Area Requirement of Gabriel Drawings

G. Liotta R. Tamassia I. G. Tollis
and Paola Vocca

Department of Computer Science
Brown University
Providence, Rhode Island 02912

CS-96-29
October 1996

Area Requirement of Gabriel Drawings *

(Extended Abstract)

Giuseppe Liotta
gl@cs.brown.edu
Center for Geometric Computing
Department of Computer Science
Brown University
Providence, RI 02912–1910, USA

Roberto Tamassia
rt@cs.brown.edu
Center for Geometric Computing
Department of Computer Science
Brown University
Providence, RI 02912–1910, USA

Ioannis G. Tollis
tollis@utdallas.edu
Department of Computer Science
The University of Texas at Dallas
Richardson, TX 75083–0688, USA

Paola Vocca
vocca@utovrm.it
Dipartimento di Matematica
University of Rome "Tor Vergata"
via della Ricerca Scientifica
00133, Rome, Italy

Abstract

In this paper we investigate the area requirement of proximity drawings and we prove an exponential lower bound. Namely, our main contribution is to show the existence of a class of Gabriel-drawable graphs that require exponential area for any Gabriel drawing and any resolution rule. Also, we extend the result to an infinite class of proximity drawings.

(September 11, 1996)

*Research supported in part by the National Science Foundation under grant CCR-9423847 and by the U.S. Army Research Office under grant 34990–MA–MUR. This research started while the fourth author was visiting the Computer Science Department of Brown University.

1 Introduction.

Proximity drawings of graphs have received increasing attention recently in the computational geometry and graph drawing communities due to the large number of applications where they play a crucial role. Such applications include pattern recognition and classification, geographic variation analysis, geographic information systems, computational geometry, computational morphology, and computer vision (see for example [29, 27, 12, 24, 31]).

A *proximity drawing* is a straight–line drawing where two vertices are adjacent if and only if they are *neighbors* according to some definition of *neighborhood*. One way of defining a neighborhood constraint between a pair of vertices is to use a *proximity region*, that is a suitable region of the plane having the two points on the boundary. Two vertices are adjacent if and only if the corresponding proximity region is *empty*, i.e., it does not contain any other vertex of the drawing. For example, two vertices u and v are considered to be neighbors if and only if the closed disk having u and v as antipodal points, is empty. Proximity drawings that obey this neighborhood constraint are known in the literature as *Gabriel drawings* ([14, 24]) and the closed disk is called *Gabriel disk*.

A different notion of proximity region is based on lunes instead of disks. In a *relative neighborhood drawing* ([30, 32]) two vertices u and v are adjacent if and only if the intersection of two open disks, one having center at u and the other at v, and with radius the distance between u and v, is empty.

Gabriel drawings and relative neighborhood drawings are just two examples of an infinite family of proximity drawings called β–drawings that have been first introduced by Kirkpatrick and Radke [18, 27] in the computational morphology context.

A *β–drawing* is a straight–line drawing such that there is an edge between a pair of vertices u and v if and only if the corresponding *β–lune* is empty. The β–lune is defined as the intersection of two disks whose radius depends on the value of the parameter β. For $\beta \geq 1$, the β–lune is the intersection of the two disks of radius $\beta d(u,v)/2$, where $d(u,v)$ is the distance between u and v, and centered at the points $(1-\beta/2)u + (\beta/2)v$ and $(\beta/2)u + (1-\beta/2)v$. In particular, for $\beta = 1$, the β–lune coincides with the Gabriel disk. Figure 1 depicts a set of β–lunes.

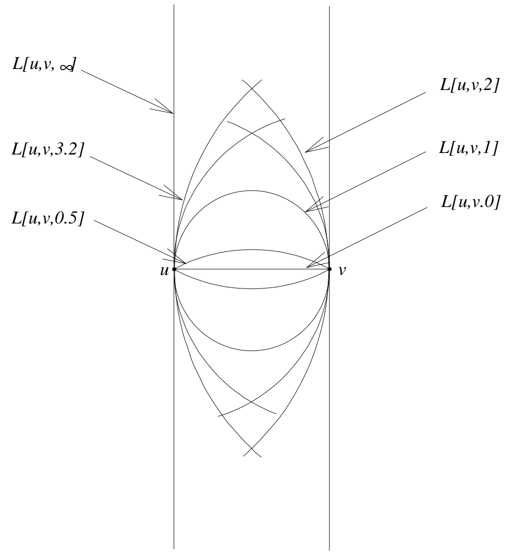


Figure 1: A set of β–lunes between vertices u and v.

A different definition of proximity drawing is given in [8]. A *weak proximity drawing* is a straight–line drawing such that if there is an edge between a pair of vertices u, v then the proximity region of u and v is empty. This definition relaxes the requirement of classical β-drawings, that for each pair of non-adjacent vertices the β-lune is not empty. In other words, if (u, v) is *not* an edge, then no requirement is placed on the proximity region of u and v. Several papers have been recently published that characterize proximity drawings and show algorithms to construct proximity drawings of different classes of graphs and different definitions of proximity [3, 23, 13, 24, 10].

In [2, 3], the problem of characterizing β–drawable trees has been addressed and an algorithm to compute Gabriel drawings and relative neighborhood drawings of trees in the plane is given. The 3–dimensional version of the same problem has been studied in [22]. Lubiw and Sleumer [23] proved that maximal outerplanar graphs are both relative neighborhood and Gabriel drawings. This result has been extended in [21] to all biconnected outerplanar graphs. Also, in [8] several algorithms to construct weak proximity drawings of different families of graphs are given. For a survey on proximity drawings see [7].

