceeding next to the other edge of F incident with u. Place the edges following e to the left of u on the horizontal line through u until another chordal edge is encountered. Without loss of generality, suppose this chordal edge separates F from a face in set A of the face partition. Label this edge with a, and draw it with positive slope in such a way that its lower endpoint has y-coordinate greater than that of v.

Now place all but the last remaining edge of F below and vertically aligned with the second endpoint of the chordal edge labeled a. Label one of these edges, namely the one corresponding to the third chordal edge of F, with the label b.

Place all the vertices of H except the endpoints of e and d to the right of v on the horizontal line through v so that their x-coordinates are less than that of the right endpoint of d. If H contains a chordal edge other than e, then label this edge with the label c; this edge separates H from some other face in set C of the face partition.

Let s_a, s_b and s_c denote the segments representing the chordal edges labeled a, b and c, respectively. Apply the algorithm of Lemma 5.3 to place the remaining vertices of the faces in sets A, B and C of the face partition in strips $S_{north}(s_a)$, $S_{west}(s_b)$ and $S_{south}(s_c)$ respectively. □

Figure 19 gives the closed rectangle of influence drawing produced by applying the algorithm in the previous lemma to the graph G given in Figure 18. Note that Figure 18 gives a straight-line drawing for G that is *not* a closed rectangle of influence drawing.

Now we move from considering outerplanar graphs with at most three leaves in the dual tree to considering outerplanar graphs with more than four leaves in the dual.

Lemma 5.5 *No biconnected outerplanar graph whose dual has more than four leaves is closed RID.*

Proof: Let F be a face of G corresponding to a leaf of the dual tree. Let (x, y) be the unique chordal edge of F. Suppose G is a closed RID graph and let $RIG[P]$ be a drawing of G. We say x and y are *aligned* in a drawing if the points that represent them determine a horizontal or vertical line. We will consider drawings in which x and y are aligned and drawings in which they are not aligned.

If (x, y) is drawn on a vertical line l, then at least one vertex of F does not lie on this line. Assume without loss of generality that F has a vertex to the left of l, and let v be a left-most vertex of F in $RIG[P]$. Let H be the open, axis-aligned half-plane that lies to