In this paper, we investigate the area requirement of proximity drawings. The finite resolution of display and printing devices requires that some constraints be placed on the drawing so that its dimensions cannot be arbitrarily scaled down. Any constraint which imposes bounds on the minimum distance between vertices and (non–incident) edges in the drawing is called a *resolution rule.* Typical resolution rules are [5, 9]: the *vertex resolution rule* which requires that any two vertices have distance at least (a constant) δ, (typically, $\delta = 1$); the *edge resolution rule* which requires that the minimum distance between any vertex and a non–incident edge is at least δ; the *angular resolution rule* which states that the vertex resolution rule is verified, and that the minimum angle between two edges incident at the same vertex is at least $\alpha(d)$, where $\alpha(d)$ is a predefined function of the maximum degree of the graph. Once the resolution rule is given, the problem of evaluating the area of a drawing naturally arises [4, 9, 16, 17, 15, 6, 19, 20, 28]. In fact, any resolution rule implies a finite minimum area for a drawing of a graph.

All known algorithms that compute (weak) proximity drawings produce representations whose area increases exponentially with the number of vertices. As a consequence, the problem of constructing proximity drawings of graphs that have small area is considered a very challenging one by several authors (see [3, 10, 24]). Additionally, the importance of this question arises in practice, by observing that several heuristics for drawing graphs often produce proximity drawings; see, for example [11].

In this paper we present the first results on area requirements of proximity drawings. Namely, we present a class of graphs whose proximity drawings require exponential area under several different definitions of proximity.

The main contributions of the paper are listed below.

1. We describe a class of graphs whose Gabriel drawing has exponential area.
2. We extend the above result to weak Gabriel drawings.
3. We show an exponential lower bound on the area of an infinite class of β–drawings, for $1 \leq \beta < \frac{1}{1-\cos 2\pi/5}$.

The rest of the paper is organized as follows. Section 2 contains basic notation and properties of Gabriel drawings. In Section 3, we describe the class of graphs; while Section 4 is completely devoted to the proof of the lower bound on the area of (weak) Gabriel drawings. Finally, in Section 5 we extend the main result to deal with a wider class of β–drawings and describe some open problems.

2 Preliminaries.

We assume familiarity with basic graph theoretic and computational geometry definitions. For more details see [1] and [26].

A *graph* $G = (V, E)$ consists of a finite non empty set $V(G)$ of *vertices*, and a set $E(G)$ of unordered pairs of vertices known as *edges*. Given an edge $e = (u, v)$, u and v are the *endpoints* of e and are said to be *adjacent* vertices.

A *simple path* of *length* k in a graph is a finite sequence $P = v_1 v_2 \dots v_k$, where $v_i \neq v_j$, for $1 \leq i < j \leq k$, and $(v_i, v_{i+1}) \in E(G)$, for $i \in \{1, \dots, k-1\}$. The vertices v_1 and v_k are the *endpoints* of the path. A *k–cycle* $C_k = v_1 v_2 \dots v_k$ is a sequence of vertices such that $P = v_1 v_2 \dots v_k v_1$ is a simple path.

A *drawing* Γ of a graph $G = (V, E)$ is a function which maps each vertex of G to a distinct point of the plane and each edge $e = (u, v)$ in G to a simple Jordan curve with endpoints the points of the plane corresponding to u and v. Γ is a *straight–line* drawing if each edge is a straight–line segment; Γ is *planar* if no two edges intersect, except possibly at their endpoints. In this paper, when it does not give rise to ambiguities, we refer to a drawing of a graph as the graph itself.

A *planar triangular graph* is an embedded planar graph so that every internal face is a 3-cycle, a *triangle*.

The *area* of a drawing Γ can be defined in several ways depending on whether we evaluate lower or upper bounds. In this paper, we define the area of Γ as the area of the smallest polygon covering Γ [9].

In our proofs, we will use several geometric objects. Let $\mathbb{R}^2$ denote the euclidean plane. Given any three distinct points $a, b, c \in \mathbb{R}^2$, $\angle abc$ denotes the counterclockwise angle between line segments $\overline{ab}$ and $\overline{bc}$; Δabc denotes the triangle whose vertices are a, b, and c.

Let P_1 be a convex pentagon. The intersections of the five diagonals of P_1 define a pentagon P_2 inside P_1 (see Figure 2). We call P_1 the *extruded pentagon* of P_2, and we denote it by $Extr(P_2)$. Conversely, P_2 is the *intruded pentagon* of P_1, denoted by $Intr(P_1)$. Let a be a vertex of P_1. Its *opposite vertex*, $op(a)$, is the vertex of $Intr(P_1)$ belonging to the region of the plane delimited by the two diagonals outgoing from a.

A *Gabriel drawing* is a planar straight–line drawing such that there is an edge between two vertices u and v if and only if the closed disk having u and v as antipodal points is empty. The closed disk is denoted as $D[u, v]$.

A *weak Gabriel drawing* is a planar straight–line drawing such that there is an edge between two vertices u and v if the closed disk having u and v as antipodal points is empty.

A given graph G is *Gabriel drawable* if it admits a Gabriel drawing.

A *maximal Gabriel graph* is a Gabriel drawable graph with the maximum number of edges.

In what follows we show some properties of (weak) Gabriel drawings.

Property 1 *A Gabriel drawing of a planar triangular graph is such that all internal faces have acute angles.*

Proof: By contradiction. Suppose a triangle $\Delta(abc)$ is such that $\angle abc \geq \frac{\pi}{2}$ then $b \in D[a, c]$. □

Property 2 [24] *In a Gabriel drawing every 3–cycle and every chordless 4–cycle is an internal face.*

Proof: By contradiction. Suppose a vertex v is drawn inside a 4–cycle $C_4 = abcd$. Then at least one of the angles $\angle bva$, $\angle cvb$, $\angle dvc$, and $\angle dva$ is greater than or equal to $\frac{\pi}{2}$. Say, $\angle bva \geq \frac{\pi}{2}$ then $b \in D[a, b]$. A similar reasoning holds for a 3–cycle.

□

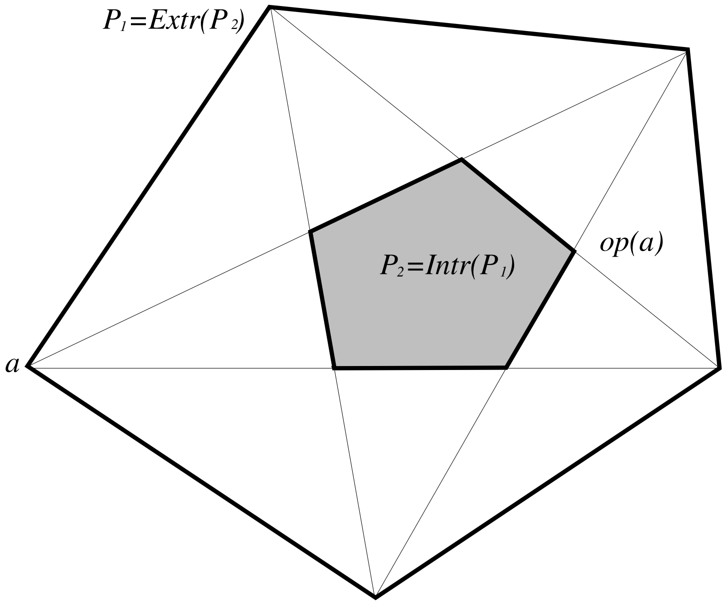


Figure 2: Extruded and Intruded Pentagons.

Property 3 *A weak Gabriel drawing of a maximal Gabriel graph is also a Gabriel drawing.*

Proof: The proof easily follows from the definitions of weak Gabriel and Gabriel drawings and from maximality condition. □

3 Description of the class of Graphs

In this section we exhibit a class of planar graphs which require exponential area in any Gabriel drawing.

Let us define the following class of graphs (see Figure 3). Graph G_1, shown in Figure 3 (a), is a 5-vertex wheel graph consisting of vertices $v_0^1, v_1^1, v_2^1, v_3^1, v_5^1$, and v^*, and edges (v^*, v_i^1) and $(v_i^1, v_{(i+1) \bmod 5}^1)$, for $i \in \{0, \dots, 4\}$. For $n \geq 2$, G_n is constructed from G_{n-1} by adding vertices $v_0^n, v_1^n, v_2^n, v_3^n, v_4^n$, and edges $(v_i^n, v_{(i+1) \bmod 5}^n)$, (v_i^n, v_i^{n-1}), and $(v_i^n, v_{(i+1) \bmod 5}^{n-1})$, where $i \in \{0, \dots, 4\}$ as shown in Figure 3(b). We refer to the 5–cycle composed by $v_0^n v_1^n v_2^n v_3^n v_4^n$ as F_5^n.

It is easy to verify that G_n is a planar triangular graph, with $5n+1$ vertices, and $15n-5$ edges. Also, G_n is triconnected.

The following two results are shown in [24] (see Figure 4 for a drawing of G_2):

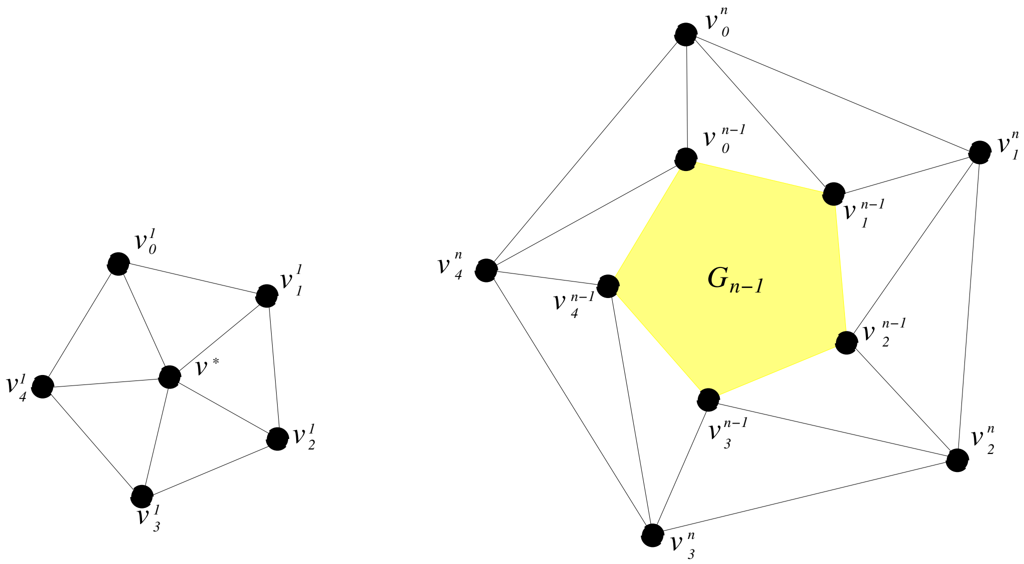
Lemma 1 [24] *G_n is Gabriel drawable.*

Lemma 2 [24] *G_n is a maximal Gabriel graph.*

Lemma 3 *In a Gabriel drawing of G_n the external face is a 5–cycle.*

Sketch of Proof: As G_n is triconnected, its planar embedding is fixed for any given choice of the external face. From Property 2, the external face of a Gabriel drawing of G_n can only be the 5–cycle $F_5^n = v_0^n v_1^n v_2^n v_3^n v_4^n$. □

Lemma 4 *In a Gabriel drawing of G_n ($n \geq 2$), for all $1 \leq i \leq n-1$, F_5^i is a strictly convex pentagon.*



(a) (b)

Figure 3: A class of graphs that require exponential area.

Sketch of Proof: We prove the lemma by contradiction. Suppose F_5^{n-1} is drawn as a concave pentagon (see Figure 5). Let $\angle v_0^{n-1}v_1^{n-1}v_2^{n-1} \geq \pi$. Since the sum of the external angles of a pentagon is equal to 7π, then there exists at least one external angle, say $\angle v_1^{n-1}v_0^{n-1}v_4^{n-1}$, such that $\angle v_1^{n-1}v_0^{n-1}v_4^{n-1} \geq \frac{3}{2}\pi$. Hence, at least one of the angles $\angle v_1^{n-1}v_0^{n-1}v_0^{n}$, $\angle v_0^{n}v_0^{n-1}v_4^{n}$, and $\angle v_4^{n}v_0^{n-1}v_4^{n-1}$ is greater than or equal to $\frac{\pi}{2}$. Thus, at least one of the triangles $\Delta(v_1^{n-1}v_0^{n-1}v_0^{n})$, $\Delta(v_0^{n}v_0^{n-1}v_4^{n})$, $\Delta(v_4^{n}v_0^{n-1}v_4^{n-1})$ has a non acute angle. This contradicts Property 1. □

4 Area Requirement

In this section, we prove that a (weak) Gabriel drawing of G_n requires exponential area. Before showing the main result, we need a preliminary technical lemma.