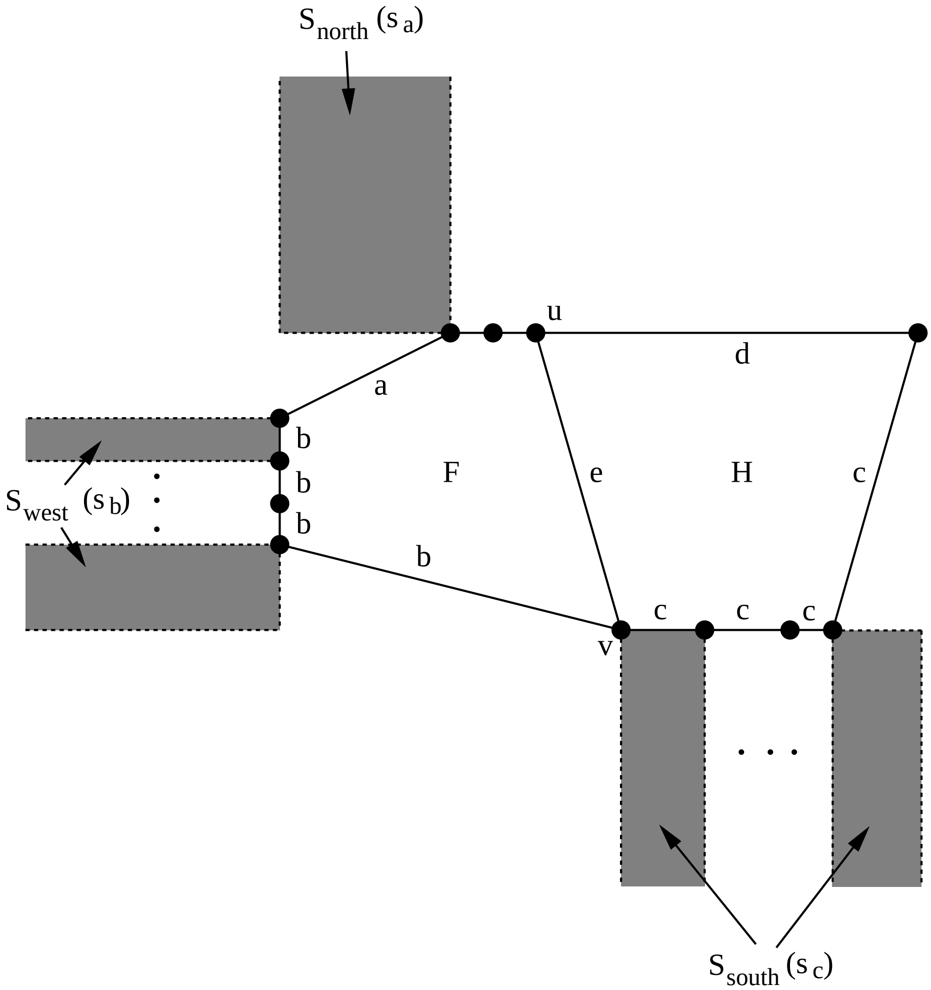


Figure 17: Face F corresponding to the dual vertex of degree 3. All possibilities for the placement of labels b and c are shown.

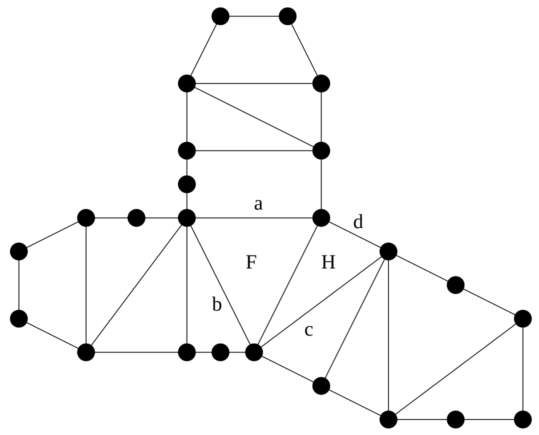


Figure 18: An outerplanar graph G whose dual tree has three leaves.

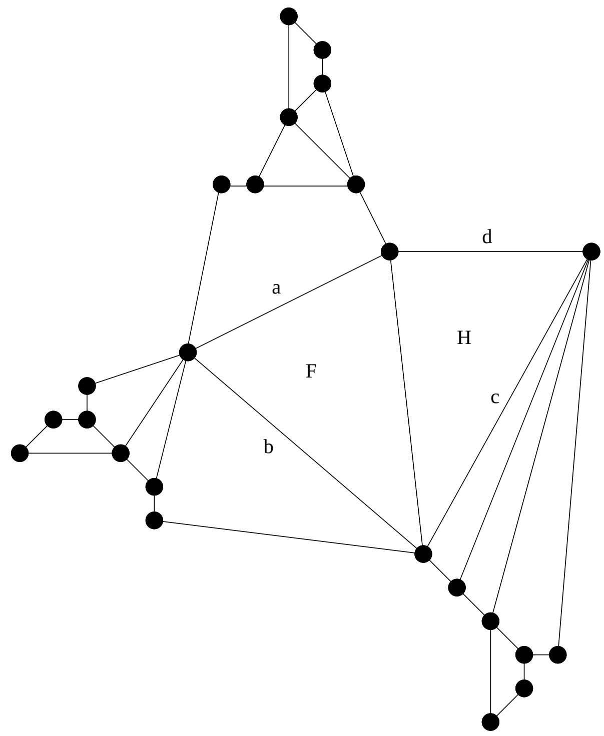


Figure 19: A closed rectangle of influence drawing of G.

the left of the vertical line through v. Suppose H contains a vertex w of G that does not belong to F. Then by Lemma 3.2, there is a path from v to w not passing through x or y. Since this is not possible, the only vertices of G that can lie in H are degree 2 vertices of F.

Similarly, if (x, y) is drawn on a horizontal line, then there is an open, axis-aligned half-plane that can only contain degree 2 vertices of F.

If x and y are not drawn aligned, the rectangle $R[x, y]$ is non-degenerate and empty. All vertices of F except x and y lie outside $R[x, y]$. Again without loss of generality, assume that F has a vertex to the left of $R[x, y]$ and let v denote a leftmost such vertex. Again the open, axis-aligned half-plane to the left of the vertical line through v can contain no vertices of G other than degree 2 vertices of F.

In the drawing $RIG[P]$ we can find for each face F corresponding to a leaf of the dual tree of G an open, axis-aligned half-plane containing only degree 2 vertices from F. Since there can be at most four such half-planes, the theorem follows. $\square$

Finally, we consider the only remaining case, namely biconnected outerplanar graphs with exactly four leaves in their duals.