Let two straight lines in general position (not parallel) l_1 and l_2 be given; let a and b two points on l_1, and c and d two points on l_2, such that a is to the left of b and c is to the left of d (see Figure 6). Suppose that the crossing point of l_1 and l_2 lays to the left of a and c. Let o be the crossing point of the line through a and d and of the line through c and b. Then we have:

Lemma 5 *The area of the triangle $\Delta(aoc)$ is smaller than the area of triangle $\Delta(bod)$.*

Sketch of Proof: Triangles $\Delta(abc)$ and $\Delta(abd)$ have the same basis $\overline{ab}$, while the height of $\Delta(abc)$ is smaller than the height of $\Delta(abd)$. Hence, the area of $\Delta(abc)$ is smaller than the area $\Delta(abd)$. Since they share $\Delta(abo)$, we have the proof. □

Theorem 1 *Given any resolution rule, both a Gabriel drawing and a weak Gabriel drawing of G_n, with $5n+1$ vertices requires area $\Omega(3^n)$.*

Sketch of Proof: Let A_n be the minimum area of a Gabriel drawing of G_n. We use induction to prove that $A_n \geq 3A_{n-1}$. Since $A_1 \geq c$, for some constant c depending on the resolution rule, this implies the claimed result.

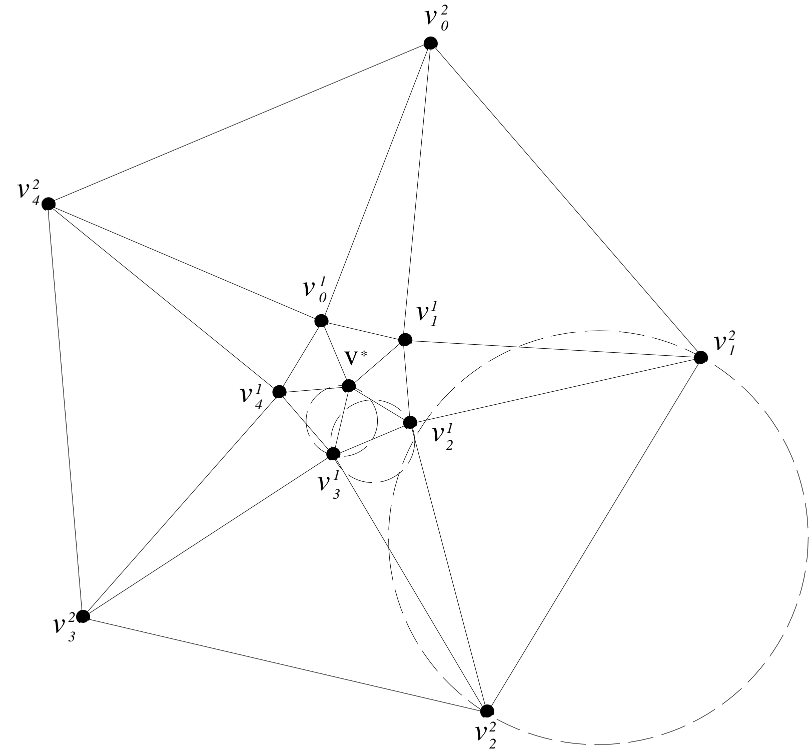


Figure 4: A Gabriel drawing of G_2.

We prove the theorem in two steps. We start by showing that $Area(\Gamma_n) \geq 2Area(\Gamma_{n-1})$. Then we prove that $Area(\Gamma_n) \geq 3Area(\Gamma_{n-1})$. For the sake of simplicity of presentation, unless stated otherwise, we assume in the following that the index $i \in \{0, \ldots, 4\}$ and all the operations on the indexes are modulo 5.

Let Γ_{n-1} be a Gabriel drawing of G_{n-1}. We show how to produce a drawing Γ_n of G_n starting from Γ_{n-1}. First, we need some more notation, described in Figure 7. Let B_i denote the region of the plane delimited by the two lines p_i and p_{i+1} through v_i^{n-1} and v_{i+1}^{n-1}, perpendicular to the edge $(v_i^{n-1}, v_{i+1}^{n-1})$. Moreover, let $HP(v_i^{n-1})$ be the the half plane delimited by the line through the edge $(v_{i-1}^{n-1}, v_i^{n-1})$ and not containing Γ_{n-1}. Finally, let $C_i = B_i \cap HP(v_i^{n-1}) \cap HP(v_{i+2}^{n-1})$.

We show that v_i^n must be placed inside C_i. Let us suppose by contradiction that v_i^n does not belong to C_i. Three cases are possible:

1. $v_i^n \notin B_i$. Let v_i^n be placed to the right of p_i. Then $\angle v_{i+1}^{n-1} v_i^{n-1} v_i^n > \frac{\pi}{2}$. Hence, from Property 1, edge (v_{i+1}^{n-1}, v_i^n) cannot be drawn. Similarly, v_i^n cannot be placed to the left of p_{i+1}.
2. $v_i^n \notin HP(v_i^{n-1})$. According to the above condition, each vertex v_i^n is placed in the corresponding B_i region. Thus, in this case, $v_i^{n-1} \in D[v_i^n, v_{i-1}^n]$. Hence, edge (v_i^n, v_{i-1}^n) cannot be drawn.
3. $v_i^n \notin HP(v_{i+2}^{n-1})$. Similar to the previous case, $v_{i+1}^{n-1} \in D[v_i^n, v_{i+1}^n]$. Hence, edge (v_i^n, v_{i+1}^n)

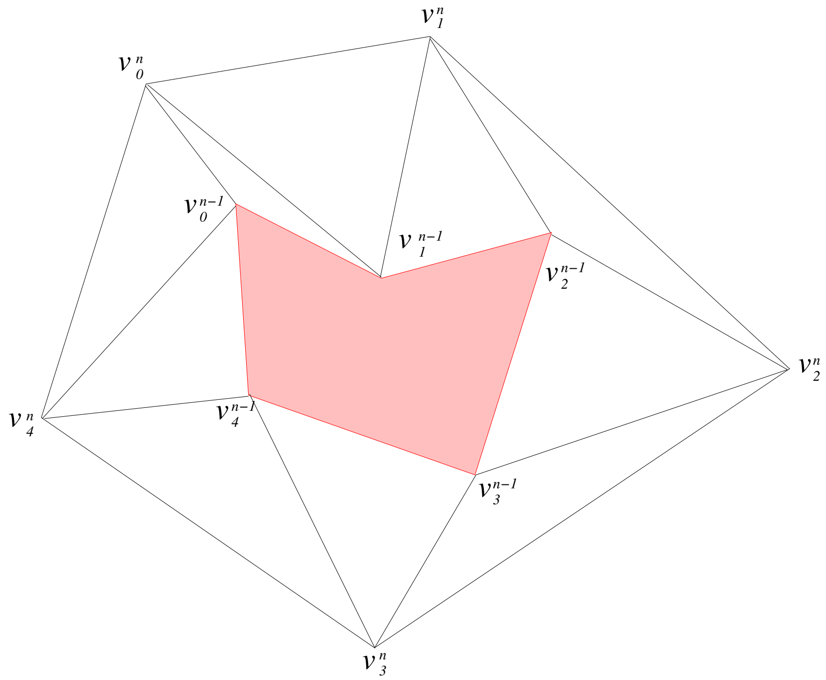


Figure 5: Illustration for Theorem 4.

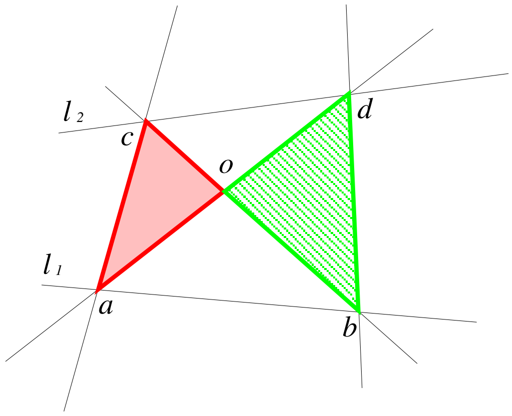


Figure 6: Illustration for Lemma 5.

cannot be drawn.

From the above discussion, it follows that a Gabriel drawing Γ_n of G_n must strictly contain $Extr(F_5^{n-1})$ (see Figure 8), where $Extr(F_5^{n-1})$ is the extruded pentagon of the external face F_5^{n-1} of G_{n-1}.

In order to prove the claim, we show that $Area(Extr(F_5^{n-1}))$ is at least three times $Area(\Gamma_{n-1})$ which implies that $Area(\Gamma_n) > 2Area(\Gamma_{n-1})$.

First we need some more notation (see Figure 8). Let $w_0^n, w_1^n, w_2^n, w_3^n, w_4^n$ be the vertices of $Extr(F_5^{n-1})$, such that $v^{n-1}_{(i+3)\mathrm{mod}5}$ is the opposite vertex of w_i^n.

We denote by P_i^n, the triangle $\Delta(v_i^{n-1}, v_{i+1}^{n-1}, w_i^n)$. Also, we denote by W_i^n, the triangle $\Delta(v_i^{n-1}, w_{i-1}^n, w_i^n)$.

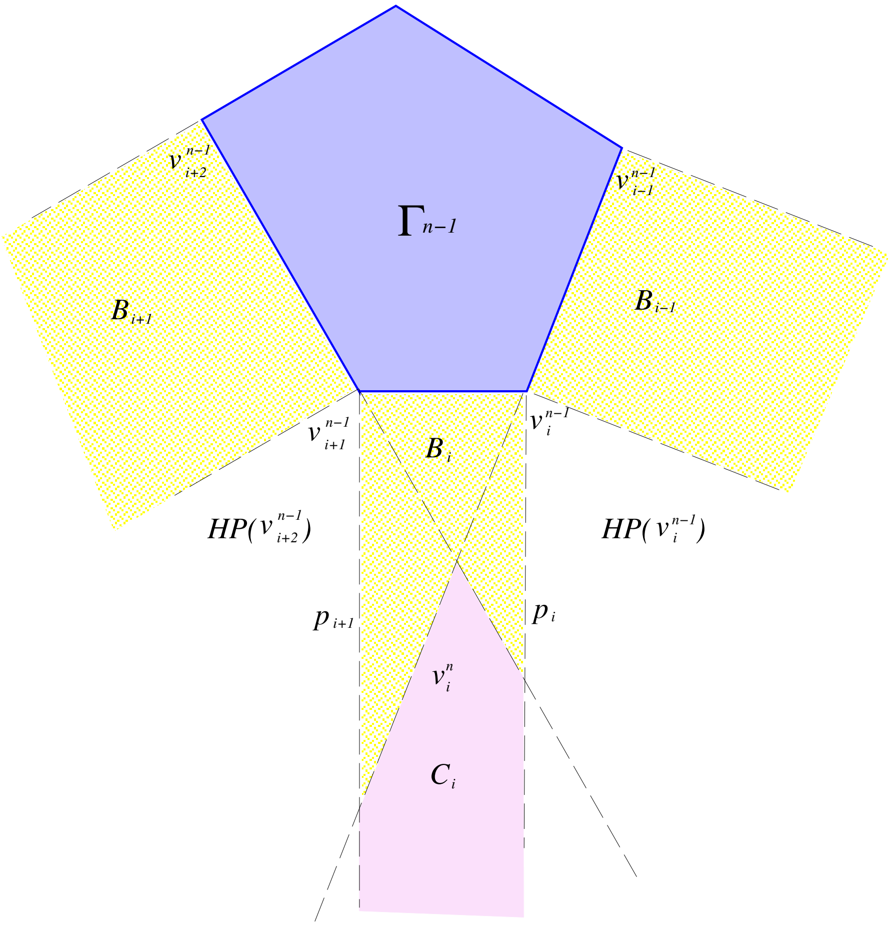


Figure 7: Construction of Γ_n starting from Γ_{n-1}.