Lemma 5.6 *Some biconnected outerplanar graphs whose duals have four leaves are closed RID.*

Proof: Figure 20 gives a closed rectangle of influence drawing of an outerplanar graph whose dual tree has four leaves. $\square$

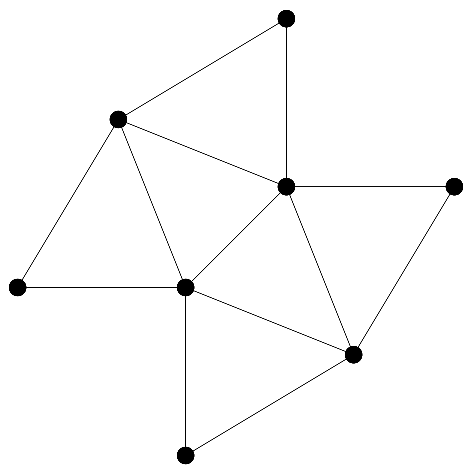


Figure 20: A closed rectangle of influence drawing of an outerplanar graph whose dual tree has four leaves.

Lemma 5.7 *Some biconnected outerplanar graphs whose duals have four leaves are not closed RID.*

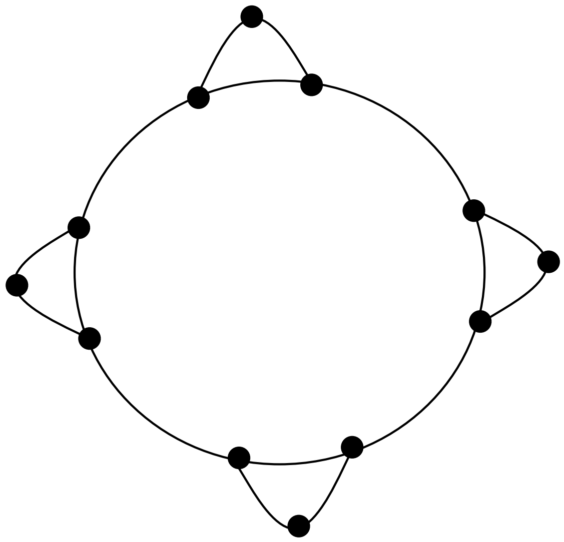


Figure 21: An outerplanar graph that is not closed RID and whose dual tree has four leaves.

Proof: Figure 21 gives an outerplanar graph with four leaves in the dual that is not closed RID. The proof that this graph does not admit a closed rectangle of influence drawing goes by a straightforward case analysis of the possible placements in the drawing for the vertices of the face that forms an 8-cycle. □

The next theorem concludes this subsection with a summary of the results.

Theorem 5.4 *If a biconnected outerplanar graph has*

- *at most three leaves in its dual, then it is a closed RID graph;*
- *exactly four leaves in its dual, then in some instances it is a closed RID graph and in other instances, it is not a closed RID graph;*
- *five or more leaves in its dual, then it is not a closed RID graph.*

5.4 Cliques

In this subsection, we consider which cliques are closed RID graphs or subgraphs of such graphs. We adopt the same notation as in Subsection 4.4.

Theorem 5.5 *K_n is closed RID if and only if $n \leq 4$.*

Proof: Let K_n be a closed RID clique, and let $RIG[P]$ be a closed rectangle of influence drawing of K_n. By the reasoning of Subsection 4.4, we may assume that the smallest axis-aligned rectangle R containing the vertices of $RIG[P]$ is non-degenerate.

Suppose that the interior $int(R)$ of R contains a vertex p of $RIG[P]$, and let w be a vertex on the left side W of R. Suppose without loss of generality that $y(w) \leq y(p)$. Let n be any vertex on N. Then $x(n) < x(p)$ or (w, n) would not be an edge of $RIG[P]$. Let e be any vertex on side E of R. Because (w, e) is an edge, e must satisfy $y(e) < y(p)$. But this contradicts the fact that (n, e) is an edge. Therefore, $int(R)$ contains no vertices, and all vertices of R lie on its sides.

Suppose some side, say N, contains two vertices n_1 and n_2, where $x(n_1) < x(n_2)$. Let w be a vertex on W. Since (w, n_2) is an edge, w must be the same point as n_1 and must lie in the top left corner of R. By a similar argument, n_2 must lie in the top right corner of R. Clearly neither W nor E can contain a second vertex. By an argument analogous to the one above, if S contained two vertices, these would have to lie in the bottom left and right corners of R. Since this is not possible, S must contain exactly one vertex, and so $n = 3$ whenever R has a side containing two vertices.

If no side of R contains two vertices, then clearly $n = 4$. This completes the proof that the condition is necessary.

It is trivial to establish the sufficiency of the condition by construction. See, for example, Theorem 5.2 for a drawing of K_3 and Theorem 5.1 for K_4. □

With the same reasoning as in Theorem 4.5 we can prove the following.