The proof of the claim proceeds in two steps. First we show that $Area(\bigcup_{i=0}^{4} P_i^n) > Area(\Gamma_{n-1})$ and then that $Area(\bigcup_{i=0}^{4} W_i^n) > Area(\Gamma_{n-1})$. Hence, the proof follows observing that $Area(Extr(F_5^{n-1})) = Area(\bigcup_{i=0}^{4} P_i^n) + Area(\bigcup_{i=0}^{4} W_i^n) + Area(\Gamma_{n-1})$.

We show that $Area(\bigcup_{i=0}^{4} P_i^n)$ is at least as large as $Area(\Gamma_{n-1})$. Let us consider the line segments having as endpoints vertices of $Extr(F_5^{n-1})$ and the corresponding opposite vertices of F_5^{n-1}. These segments subdivide each P_i^n in two triangles. With reference to Figure 9(a), let us consider the two triangles $\Delta(COD)$ and $\Delta(AOB)$. Due to Lemma 4, line segments ED and EB and points C, D, A, and B satisfy the conditions of Lemma 5, hence, $Area(\Delta(COD))$ is greater than $Area(\Delta(AOB))$. A similar reasoning holds for all P_i^n. It is well known that in any arrangement of lines, every internal region is a simply connected region. Thus, $Area(\bigcup_{i=0}^{4} P_i^n) > Area(\Gamma_{n-1})$.

As a next step, prove that $Area(\bigcup_{i=0}^{4} W_i^n) > Area(\Gamma_{n-1})$. Consider the pentagon $Intr(F_5^{n-1})$. We extend the notation to the intruded pentagon, so defining the P_i^{n-1}'s and W_i^{n-1}'s regions.

As above, applying Lemmas 4 and 5, it is possible to derive that $Area(W_i^n) > Area(W_i^{n-1}) + Area(P_i^{n-1}) + Area(W_{i+1}^{n-1})$. Hence, $Area(\bigcup_{i=0}^{4} W_i^n) > Area(\bigcup_{i=0}^{4} P_i^{n-1}) + 2Area(\bigcup_{i=0}^{4} W_i^{n-1})$. As $\bigcup_{i=0}^{4} W_i^{n-1}$ is covered twice it is possible to recursively apply the above strategy, to $Intr(Intr(F_5^{n-1}))$.

From the first part of the proof we have that $Area(Extr(F_5^{n-1}))$ is at least twice $Area(\Gamma_{n-1})$. Hence, the recursion ends after a finite number of steps, proving that $Area(\bigcup_{i=0}^{4} W_i^n) >$

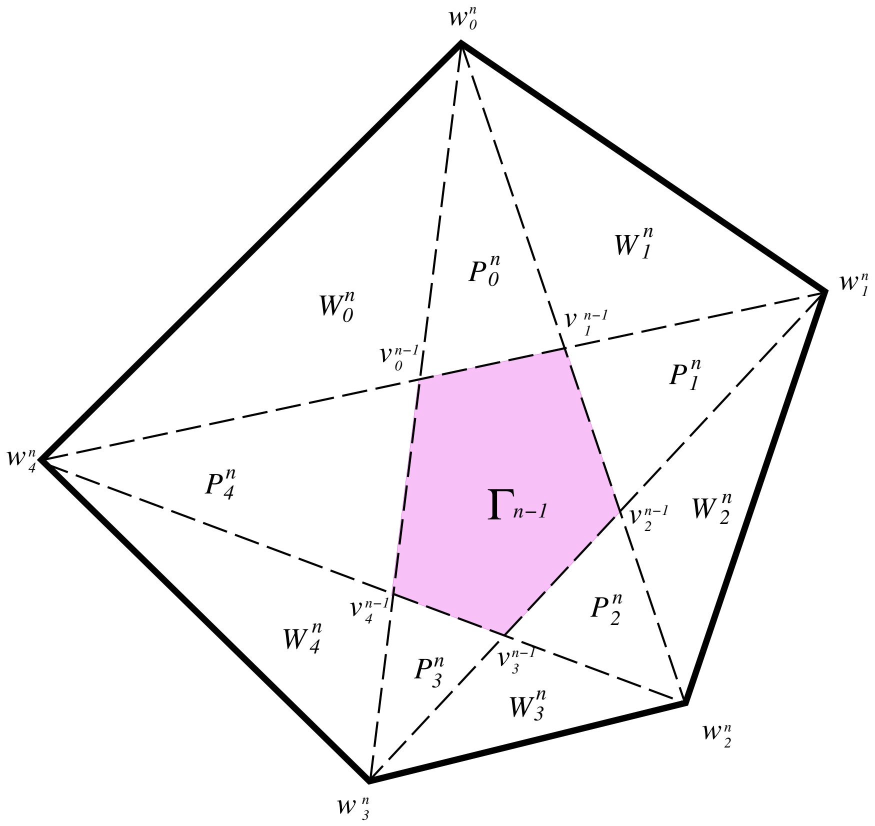


Figure 8: Extruded pentagon of F_5^{n-1}.

$Area(\Gamma_{n-1})$.

As $\bigcup_{i=0}^{4} W_i^n$ and $\bigcup_{i=0}^{4} P_i^n$ are disjoint, combining the two results we prove the first part of the claim.

The proof for the weak Gabriel drawings follows in a straightforward manner from Property 3 and Lemma 2.