Theorem 5.6 *K_n may appear as a proper subgraph of a closed RID graph if and only if $n \leq 4$.*

6 Conclusions and Open Problems

This paper has initiated the study of the rectangle of influence drawability problem. We have both provided combinatorial characterizations of several families of graphs that admit (open or closed) rectangle of influence drawings and have presented various drawing algorithms. All the algorithms can be implemented so that they (1) produce drawings with all vertices placed at intersection points of an integer grid of size $O(n^2)$, (2) perform arithmetic operations on integers only, and (3) run in $O(n)$ time, where n is the number of vertices of the input graph.

The paper naturally leads to several questions. We list here the ones that, in our opinion, are the most relevant.

1. Except for the classes of graphs studied in the previous sections, i.e. cycles, wheels, trees, outerplanar graphs, and cliques, very little is known about which graphs admit rectangle of influence drawings. For example, it would be interesting to characterize (open or closed) RID planar graphs. A preliminary result is in [12], where it is shown that all interior faces in a planar open rectangle of influence drawing are triangles.

2. The characterization of which biconnected outerplanar graphs are closed RID is only partial. If the dual of the input graph has three leaves the graph is a closed RID; if the dual has five or more leaves, then the graph is not a closed RID. As for outerplanar graphs whose duals have four leaves, some are closed RID, some others are not. It would be interesting to complete this characterization.

3. Rectangle of influence drawings are particularly appealing proximity drawings because of their small area. However, both a theoretic and an experimental study of other quality measures (for example the angular resolution and the screen-ratio) of such drawings should be performed. A taxonomy of the main quality measures to evaluate when comparing drawings of graphs is in [28, 6].

4. Finally, it would be a challenging problem to efficiently recognize rectangle of influence drawings. In [24] it is shown that given a set P of n points, both $RIG(P)$ and $RIG[P]$ can be computed in $O(n \log n + e)$, where e is the number of edges in the output. The question is whether one can verify if a given straight-line drawing is an (open or closed) rectangle of influence drawing in less than $O(n \log n + e)$ time. Note that an immediate consequence of Theorem 4.2 is that a polygonal path in the plane is an open rectangle of influence drawing if and only if it is strictly monotone in both the x- and the y-directions, or else it is a simple path that is purely horizontal or vertical.

Acknowledgments

Some of the results of this paper were obtained while the authors were participating the International Workshop on Geometric and Visibility Representations of Graphs, co-organized by D. Rappaport and S. Whitesides and held at the Bellairs Research Institute of McGill University, February 11-17, 1994. We are grateful to the other participants of the Workshop for many useful discussions.

References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications.* Elsevier Science, New York, New York, 1976.

[2] P. Bose, W. Lenhart, and G. Liotta. Characterizing Proximity Trees. *Algorithmica: Special Issue on Graph Drawing* (to appear). Also available as TR-SOCS 93.9, School of Computer Science, McGill Univ., 1993.

[3] P. Bose, G. Di Battista, W. Lenhart, and G. Liotta. Proximity Constraints and Representable Trees. *Graph Drawing, Proc. of the DIMACS International Workshop GD'94*, New Jersey, USA, October 1994, LNCS **894**, R. Tamassia and I.G. Tollis *eds.*, Springer-Verlag, 1995, pp. 340-351.

[4] R. J. Cimikowski. Properties of Some Euclidean Proximity Graphs. *Pattern Recognition Letters,* **13**, 1992, pp. 417-423.

[5] M. de Berg, S. Carlsson, and M.H. Overmars. A General Approach to Dominance in the Plane. *Journal of Algorithms,* **13**, 1992, pp. 274-296.

[6] G. Di Battista, P. Eades, R. Tamassia and I.G. Tollis. Algorithms for Automatic Graph Drawing: An Annotated Bibliography. *Computational Geometry: Theory and Applications* **4**, 1994, pp. 235-282.

[7] G. Di Battista, W. Lenhart, and G. Liotta. Proximity Drawability: a Survey. *Graph Drawing, Proc. of the DIMACS International Workshop GD'94*, New Jersey, USA, October 1994, LNCS **894**, R. Tamassia and I.G. Tollis *eds.*, Springer-Verlag, 1995, pp. 328-339.

[8] G. Di Battista and L. Vismara. Angles of Planar Triangular Graphs. *Proceedings ACM Symposium on Theory of Computing,* 1993, pp. 431-437.

[9] M. B. Dillencourt and W. D. Smith. Graph-Theoretical Conditions for Inscribability and Delaunay Realizability. *Proceedings of the 6th Canadian Conference on Computational Geometry*, Saskatoon, Canada, 1994, pp.287-292.

[10] P. Eades and S. Whitesides. The Realization Problem for Euclidean Minimum Spanning Tree is NP-hard. *Proceedings 10th ACM Symposium on Computational Geometry*, 1994, pp. 49-56.

[11] P. Eades and S. Whitesides. Nearest Neighbor Graph Realizability is NP-hard. *Proc. Latin'95*, Valparaiso, Chile, 1995, LNCS **911**, R. Baeza-Yates, E. Goles, P. Poblete *eds.*, Springer-Verlag, 1995, pp. 245-256.

[12] H. ElGindy, G. Liotta, A. Lubiw, G. Liotta, H. Meijer, S.H. Whitesides. Recognizing Rectangle of Influence Drawable Graphs. *Graph Drawing, Proc. of the DIMACS International Workshop GD'94*, New Jersey, USA, October 1994, LNCS **894**, R. Tamassia and I.G. Tollis *eds.*, Springer-Verlag, 1995, pp. 352-363.

[13] P. Erdös and A. Szekeres. A Combinatorial Property in Geometry. *Compositio Mathematica*, **2**, 1935, pp. 463-470.

[14] H. de Fraysseix, J. Pach, and R. Pollack. Small Sets Supporting Fary Embeddings of Planar Graphs. *Proceedings ACM Symposium on Theory of Computing*, 1988, pp. 426-433.

[15] H. de Fraysseix, J. Pach, and R. Pollack. How to Draw a Planar Graph on a Grid. *Combinatorica*, **10**, 1990, pp. 41-51.

[16] M. Ichino and J. Sklansky. The Relative Neighborhood Graph for Mixed Feature Variables. *Pattern Recognition*, **18**, 1985, pp. 161-167.

[17] J. W. Jaromczyk and G. T. Toussaint. Relative Neighborhood Graphs and Their Relatives. *Proceedings of the IEEE*, **80**, 1992, pp. 1502-1517.

[18] G. Kant. Drawing Planar Graphs Using the *lmc*-ordering. *Proceedings IEEE Foundations of Computer Science '92*, 1992, pp. 101-110.

[19] D. G. Kirkpatrick and J. D. Radke. A Framework for Computational Morphology. *Computational Geometry*, G. T. Toussaint *ed.*, Elsevier, Amsterdam, 1985, pp. 217-248.

[20] P. M. Lankford. Regionalization: Theory and Alternative Algorithms. *Geographical Analysis*, **1**, 1969, pp. 196-212.

[21] G. Liotta and G. Di Battista. Computing Proximity Drawings of Trees in the 3-Dimensional Space. *Proc. of the International Workshop on Algorithms and Data structures, WADS'95*, Kingston, Canada, August 1995, LNCS **995**, S. L. Akl, F. Dehne, J. R. Sack, and N. Santoro *eds.*, Springer-Verlag, 1995, pp. 239-250.

[22] A. Lubiw and N. Sleumer. All Maximal Outerplanar Graphs are Relative Neighborhood Graphs. *Proceedings 5th Canadian Conference on Computational Geometry*, Waterloo, Canada, 1993, pp. 198-203.

[23] D. W. Matula and R. R. Sokal. Properties of Gabriel Graphs Relevant to Geographic Variation Research and the Clustering of Points in the Plane. *Geographical Analysis*, **12**, 1980, pp. 205-222.

[24] M. H. Overmars and D. Wood. On Rectangular Visibility. *Journal of Algorithms,* **9**, 1988, pp. 372-390.

[25] M. S. Paterson and F.F. Yao. On Nearest-Neighbor Graphs. *Proceedings Annual International Colloqium on Automata, Languages and Programming, ICALP'92*, LNCS **623**, Springer-Verlag, 1992, pp. 416-426.

[26] F. P. Preparata and M. I. Shamos. *Computational Geometry: an Introduction.* Springer-Verlag, New York, 1985.

[27] J. D. Radke. On the Shape of a Set of Points. *Computational Morphology*, G. T. Toussaint *ed.*, Elsevier, Amsterdam, 1988, pp. 105-136.

[28] R. Tamassia, G. Di Battista, and C. Batini. Automatic Graph Drawing and Readability of Diagrams. *IEEE Transactions on Systems, Man and Cybernetics*, **18**, 1988, pp. 61-79.

[29] G. T. Toussaint. The Relative Neighborhood Graph of a Finite Planar Set. *Pattern Recognition*, **12**, 1980, pp. 261-268.