□

5 Extensions and Open Problems

In this section we extend the result of Theorem 1 to an infinite family of (β)–drawings.

Theorem 2 *Given any resolution rule, both a β–drawing and a weak β–drawing of G_n, with $5n+1$ vertices requires area $\Omega(3^n)$, for $1 \leq \beta < \frac{1}{1-\cos 2\pi/5}$.*

Sketch of Proof: We first show that G_n is β-drawable for $1 \leq \beta < \frac{1}{1-\cos 2\pi/5}$.

Let us consider the following drawing of G_n. Draw v^* as the origin point and the n sets of vertices $v_0^i, v_1^i, v_2^i, v_3^i, v_4^i$, for $i \in \{1, \ldots, n\}$, as equally spaced points on the boundaries of concentric circles, each circle having $k > 1/(\cos \pi/5 - (\sin \pi/5)\sqrt{2\beta - 1})$ times the diameter of the preceding one, and each set of five points are $36°$ out of phase with respect to their predecessors. Using basic geometry, it is readily verified that the obtained drawing is a (weak) β–drawing.

Based on the fact that a β–lune contains the Gabriel disk, for $1 \leq \beta < \frac{1}{1-\cos 2\pi/5}$, it can be proved that the β–drawing is also a Gabriel drawing. Hence, the claim follows from Theorem 1. □

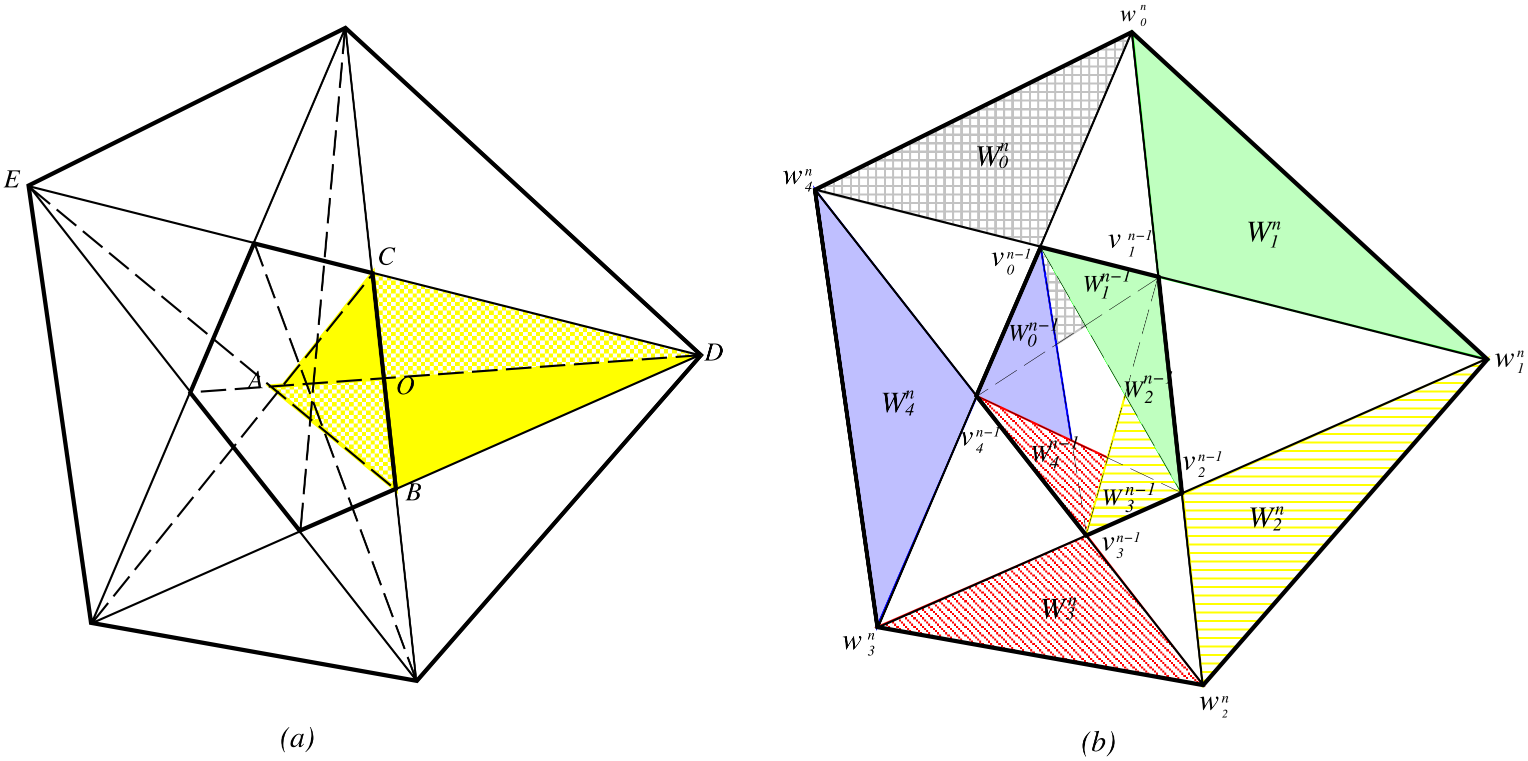


Figure 9: Covering of Γ_{n-1}.

Several problems remain open in this area: (1) Study the area of proximity drawings using different definitions of proximity. For example, we find interesting to investigate the area of relative neighborhood drawings, and minimum spanning trees. For what concerns minimum spanning trees, note that Monma and Suri [25] conjectured an exponential lower bound on the area requirement. (2) Motivated by our exponential lower bounds, it is interesting to investigate classes of graphs that admit a (weak) Gabriel drawing with polynomial area.

References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications.* North-Holland, New York, NY, 1976.

[2] P. Bose, G. Di Battista, W. Lenhart, and G. Liotta. Proximity constraints and representable trees. In R. Tamassia and I. G. Tollis, editors, *Graph Drawing (Proc. GD '94)*, volume 894 of *Lecture Notes in Computer Science*, pages 340–351. Springer-Verlag, 1995.

[3] P. Bose, W. Lenhart, and G. Liotta. Characterizing proximity trees. *Algorithmica.* (special issue on Graph Drawing, edited by G. Di Battista and R. Tamassia, to appear).

[4] R. P. Brent and H. T. Kung. On the area of binary tree layouts. *Inform. Process. Lett.*, 11:521–534, 1980.

[5] Marek Chrobak, Michael T. Goodrich, and Roberto Tamassia. Convex drawings of graphs in two and three dimensions. In *Proc. 12th Annu. ACM Sympos. Comput. Geom.*, pages 319–328, 1996.

[6] P. Crescenzi, G. Di Battista, and A. Piperno. A note on optimal area algorithms for upward drawings of binary trees. *Comput. Geom. Theory Appl.*, 2:187–200, 1992.

[7] G. Di Battista, W. Lenhart, and G. Liotta. Proximity drawability: a survey. In R. Tamassia and I. G. Tollis, editors, *Graph Drawing (Proc. GD '94)*, volume 894 of *Lecture Notes in Computer Science*, pages 328–339. Springer-Verlag, 1995.

[8] G. Di Battista, G. Liotta, and S. H. Whitesides. The strength of weak proximity. In F. J. Brandenburg, editor, *Graph Drawing (Proc. GD '95)*, volume 1027 of *Lecture Notes in Computer Science*, pages 178–189. Springer-Verlag, 1996.

[9] G. Di Battista, R. Tamassia, and I. G. Tollis. Area requirement and symmetry display of planar upward drawings. *Discrete Comput. Geom.*, 7:381–401, 1992.

[10] P. Eades and S. Whitesides. The realization problem for Euclidean minimum spanning trees is NP-hard. *Algorithmica.* (special issue on Graph Drawing, edited by G. Di Battista and R. Tamassia, to appear).

[11] P. D. Eades. Drawing free trees. *Bulletin of the Institute for Combinatorics and its Applications*, 5:10–36, 1992.

[12] H. Edelsbrunner, D. G. Kirkpatrick, and R. Seidel. On the shape of a set of points in the plane. *IEEE Trans. Inform. Theory*, IT-29:551–559, 1983.

[13] H. ElGindy, G. Liotta, A. Lubiw, H. Meijer, and S. H. Whitesides. Recognizing rectangle of influence drawable graphs. In R. Tamassia and I. G. Tollis, editors, *Graph Drawing (Proc. GD '94)*, volume 894 of *Lecture Notes in Computer Science*, pages 352–363. Springer-Verlag, 1995.

[14] K. R. Gabriel and R. R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Zoology*, 18:259–278, 1969.

[15] A. Garg, M. T. Goodrich, and R. Tamassia. Area-optimal upward tree drawings. *Internat. J. Comput. Geom. Appl.* to appear.

[16] A. Garg, M. T. Goodrich, and R. Tamassia. Area-efficient upward tree drawings. In *Proc. 9th Annu. ACM Sympos. Comput. Geom.*, pages 359–368, 1993.

[17] G. Kant, G. Liotta, R. Tamassia, and I. Tollis. Area requirement of visibility representations of trees. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 192–197, Waterloo, Canada, 1993.

[18] D. G. Kirkpatrick and J. D. Radke. A framework for computational morphology. In G. T. Toussaint, editor, *Computational Geometry*, pages 217–248. North-Holland, Amsterdam, Netherlands, 1985.

[19] M. R. Kramer and J. van Leeuwen. The complexity of wire-routing and finding minimum area layouts for arbitrary VLSI circuits. In F. P. Preparata, editor, *Advances in Computing Research*, volume 2, pages 129–146. JAI Press, Greenwich, Conn., 1985.

[20] C. E. Leiserson. Area-efficient graph layouts (for VLSI). In *Proc. 21st Annu. IEEE Sympos. Found. Comput. Sci.*, pages 270–281, 1980.

[21] W. Lenhart and G. Liotta. Proximity drawings of outerplanar graphs. In *Graph Drawing (Proc. GD '96)*, Lecture Notes in Computer Science. Springer-Verlag, 1996.

[22] Giuseppe Liotta and Giuseppe Di Battista. Computing proximity drawings of trees in the 3-dimemsional space. In *Proc. 4th Workshop Algorithms Data Struct.*, volume 955 of *Lecture Notes in Computer Science*, pages 239–250. Springer-Verlag, 1995.

[23] A. Lubiw and N. Sleumer. Maximal outerplanar graphs are relative neighborhood graphs. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 198–203, Waterloo, Canada, 1993.

[24] D. W. Matula and R. R. Sokal. Properties of Gabriel graphs relevant to geographic variation

research and clustering of points in the plane. *Geogr. Anal.*, 12:205–222, 1980.

[25] C. Monma and S. Suri. Transitions in geometric minimum spanning trees. *Discrete Comput. Geom.*, 8:265–293, 1992.

[26] F. P. Preparata and M. I. Shamos. *Computational Geometry: An Introduction.* Springer-Verlag, New York, NY, 1985.

[27] J. D. Radke. On the shape of a set of points. In G. T. Toussaint, editor, *Computational Morphology*, pages 105–136. North-Holland, Amsterdam, Netherlands, 1988.

[28] C. D. Thompson. Area-time complexity for VLSI. In *Proc. 11th Annu. ACM Sympos. Theory Comput.*, pages 81–88, 1979.

[29] G. T. Toussaint. Pattern recognition and geometrical complexity. In *Proc. 5th IEEE Internat. Conf. Pattern Recogn.*, pages 1324–1347, 1980.

[30] G. T. Toussaint. The relative neighbourhood graph of a finite planar set. *Pattern Recogn.*, 12:261–268, 1980.

[31] G. T. Toussaint. A graph-theoretical primal sketch. In G. T. Toussaint, editor, *Computational Morphology*, pages 229–260. North-Holland, Amsterdam, Netherlands, 1988.

[32] R. B. Urquhart. Graph theoretical clustering based on limited neighbourhood sets. *Pattern Recogn.*, 15:173–187, 1982.